Simplest Quantum Mechanics
Novel Introduction to Orthogonal Polynomials of a Discrete Variable in the Askey Scheme

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Outline

Background

Similarity:

- Quantum Mechanics (QM),
- Eigenvalue problem of hermitian matrices – Much simpler

QM: Self-adjoint operator, $\mathcal{H}\psi(x) = \mathcal{E}\psi(x)$, $\mathcal{H}^\dagger = \mathcal{H}$,

Exactly solvable 1-d QM, $\mathcal{H} \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + V(x)$

$\Longrightarrow$ **Classical Orthogonal Polynomials** as eigenfunctions, Hermite, Laguerre and Jacobi

Q1: *Is there a special class of hermitian matrices corresponding to the 1-d QM?* **Yes, Simplest QM**

Q2: *Are there many* exactly solvable *ones whose eigenvectors define orthogonal polynomials?* **Yes, Many**

Criticism

Analysis of Linear Operators in Hilbert Space (= finite or infinite matrices) is the main theme of Functional Analysis. However, existing literature (Stone, Kolmogorov, Dunford-Schwartz, Simon etc) failed to tackle this very interesting problem (*i.e.* analysis of Jacobi operators and simplest Quantum Mechanics). The strategy of the discipline (Functional Analysis) should be criticised. See Comments later.

Exactly Solvable 1-d QM

- $V(x) = x^2 - 1,\ -\infty < x < \infty \Longrightarrow \mathcal{E}(n) = 2n,$
  $\psi_n(x) = e^{-x^2/2} H_n(x),\ n = 0, 1, \ldots,$ Hermite polynomial
- $V(x) = x^2 + g(g-1)/x^2 - (1+2g),\ g > 1,\ 0 < x < \infty \Longrightarrow$
  $\mathcal{E}(n) = 4n,\ \psi_n(x) = e^{-x^2/2} x^g L_n^{(g-1/2)}(x^2),\ n = 0, 1, \ldots,$
  Laguerre polynomial
- $V(x) = g(g-1)/\sin^2 x + h(h-1)/\cos^2 x - (g+h)^2,$
  $g, h > 1,\ 0 < x < \pi/2 \Longrightarrow \mathcal{E}(n) = 4n(n+g+h),$
  $\psi_n(x) = (\sin x)^g (\cos x)^h P_n^{(g-1/2,h-1/2)}(\cos 2x),\ n = 0, 1, \ldots,$
  Jacobi polynomial

Factorised Hamiltonian (hermite matrix)

denote the indices of the matrices by $x, y = 0, 1, \ldots x_{\text{max}}$, $x_{\text{max}} = N$ (finite) or $x_{\text{max}} = \infty$.
Matrix eigenvalue problem

$$\sum_{y=0}^{x_{\text{max}}} \mathcal{H}_{x,y} v_y = \lambda v_x, \quad \text{or} \quad \mathcal{H} v(x) = \lambda v(x), \quad \sum_{y=0}^{x_{\text{max}}} \mathcal{H}_{x,y} v(y) = \lambda v(x),$$

Assume: discrete eigenvalues only, bounded from below spectrum

$$\mathcal{H}\phi_n(x) = \mathcal{E}(n)\phi_n(x), \ 0 = \mathcal{E}(0) \le \mathcal{E}(1) \le \mathcal{E}(2) \le \cdots .$$

$\mathcal{H}$: Positive semi-definite$\Rightarrow$ $\mathcal{H} = \mathcal{A}^\dagger \mathcal{A}, \quad 0 = \det(\mathcal{A}) = \det(\mathcal{H})$

Factorised Hamiltonian in 1-d QM

groundstate eigenfunction $\phi_0(x)$, having no zero

$$-\frac{d^2\phi_0(x)}{dx^2} + V(x)\phi_0(x) = 0$$

$$\Rightarrow V(x) = \frac{\partial_x^2\phi_0(x)}{\phi_0(x)} = \partial_x\left(\frac{\partial_x\phi_0(x)}{\phi_0(x)}\right) + \left(\frac{\partial_x\phi_0(x)}{\phi_0(x)}\right)^2$$

Factorised Hamiltonian

$$\mathcal{H} = \mathcal{A}^\dagger\mathcal{A} = -\frac{d^2}{dx^2} + V(x), \quad \mathcal{A}\phi_0(x) = 0 \ \Rightarrow \mathcal{H}\phi_0(x) = 0,$$

$$\mathcal{A} \stackrel{\text{def}}{=} \frac{d}{dx} - \frac{\partial_x\phi_0(x)}{\phi_0(x)}, \quad \mathcal{A}^\dagger = -\frac{d}{dx} - \frac{\partial_x\phi_0(x)}{\phi_0(x)}$$

Matrix Analogue of 1-d QM Factorised Hamiltonian

$$\mathcal{A}=\sqrt{B(x)}-e^{\partial}\sqrt{D(x)},\ \mathcal{A}^{\dagger}=\sqrt{B(x)}-\sqrt{D(x)}\,e^{-\partial},$$
$$\mathcal{A}_{x,y}=\sqrt{B(x)}\,\delta_{x,y}-\sqrt{D(x+1)}\,\delta_{x+1,y},$$
$$(\mathcal{A}^{\dagger})_{x,y}=\sqrt{B(x)}\,\delta_{x,y}-\sqrt{D(x)}\,\delta_{x-1,y},\ \ B(x),D(x)>0,$$

$\mathcal{H}$: Tri-Diagonal Real Symmetric Simple Spectrum

$$\mathcal{H}=B(x)+D(x)-\sqrt{B(x)}\,e^{\partial}\sqrt{D(x)}-\sqrt{D(x)}\,e^{-\partial}\sqrt{B(x)},$$
$$\mathcal{H}_{x,y}=\big(B(x)+D(x)\big)\delta_{x,y}-\sqrt{B(x)D(x+1)}\,\delta_{x+1,y}$$
$$-\sqrt{B(x-1)D(x)}\,\delta_{x-1,y}$$

Non appearance of $\psi(-1)$ or $\psi(N+1)\Rightarrow D(0)=0,\ B(N)=0$

Explicit Form of the Tri-Diagonal Hamiltonian (Jacobi Matrix)

$$\mathcal{H} = \begin{pmatrix} B(0) & -\sqrt{B(0)D(1)} & 0 & \cdots & \cdots & 0 \\ -\sqrt{B(0)D(1)} & B(1)+D(1) & -\sqrt{B(1)D(2)} & 0 & \cdots & \vdots \\ 0 & -\sqrt{B(1)D(2)} & B(2)+D(2) & -\sqrt{B(2)D(3)} & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & -\sqrt{B(N-2)D(N-1)} & B(N-1)+D(N-1) & -\sqrt{B(N-1)D(N)} \\ 0 & \cdots & \cdots & 0 & -\sqrt{B(N-1)D(N)} & D(N) \end{pmatrix}$$

Explicit Form of the Factorisation Matrix $\mathcal{A}$

$$
\mathcal{A} = \begin{pmatrix}
\sqrt{B(0)} & -\sqrt{D(1)} & 0 & \cdots & \cdots & 0 \\
0 & \sqrt{B(1)} & -\sqrt{D(2)} & 0 & \cdots & \vdots \\
0 & 0 & \sqrt{B(2)} & -\sqrt{D(3)} & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & \sqrt{B(N-1)} & -\sqrt{D(N)} \\
0 & \cdots & \cdots & 0 & 0 & 0
\end{pmatrix}
$$

Explicit Form of the Factorisation Matrix $\mathcal{A}^\dagger$

$$\mathcal{A}^\dagger = \begin{pmatrix} \sqrt{B(0)} & 0 & 0 & \cdots & \cdots & 0 \\ -\sqrt{D(1)} & \sqrt{B(1)} & 0 & 0 & \cdots & \vdots \\ 0 & -\sqrt{D(2)} & \sqrt{B(2)} & 0 & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & -\sqrt{D(N-1)} & \sqrt{B(N-1)} & 0 \\ 0 & \cdots & \cdots & 0 & -\sqrt{D(N)} & 0 \end{pmatrix}$$

Eigenvectors

groundstate eigenvector $\phi_0(x)$, $\mathcal{H}\phi_0(x) = 0$,

$$\mathcal{A}\phi_0(x) = 0 \Rightarrow \sqrt{B(x)}\phi_0(x) - \sqrt{D(x+1)}\phi_0(x+1) = 0,$$
$$\Rightarrow \phi_0(x) = \prod_{y=0}^{x-1}\sqrt{\frac{B(y)}{D(y+1)}}, \quad x = 1, 2, \ldots, \quad \phi_0(0) = 1$$

Factorised eigenvectors $\phi_n(x) = \phi_0(x)\check{P}_n(x)$

$$\left(B(x) + D(x)\right)\phi_0(x)\check{P}_n(x) - \sqrt{B(x)D(x+1)}\phi_0(x+1)\check{P}_n(x+1)$$
$$- \sqrt{B(x-1)D(x)}\phi_0(x-1)\check{P}_n(x-1) = \mathcal{E}(n)\phi_0(x)\check{P}_n(x).$$

⇓

Difference Eq. for Eigenvectors, Triangularity=Solvability

$$B(x)\left(\check{P}_n(x)-\check{P}_n(x+1)\right)+D(x)\left(\check{P}_n(x)-\check{P}_n(x-1)\right)=\mathcal{E}(n)\check{P}_n(x),$$

$$\Rightarrow \widetilde{\mathcal{H}}\check{P}_n(x)=\mathcal{E}(n)\check{P}_n(x),\qquad \widetilde{\mathcal{H}}1=0,$$

$$\widetilde{\mathcal{H}}\stackrel{\text{def}}{=}\phi_0(x)^{-1}\cdot\mathcal{H}\cdot\phi_0(x)=B(x)(1-e^{\partial})+D(x)(1-e^{-\partial})$$

Triangularity w.r.t. special basis, $1,\ \eta(x),\ \eta(x)^2,\ \ldots,\eta(x)^n,\ \ldots$

$$\widetilde{\mathcal{H}}\eta(x)^n=\mathcal{E}(n)\eta(x)^n+\text{lower degrees in }\eta(x)$$

- Gram-Schmidt $\Rightarrow \{P_n(\eta(x))\}$

$\eta(x)$: sinusoidal coordinate, $\eta(x)\to a\eta(x)+b$, B.C. $\eta(0)=0$

(i) $\eta(x)=x$, (ii) $\eta(x)=x(x+d)$,

(iii) $\eta(x)=1-q^x$, (iv) $\eta(x)=q^{-x}-1$, $0<q<1$,

(v) $\eta(x)=(q^{-x}-1)(1-dq^x)$,

Explicit Example 1

Krawtchouk Finite matrix: $B(x) = p(N-x)$, $D(x) = (1-p)x$, $0 < p < 1$, $\eta(x) = x$
Triangular:

$$\begin{aligned}\widetilde{\mathcal{H}}x^n &= p(N-x)\big(x^n - (x+1)^n\big) + (1-p)x\big(x^n - (x-1)^n\big)\\ &= nx^n + \text{lower degrees}, \quad \mathcal{E}(n) = n.\end{aligned}$$

eigenpolynomials

$$P_n(x;p,N) = {}_2F_1\left(\begin{matrix}-n,\ -x\\ -N\end{matrix}\ \Big|\ \frac{1}{p}\right), \quad P_0 = 1,\ P_1 = \frac{pN-x}{pN},$$

$$P_2 = \frac{p^2N(N-1) + (2p(1-N)-1)x + x^2}{p^2N(N-1)}, \ldots.$$

Explicit Example 1-2

Krawtchouk orthogonality weightfunction

$$\phi_0^2(x) = \prod_{y=0}^{x-1} \frac{p}{1-p}\frac{N-y}{y+1} = \left(\frac{p}{1-p}\right)^x \frac{N!}{x!(N-x)!}$$

$$= \left(\frac{p}{1-p}\right)^x \frac{N!}{\Gamma(x+1)\Gamma(N+1-x)} \ \text{: Binomial distribution}$$

Trace formula

$$\frac{N}{2}(N+1) = \sum_{n=0}^{N} \mathcal{E}(n) = \mathrm{Tr}(\mathcal{H}) = \sum_{x=0}^{N} (B(x) + D(x))$$

Explicit Example 2

Meixner ∞ matrix $B(x) = \dfrac{c}{1-c}(x+\beta)$, $D(x) = \dfrac{x}{1-c}$, $\beta > 0$, $0 < c < 1$, $\eta(x) = x$

$$(1-c)\widetilde{\mathcal{H}}x^n = c(x+\beta)\big(x^n - (x+1)^n\big) + x\big(x^n - (x-1)^n\big)$$
$$= (1-c)nx^n + \text{lower degrees}, \quad \mathcal{E}(n) = n$$

$$P_n(x;\beta,c) = {}_2F_1\left(\begin{matrix}-n,\ -x\\ \beta\end{matrix}\ \Big|\ 1-\frac{1}{c}\right), \quad P_0 = 1,\ P_1 = \frac{\beta c - (1-c)x}{\beta c},$$

$$P_2 = \frac{\beta(\beta+1)c^2 + (-1+c)(1+c+2\beta c)x + (-1+c)^2x^2}{\beta(\beta+1)c^2}, \ldots$$

$$\phi_0^2(x) = \prod_{y=0}^{x-1}\frac{c(y+\beta)}{y+1} = \frac{(\beta)_x c^x}{x!}, \quad \sum_{x=0}^{\infty}\phi_0^2(x) = \frac{1}{(1-c)^\beta}$$

Explicit Example 3

Racah Finite, one of the parameters a, b, c is identified with $-N$

$$B(x) = -\frac{(x+a)(x+b)(x+c)(x+d)}{(2x+d)(2x+1+d)},$$

$$D(x) = -\frac{(x+d-a)(x+d-b)(x+d-c)x}{(2x-1+d)(2x+d)},$$

$$\eta(x) = x(x+d), \quad \mathcal{E}(n) = n(n+\tilde{d}), \quad \tilde{d} \stackrel{\text{def}}{=} a+b+c-d-1,$$

$$\phi_0^2(x) = \frac{(a,b,c,d)_x}{(1+d-a, 1+d-b, 1+d-c, 1)_x} \, \frac{2x+d}{d}.$$

Explicit Example 4

q-**Racah** Finite, one of the parameters a, b, c is identified with q^{-N}

$$B(x) = -\frac{(1-aq^x)(1-bq^x)(1-cq^x)(1-dq^x)}{(1-dq^{2x})(1-dq^{2x+1})}, \quad \tilde{d} \stackrel{\text{def}}{=} abc/(dq)$$

$$D(x) = -\tilde{d}\,\frac{(1-a^{-1}dq^x)(1-b^{-1}dq^x)(1-c^{-1}dq^x)(1-q^x)}{(1-dq^{2x-1})(1-dq^{2x})},$$

$$\eta(x) = (q^{-x}-1)(1-dq^x), \quad \mathcal{E}(n) = (q^{-n}-1)(1-\tilde{d}q^n),$$

$$\phi_0^2(x) = \frac{(a,b,c,d\,;q)_x}{(a^{-1}dq, b^{-1}dq, c^{-1}dq, q\,;q)_x\,\tilde{d}^x}\,\frac{1-dq^{2x}}{1-d}$$

Orthogonality Relation

$$\sum_x \phi_0^2(x) P_n(\eta(x)) P_m(\eta(x)) = \frac{1}{d_n^2}\,\delta_{n,m},, \quad d_n > 0$$

orthonormality

$$\sum_x d_n\phi_0(x)P_n(\eta(x)) \cdot d_m\phi_0(x)P_m(\eta(x)) = \delta_{n,m}$$

completeness relation

$$\sum_n d_n\phi_0(x)P_n(\eta(x)) \cdot d_n\phi_0(y)P_n(\eta(y)) = \delta_{x,y}$$

$$\Rightarrow \sum_n d_n^2 P_n(\eta(x)) P_n(\eta(y)) = \frac{1}{\phi_0^2(x)}\,\delta_{x,y}$$

3 term Recurrence for P_n= Difference Eq. in n

3 term Recurrence for P_n

$$\eta P_n(\eta) = A_n P_{n+1}(\eta) + B_n P_n(\eta) + C_n P_{n-1}(\eta), \quad n = 0, 1, \ldots,$$

$P_n(0) = 1 \Rightarrow B_n = -(A_n + C_n)$
Difference eq. for P_n in n variable:

$$-A_n\big(P_n(\eta) - P_{n+1}(\eta)\big) - C_n\big(P_n(\eta) - P_{n-1}(\eta)\big) = \eta P_n(\eta),$$
$$\Big[-A_n(1 - e^{\partial_n}) - C_n(1 - e^{-\partial_n})\Big] P_n(\eta) = \eta P_n(\eta)$$

Dual Polynomial

Rewrite the eigenvalue problem $\widetilde{\mathcal{H}}\psi(x) = \mathcal{E}\psi(x)$ into an explicit matrix form, with $\psi(x) \to {}^t(Q_0, Q_1, \ldots, Q_x, \ldots)$

$$\sum_y \widetilde{\mathcal{H}}_{x,y} Q_y = \mathcal{E} Q_x, \quad x, y = 0, 1, \ldots, (N), \ldots$$

$\Downarrow$ gives 3 term recurrence for $Q_x(\mathcal{E})$

$$\mathcal{E} Q_x(\mathcal{E}) = B(x)\big(Q_x(\mathcal{E}) - Q_{x+1}(\mathcal{E})\big) + D(x)\big(Q_x(\mathcal{E}) - Q_{x-1}(\mathcal{E})\big),$$

Starting from $Q_0 \equiv 1$, $Q_x(\mathcal{E})$ determined as a degree x polynomial in $\mathcal{E}$, with $Q_x(0) = 1$, $x = 0, 1, \ldots, (N), \ldots$. $x = N$ gives the characteristic eq. $\mathcal{E} Q_N(\mathcal{E}) = D(N)\big(Q_N(\mathcal{E}) - Q_{N-1}(\mathcal{E})\big)$. Replacing $\mathcal{E} \to \mathcal{E}(n)$ gives explicit form of another eigenpoly (Leonard, Terwilliger, ...)

$$\sum_y \widetilde{\mathcal{H}}_{x,y} Q_y(\mathcal{E}(n)) = \mathcal{E}(n) Q_x(\mathcal{E}(n)), \quad x = 0, 1, \ldots, (N), \ldots$$

Dual Polynomial 2

By B.C. $P_n(0) = 1 = Q_x(0)$, the two polynomials, $\{P_n(\eta)\}$ and its dual polynomial $\{Q_x(\mathcal{E})\}$, coincide at the integer lattice points:

$$P_n(\eta(x)) = Q_x(\mathcal{E}(n)), \quad n = 0, 1, \ldots, (N), \ldots, \quad x = 0, 1, \ldots, (N), \ldots .$$

$$\text{(i)}: \ \mathcal{E}(n) = n, \qquad \text{(ii)}: \ \mathcal{E}(n) = n(n+\alpha),$$
$$\text{(iii)}: \ \mathcal{E}(n) = 1 - q^n, \qquad \text{(iv)}: \ \mathcal{E}(n) = q^{-n} - 1,$$
$$\text{(v)}: \ \mathcal{E}(n) = (q^{-n} - 1)(1 - \alpha q^n),$$

Dual correspondence: $\displaystyle\sum_n d_n^2 Q_x(\mathcal{E}(n)) Q_y(\mathcal{E}(n)) = \delta_{x,y}/\phi_0^2(x)$,

$$x \leftrightarrow n, \quad \eta(x) \leftrightarrow \mathcal{E}(n), \quad \eta(0) = 0 \leftrightarrow \mathcal{E}(0) = 0,$$

$$B(x) \leftrightarrow -A_n, \quad D(x) \leftrightarrow -C_n, \quad \frac{\phi_0(x)}{\phi_0(0)} \leftrightarrow \frac{d_n}{d_0}$$

Intertwining Relations

$$\mathcal{H} \equiv \mathcal{H}^{[0]} = \mathcal{A}^\dagger \mathcal{A}, \quad \mathcal{H}^{[0]}\phi_n^{[0]}(x) = \mathcal{E}(n)\phi_n^{[0]}(x), \quad n = 0, 1, 2, \ldots,$$

new Hamiltonian (Jacobi matrix) $\mathcal{H}^{[1]} \stackrel{\text{def}}{=} \mathcal{A}\mathcal{A}^\dagger$,

$$\mathcal{A}\mathcal{H}^{[0]} = \mathcal{A}\mathcal{A}^\dagger\mathcal{A} = \mathcal{H}^{[1]}\mathcal{A}, \qquad \mathcal{A}^\dagger\mathcal{H}^{[1]} = \mathcal{A}^\dagger\mathcal{A}\mathcal{A}^\dagger = \mathcal{H}^{[0]}\mathcal{A}^\dagger$$

iso-spectrality except for $\phi_0^{[0]}(x)$

$$\mathcal{H}^{[0]}\phi_n^{[0]}(x) = \mathcal{E}(n)\phi_n^{[0]}(x) \quad (n = 0, 1, \ldots), \quad \mathcal{A}\phi_0^{[0]}(x) = 0,$$

$$\mathcal{H}^{[1]}\phi_n^{[1]}(x) = \mathcal{E}(n)\phi_n^{[1]}(x) \quad (n = 1, 2, \ldots),$$

$$\phi_n^{[1]}(x) \stackrel{\text{def}}{=} \mathcal{A}\phi_n^{[0]}(x), \quad \phi_n^{[0]}(x) = \frac{\mathcal{A}^\dagger}{\mathcal{E}(n)}\phi_n^{[1]}(x) \quad (n = 1, 2, \ldots),$$

$$(\phi_n^{[1]}, \phi_m^{[1]}) = \mathcal{E}(n)(\phi_n^{[0]}, \phi_m^{[0]}) \quad (n, m = 1, 2, \ldots)$$

Intertwining Relations 2

$\mathcal{H}^{[1]} - \mathcal{E}(1)$: positive semi-definite, $\mathcal{A}^{[1]}\phi_1^{[1]}(x) = 0$

$$\mathcal{H}^{[1]} = \mathcal{A}^{[1]\dagger}\mathcal{A}^{[1]} + \mathcal{E}(1),\ \ B^{[1]}(x), D^{[1]}(x) > 0,\ \ D^{[1]}(0) = 0 = B^{[1]}(N-1),$$

$$\mathcal{A}^{[1]} \stackrel{\text{def}}{=} \sqrt{B^{[1]}(x)} - e^{\partial}\sqrt{D^{[1]}(x)}, \quad \mathcal{A}^{[1]\dagger} = \sqrt{B^{[1]}(x)} - \sqrt{D^{[1]}(x)}\,e^{-\partial},$$

new iso-spectral Hamiltonian $\mathcal{H}^{[2]} \stackrel{\text{def}}{=} \mathcal{A}^{[1]}\mathcal{A}^{[1]\dagger} + \mathcal{E}(1)$,

$$\mathcal{H}^{[2]}\phi_n^{[2]}(x) = \mathcal{E}(n)\phi_n^{[2]}(x) \qquad (n = 2, 3, \ldots),$$

$$\phi_n^{[2]}(x) \stackrel{\text{def}}{=} \mathcal{A}^{[1]}\phi_n^{[1]}(x), \quad \phi_n^{[1]}(x) = \frac{\mathcal{A}^{[1]\dagger}}{\mathcal{E}(n) - \mathcal{E}(1)}\phi_n^{[2]}(x) \quad (n = 2, 3, \ldots),$$

$$(\phi_n^{[2]}, \phi_m^{[2]}) = \big(\mathcal{E}(n) - \mathcal{E}(1)\big)(\phi_n^{[1]}, \phi_m^{[1]}) \qquad (n, m = 2, 3, \ldots),$$

$$\mathcal{H}^{[2]} = \mathcal{A}^{[2]\dagger}\mathcal{A}^{[2]} + \mathcal{E}(2), \quad \mathcal{A}^{[2]}\phi_2^{[2]}(x) = 0$$

Crum's Sequence

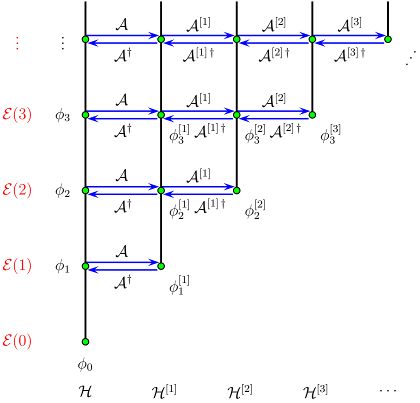

Figure: Fundamental structure of the solution space of 1-d QM.

Crum's Theorem

Theorem

(Crum [5]) *For a given Hamiltonian system* $\{\mathcal{H}^{[0]}, \mathcal{E}(n), \phi_n^{[0]}(x)\}$ *there exist as many associated Hamiltonian systems* $\{\mathcal{H}^{[1]}, \mathcal{E}(n), \phi_n^{[1]}(x)\}$, $\{\mathcal{H}^{[2]}, \mathcal{E}(n), \phi_n^{[2]}(x)\}$, *. . . , as the number of discrete eigenvalues. They share the eigenvalues* $\{\mathcal{E}(n)\}$ *of the original Hamiltonian and the eigenvectors of* $\mathcal{H}^{[j]}$ *and* $\mathcal{H}^{[j+1]}$ *are related by* $\mathcal{A}^{[j]}$ *and* $\mathcal{A}^{[j]\dagger}$.

Shape Invariance

set of parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$, $q^{\boldsymbol{\lambda}} = (a, b, \ldots)$, $\mathcal{H}(\boldsymbol{\lambda})$, $\mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{E}(n; \boldsymbol{\lambda})$, $\eta(x; \boldsymbol{\lambda})$, $\phi_n(x; \boldsymbol{\lambda})$, $P_n(\eta(x; \boldsymbol{\lambda}); \boldsymbol{\lambda})$ etc

Shape invariance: $\mathcal{A}^{[1]}(\boldsymbol{\lambda}) \propto \mathcal{A}^{[0]}(\boldsymbol{\lambda} + \boldsymbol{\delta})$

$$\Leftrightarrow \mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \kappa \mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta})^\dagger \mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}(1; \boldsymbol{\lambda}), \quad \kappa > 0,$$

everything is expressed by the first excited energy $\mathcal{E}(1; \boldsymbol{\lambda})$, $\mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{A}(\boldsymbol{\lambda})^\dagger$ and the groundstate vector $\phi_0(x; \boldsymbol{\lambda})$ at variously shifted values of $\boldsymbol{\lambda}$, universal Eigenvalue formula

$$\mathcal{E}(n; \boldsymbol{\lambda}) = \sum_{s=0}^{n-1} \kappa^s \mathcal{E}(1; \boldsymbol{\lambda}^{[s]}), \qquad \boldsymbol{\lambda}^{[s]} \stackrel{\text{def}}{=} \boldsymbol{\lambda} + s\boldsymbol{\delta},$$

$$\phi_n(x; \boldsymbol{\lambda}) \propto \mathcal{A}(\boldsymbol{\lambda}^{[0]})^\dagger \mathcal{A}(\boldsymbol{\lambda}^{[1]})^\dagger \mathcal{A}(\boldsymbol{\lambda}^{[2]})^\dagger \cdots \mathcal{A}(\boldsymbol{\lambda}^{[n-1]})^\dagger \phi_0(x; \boldsymbol{\lambda}^{[n]})$$

Universal Rodrigues Formula

auxiliary function $\varphi(x)$: $\varphi(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \dfrac{\eta(x+1;\boldsymbol{\lambda})-\eta(x;\boldsymbol{\lambda})}{\eta(1;\boldsymbol{\lambda})}$,
$\varphi(0;\boldsymbol{\lambda})=1$,

$$\mathcal{A}(\boldsymbol{\lambda})^{\dagger} \propto \phi_0(x;\boldsymbol{\lambda})^{-1}\cdot\mathcal{D}(\boldsymbol{\lambda})\cdot\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta}), \quad \mathcal{D}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (1-e^{-\partial})\varphi(x;\boldsymbol{\lambda})^{-1}$$

by universal normalisation $P_n(0;\boldsymbol{\lambda})=1 \Rightarrow$ Universal Rodrigues Formula

$$P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}) = \phi_0(x;\boldsymbol{\lambda})^{-2}\cdot\prod_{j=0}^{n-1}\mathcal{D}(\boldsymbol{\lambda}+j\boldsymbol{\delta})\cdot\phi_0^2(x;\boldsymbol{\lambda}+n\boldsymbol{\delta})$$

Heisenberg Equation

Heisenberg Equation

$$\frac{\partial}{\partial t}\hat{\mathbf{A}}(t) = i\,[\mathcal{H}, \hat{\mathbf{A}}(t)]$$

formal solution, $(\mathrm{ad}\,\mathcal{H})X \stackrel{\mathrm{def}}{=} [\mathcal{H}, X]$

$$\hat{\mathbf{A}}(t) = e^{it\mathcal{H}}\hat{\mathbf{A}}(0)e^{-it\mathcal{H}} = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}(\mathrm{ad}\,\mathcal{H})^n\hat{\mathbf{A}}(0),$$

closed form expression mostly intractable, but for the sinusoidal coordinate $\eta(x)$ it is always possible for exactly solvable QM

$$e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}}$$

Closure Relation

$$(\mathrm{ad}\,\mathcal{H})^2\eta(x) = \eta(x)\,R_0(\mathcal{H}) + (\mathrm{ad}\,\mathcal{H})\eta(x)\,R_1(\mathcal{H}) + R_{-1}(\mathcal{H})$$

$$(\mathrm{ad}\,\widetilde{\mathcal{H}})^2\eta(x) = \eta(x)\,R_0(\widetilde{\mathcal{H}}) + (\mathrm{ad}\,\widetilde{\mathcal{H}})\eta(x)\,R_1(\widetilde{\mathcal{H}}) + R_{-1}(\widetilde{\mathcal{H}})$$

l.h.s. $e^{2\partial}$, e^{∂}, 1, $e^{-\partial}$, $e^{-2\partial}$ and $\mathcal{H}$ contains $e^{\pm\partial}$

$$R_1(z) = r_1^{(1)}z + r_1^{(0)},\ \ R_0(z) = r_0^{(2)}z^2 + r_0^{(1)}z + r_0^{(0)},$$

$$R_{-1}(z) = r_{-1}^{(2)}z^2 + r_{-1}^{(1)}z + r_{-1}^{(0)}$$

$$(\mathrm{ad}\,\mathcal{H})^3\eta(x) = [\mathcal{H},\eta(x)]R_0(\mathcal{H}) + [\mathcal{H},[\mathcal{H},\eta(x)]]\,R_1(\mathcal{H})$$

$$= \eta(x)\,R_0(\mathcal{H})R_1(\mathcal{H}) + (\mathrm{ad}\,\mathcal{H})\eta(x)\,\big(R_1(\mathcal{H})^2 + R_0(\mathcal{H})\big) + R_{-1}(\mathcal{H})R_1(\mathcal{H})$$

$\Rightarrow$ $(\mathrm{ad}\,\mathcal{H})^n\eta(x)$ is a combination of $\eta(x)$ and $[\mathcal{H},\eta(x)]$ with $\mathcal{H}$ depending coefficients

Heisenberg Operator Solution

$$
\begin{aligned}
e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}} &= \sum_{n=0}^{\infty}\frac{(it)^n}{n!}(\mathrm{ad}\,\mathcal{H})^n\eta(x)\\
&= [\mathcal{H},\eta(x)]\frac{e^{i\alpha_+(\mathcal{H})t}-e^{i\alpha_-(\mathcal{H})t}}{\alpha_+(\mathcal{H})-\alpha_-(\mathcal{H})} - R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}\\
&\quad + \left(\eta(x)+R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}\right)\frac{-\alpha_-(\mathcal{H})e^{i\alpha_+(\mathcal{H})t}+\alpha_+(\mathcal{H})e^{i\alpha_-(\mathcal{H})t}}{\alpha_+(\mathcal{H})-\alpha_-(\mathcal{H})}
\end{aligned}
$$

$$
\alpha_\pm(\mathcal{H}) \stackrel{\text{def}}{=} \tfrac{1}{2}\left(R_1(\mathcal{H}) \pm \sqrt{R_1(\mathcal{H})^2+4R_0(\mathcal{H})}\right),
$$

overdetermined equations for the eigenvalues:

$$
\mathcal{E}(n+1;\boldsymbol{\lambda})-\mathcal{E}(n;\boldsymbol{\lambda}) = \alpha_+(\mathcal{E}(n;\boldsymbol{\lambda})),\ \ \mathcal{E}(n-1;\boldsymbol{\lambda})-\mathcal{E}(n;\boldsymbol{\lambda}) = \alpha_-(\mathcal{E}(n;\boldsymbol{\lambda}))
$$

Creation & Annihilation Operators

$\phi_n(x) = \phi_0(x)P_n(\eta(x))$ and three term recurrence for P_n:

$$\eta(x)\phi_n(x) = A_n\phi_{n+1}(x) + B_n\phi_n(x) + C_n\phi_{n-1}(x) \quad (n \geq 0)$$

$$\begin{aligned} &e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}}\phi_n(x) \\ &= e^{it(\mathcal{E}(n+1)-\mathcal{E}(n))}A_n\phi_{n+1}(x) + B_n\phi_n(x) + e^{it(\mathcal{E}(n-1)-\mathcal{E}(n))}C_n\phi_{n-1}(x) \\ &= e^{it\alpha_+(\mathcal{E}(n))}A_n\phi_{n+1}(x) + B_n\phi_n(x) + e^{it\alpha_-(\mathcal{E}(n))}C_n\phi_{n-1}(x) \end{aligned}$$

Creation & Annihilation Operators $\phi_n(x) = {a^{(+)}}^n\phi_0(x)$

$$\begin{aligned} a^{(\pm)} &\stackrel{\text{def}}{=} \pm\Big([\mathcal{H}, \eta(x)] - \tilde{\eta}(x)\alpha_\mp(\mathcal{H})\Big)\big(\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})\big)^{-1} \\ &= \pm\big(\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})\big)^{-1}\Big([\mathcal{H}, \eta(x)] + \alpha_\pm(\mathcal{H})\tilde{\eta}(x)\Big), \end{aligned}$$

Determination of A_n and C_n

Universal formula for the coefficients of the three term recurrence A_n and C_n:

$$A_n = \frac{\frac{R_{-1}(\mathcal{E}(n))}{R_0(\mathcal{E}(n))}\big(\mathcal{E}(n)-\mathcal{E}(n-1)\big)+\eta(1)\big(\mathcal{E}(n)-B(0)\big)}{\mathcal{E}(n+1)-\mathcal{E}(n-1)},$$

$$C_n = \frac{\frac{R_{-1}(\mathcal{E}(n))}{R_0(\mathcal{E}(n))}\big(\mathcal{E}(n)-\mathcal{E}(n+1)\big)+\eta(1)\big(\mathcal{E}(n)-B(0)\big)}{\mathcal{E}(n-1)-\mathcal{E}(n+1)},$$

due to universal normalisation $P_n(0)=1$

Birth and Death Equation

- Birth and Death (BD) process: typical stationary Markov process with 1-d discrete state space
- transition only between nearest neighbours $\leftrightarrow$ tri-diagonal matrix:

$$\frac{\partial}{\partial t}\mathcal{P}(x;t) = (L_{\text{BD}}\mathcal{P})(x;t), \quad \mathcal{P}(x;t) \geq 0, \quad \sum_{x}\mathcal{P}(x;t) = 1$$

L_{BD}: *inverse similarity transformation* of the Hamiltonian $\mathcal{H}$ of the Simplest QM:

$$L_{\text{BD}} \stackrel{\text{def}}{=} -\phi_0 \circ \mathcal{H} \circ \phi_0^{-1} = (e^{-\partial} - 1)B(x) + (e^{\partial} - 1)D(x),$$
$$L_{\text{BD}}\phi_0(x)\phi_n(x) = -\mathcal{E}(n)\phi_0(x)\phi_n(x) \quad (n = 0, 1, \ldots).$$

universal solution for Initial Value Problem

Birth and Death eq. $B(x)$: Birth rate, $D(x)$: Death rate

$$\frac{\partial}{\partial t}\mathcal{P}(x;t) = B(x-1)\mathcal{P}(x-1;t) + D(x+1)\mathcal{P}(x+1;t) - (B(x)+D(x))\mathcal{P}(x;t)$$

initial probability distribution $\mathcal{P}(x;0) \geq 0$ ($\sum_x \mathcal{P}(x;0) = 1$):

$$\mathcal{P}(x;t) = \hat{\phi}_0(x) \sum_{n=0}^{n_{\max}} c_n \, e^{-\mathcal{E}(n)t} \hat{\phi}_n(x) \quad (t>0),$$

$$\mathcal{P}(x;0) = \hat{\phi}_0(x) \sum_{n=0}^{n_{\max}} c_n \hat{\phi}_n(x), \quad c_n = \big(\hat{\phi}_n(x), \hat{\phi}_0(x)^{-1}\mathcal{P}(x;0)\big), \;\; c_0 = 1$$

universal formula for Transition Probability

concentrated initial distribution at y $\mathcal{P}(x;0) = \delta_{xy}$:

$$\mathcal{P}(x,y;t) = \hat{\phi}_0(x)\Bigl(\sum_{n=0}^{n_{\text{max}}} e^{-\mathcal{E}(n)t}\hat{\phi}_n(x)\hat{\phi}_n(y)\Bigr)\hat{\phi}_0(y)^{-1} \quad (t > 0),$$

Chapman-Kolmogorov equation is satisfied $\Leftarrow$ orthonormality condition

$$\mathcal{P}(x,y;t) = \sum_{z=0}^{x_{\text{max}}} \mathcal{P}(x,z;t-t')\mathcal{P}(z,y;t') \quad (0 < t' < t)$$

Comments

In Functional Analysis, Jacobi matrices are discussed mainly via *Moment Problems*. Given a measure $d\mu(x)$, construct normalised orthogonal polynomial $\{P_n(x)\}$:

$$\langle P_n, P_m \rangle \stackrel{\text{def}}{=} \int P_n(x) P_m(x) d\mu(x) = \delta_{n,m},$$

with a positive leading term $P_n(x) = k_n x^n + \ldots$, $k_n > 0$. A Jacobi matrix is defined via three term recurrence

$$xP_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x),$$
$$a_n = \langle P_{n+1}, xP_n \rangle > 0, \quad b_n = \langle P_n, xP_n \rangle,$$

$\{P_n(x)\}$ are Not eigenvectors of the Jacobi matrix J.

Comments, continued

$$J = \begin{pmatrix} b_0 & a_0 & 0 & \cdots & \cdots & 0 \\ a_0 & b_1 & a_1 & 0 & \cdots & \vdots \\ 0 & a_1 & b_2 & a_2 & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & a_{N-2} & b_{N-1} & a_{N-1} \\ 0 & \cdots & \cdots & 0 & a_{N-1} & b_N \end{pmatrix}$$

- (Hamburger, Stieltjes) Moment problems ($m_n = \int x^n d\mu(x)$) are *too narrowly* posed. More general moments of 'sinusoidal coordinates' $M_n = \int \eta(x)^n d\mu(x)$ needed .
- Simplest QM measures $d\mu(x) = \sum_{y=0} \phi_0(y)^2 \delta_{x,y}$: always *indeterminate* $\phi_0(x)^2 \to \phi_0(x)^2 + \sin \pi x \cdot F(\sin \pi x)$.

New Developments in Orthogonal Polynomials Theory

- polynomials having Jackson integral measures can be formulated within Simplest QM
- unified quantum mechanical formulation covering the other polynomials in the Askey scheme including the Wilson, Askey-Wilson polynomials has been constructed
- ∞ many orthogonal polynomials satisfying 2nd order differential eqs are found, '08-present, Gomez-Ullate, et al, Quesne, Odake-Sasaki, called exceptional, multi-indexed Laguerre & Jacobi polynomials
- there are 'holes' in the degrees and three term recurrence does not hold (Bochner's Thr avoided), but they form complete set
- corresponding exceptional and multi-indexed Racah, q-Racah, Wilson, Askey-Wilson polynomials have been constructed

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