Perturbations around the zeros of classical orthogonal polynomials

Ryu SASAKI

Department of Physics, Shinshu University

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Outline

Background

Zeros of orthogonal polynomials:
many interesting properties explored over more than a century
in particular, for the classical orthogonal polynomials:
Hermite, Laguerre, Jacobi, Wilson, Askey-Wilson, Racah, q-Racah
etc, (Askey scheme of hypergeometric orthogonal polynomials)
They satisfy second order differential or difference eq. and the
three term recurrence relations
Exactly Solvable Quantum Mechanics provides
universal framework of classical orthogonal polynomials;
ordinary Schrödinger eq.
difference Schrödinger eq. with pure imaginary or real shifts

this Talk

- start with degree $\mathcal{N}$ solutions of a time dependent Schrödinger eq. for a classical orthogonal polynomial.
- a linear matrix equation describing perturbations around the $\mathcal{N}$ zeros of the polynomial is derived.
- The $\mathcal{N} \times \mathcal{N}$ matrix $\mathcal{M}$ has remarkable Diophantine properties.
  - the eigenvalues are independent of the zeros.
  - the corresponding eigenvectors provide the representations of the lower degree $(0, 1, \ldots, \mathcal{N} - 1)$ polynomials in terms of the zeros of the degree $\mathcal{N}$polynomial.
- universal results for all the classical orthogonal polynomials
- Basic facts: For polynomials (degree $\mathcal{N}$ or less), the differential and difference operators can be represented algebraically, in terms of $\mathcal{N} \times \mathcal{N}$ matrices

Exactly solvable Schrödinger eqs.

- Time dependent Schrödinger equation

$$i\frac{\partial\Psi(x,t)}{\partial t} = \mathcal{H}\Psi(x,t)$$

exactly solvable, if the eigenvalue problem of the Schrödinger operator $\mathcal{H}$ is exactly solvable:

$$\mathcal{H}\phi_n(x) = \mathcal{E}(n)\phi_n(x), \quad n = 0, 1, \ldots,$$

- the general solution

$$\Psi(x,t) = \sum_{n=0}^{\infty} c_n e^{-i\mathcal{E}(n)t}\phi_n(x),$$

$\{c_n\}$: constants of integration.

Exactly solvable Schrödinger eqs 2

- factorised eigenfunctions

$$\phi_n(x) = \phi_0(x) P_n\left(\eta(x)\right)$$

$\phi_0(x)$ is the ground state wave function, $\phi_0(x)^2$ provides the orthogonality weight function for the polynomial $P_n\left(\eta(x)\right)$ of degree n, $\eta(x)$ is called the sinusoidal coordinate

- similarity transformed Schrödinger operator $\widetilde{\mathcal{H}}$:

$$\widetilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0(x)^{-1} \circ \mathcal{H} \circ \phi_0(x),$$

governs the classical polynomials $\{P_n\left(\eta(x)\right)\}$:

$$\widetilde{\mathcal{H}} P_n\left(\eta(x)\right) = \mathcal{E}(n) P_n\left(\eta(x)\right), \quad n = 0, 1, \ldots .$$

Polynomial solutions

- restrict the general solution to those having *degrees up to* $\mathcal{N}$:

$$\Psi_{\mathcal{N}}(x,t)=\sum_{n=0}^{\mathcal{N}}c_n e^{-i\mathcal{E}(n)t}\phi_n(x)=e^{-i\mathcal{E}(\mathcal{N})t}\phi_0(x)\psi_{\mathcal{N}}(x,t)$$

$$\psi_{\mathcal{N}}(x,t)\stackrel{\text{def}}{=}\sum_{n=0}^{\mathcal{N}}c_n e^{i(\mathcal{E}(\mathcal{N})-\mathcal{E}(n))t}P_n\left(\eta(x)\right)$$

- $c_{\mathcal{N}}$ for monic: $c_{\mathcal{N}}P_{\mathcal{N}}\left(\eta(x)\right)=\prod_{n=1}^{\mathcal{N}}\left(\eta(x)-\eta(x_n)\right)$,

$$(\star)\quad \frac{\partial\psi_{\mathcal{N}}(x,t)}{\partial t}=-i\widetilde{\mathcal{H}}_{\mathcal{N}}\psi_{\mathcal{N}}(x,t),\qquad \widetilde{\mathcal{H}}_{\mathcal{N}}\stackrel{\text{def}}{=}\widetilde{\mathcal{H}}-\mathcal{E}(\mathcal{N}),$$

$$\psi_{\mathcal{N}}(x,t)=\prod_{n=1}^{\mathcal{N}}\left(\eta(x)-\eta(x_n(t))\right),$$

infinitesimal oscillations around the zeros of $P_{\mathcal{N}}(\eta(x))$

- focus on the infinitesimal oscillations:

$$x_n(t) = x_n + \epsilon\gamma_n(t), \quad 0 < \epsilon \ll 1, \quad n = 1, \ldots, \mathcal{N}$$

- leading to

$$\psi_{\mathcal{N}}(x,t) = \prod_{n=1}^{\mathcal{N}} \left(\eta(x) - \eta(x_n)\right) - \epsilon \sum_{n=1}^{\mathcal{N}} \gamma_n(t)\dot{\eta}(x_n) \prod_{j\neq n}^{\mathcal{N}} \left(\eta(x) - \eta(x_j)\right) + O(\epsilon^2),$$

$$\text{with} \quad \dot{\eta}(x) \stackrel{\text{def}}{=} \frac{d\eta(x)}{dx}$$

time evolution eq. for $\psi_{\mathcal{N}}(x,t)$

- l.h.s. of $(\star)$ is a degree $\mathcal{N}-1$ polynomial in $\eta(x)$:

$$-\epsilon\sum_{n=1}^{\mathcal{N}}\frac{d\gamma_n(t)}{dt}\dot{\eta}(x_n)\prod_{j\neq n}^{\mathcal{N}}\bigl(\eta(x)-\eta(x_j)\bigr)+O(\epsilon^2).$$

- r.h.s. is also a degree $\mathcal{N}-1$ polynomial in $\eta(x)$:

$$i\epsilon\sum_{m=1}^{\mathcal{N}}\gamma_m(t)\dot{\eta}(x_m)\widetilde{\mathcal{H}}_{\mathcal{N}}\prod_{j\neq m}^{\mathcal{N}}\bigl(\eta(x)-\eta(x_j)\bigr)+O(\epsilon^2),$$

since $\widetilde{\mathcal{H}}_{\mathcal{N}}P_{\mathcal{N}}(\eta(x))=0$, $(\triangle)$.

time evolution eq. for $\psi_{\mathcal{N}}(x,t)$ 2

$(\star)$ reads at the leading order of ϵ:

$$\frac{d\gamma_n(t)}{dt}\dot{\eta}(x_n)\prod_{j\neq n}^{\mathcal{N}}\bigl(\eta(x_n)-\eta(x_j)\bigr)$$

$$=-i\sum_{m=1}^{\mathcal{N}}\gamma_m(t)\dot{\eta}(x_m)\left(\widetilde{\mathcal{H}}_{\mathcal{N}}\prod_{j\neq m}^{\mathcal{N}}\bigl(\eta(x)-\eta(x_j)\bigr)\right)\Bigg|_{x=x_n},\quad n=1,\ldots,\mathcal{N},$$

in matrix form

$$\frac{d\gamma_n(t)}{dt}=i\sum_{m=1}^{\mathcal{N}}\mathcal{M}_{n\,m}\gamma_m(t),\quad n=1,\ldots,\mathcal{N},$$

$$\mathcal{M}_{n\,m}\stackrel{\text{def}}{=}-\frac{\dot{\eta}(x_m)\left(\widetilde{\mathcal{H}}_{\mathcal{N}}\prod_{j\neq m}^{\mathcal{N}}\bigl(\eta(x)-\eta(x_j)\bigr)\right)\Big|_{x=x_n}}{\dot{\eta}(x_n)\prod_{j\neq n}^{\mathcal{N}}\bigl(\eta(x_n)-\eta(x_j)\bigr)}$$

Main Results

- **Theorem 1** The eigenvalues of $\mathcal{M}$ are $\mathcal{E}(\mathcal{N})-\mathcal{E}(m)$, $m=0,1,\ldots,\mathcal{N}-1$. They depend on the basic parameters of $\widetilde{\mathcal{H}}$ but do not depend on the zeros $\{\eta(x_n)\}$ directly.
- **Theorem 2** The corresponding eigenvectors $\{\mathrm{v}_n^{(m)}\}$ of $\mathcal{M}$,

$$\sum_{\ell=1}^{\mathcal{N}} \mathcal{M}_{n\,\ell}\mathrm{v}_\ell^{(m)} = (\mathcal{E}(\mathcal{N})-\mathcal{E}(m))\mathrm{v}_n^{(m)}$$

yield the representations of the lower degree polynomials $\{P_m(\eta)\}$, $m=0,1,\ldots,\mathcal{N}-1$, in terms of the zeros $\{\eta(x_n)\}$ of $P_{\mathcal{N}}(\eta(x))$:

$$\sum_{n=1}^{\mathcal{N}} \dot{\eta}(x_n)\mathrm{v}_n^{(m)} \prod_{j\neq n}^{\mathcal{N}} (\eta-\eta(x_j)) \propto P_m(\eta), \quad m=0,1,\ldots,\mathcal{N}-1.$$

Lemmas

- well-known **Lemma** The Lagrangian interpolation of a polynomial $Q(x)$ $(\deg Q = m)$ by a higher degree polynomial $\widetilde{Q}(x)$, $(\deg \widetilde{Q} = \mathcal{N} > m)$ is exact:

$$Q(x) = \sum_{n=1}^{\mathcal{N}} \frac{Q(x_n)}{\widetilde{Q}'(x_n)} \cdot \left(\frac{\widetilde{Q}(x)}{x - x_n} \right),$$

$$\widetilde{Q}(x) \stackrel{\text{def}}{=} d_{\mathcal{N}} \prod_{n=1}^{\mathcal{N}} (x - x_n), \quad \widetilde{Q}'(x_n) = d_{\mathcal{N}} \prod_{j \neq n}^{\mathcal{N}} (x_n - x_j).$$

- **Theorem 2** $\Rightarrow$ **Corollary**

$$\mathrm{v}_n^{(m)} \propto \frac{P_m(\eta(x_n))}{\dot{\eta}(x_n) P'_{\mathcal{N}}(\eta(x_n))} = \frac{P_m(\eta(x_n))}{\left(\frac{dP_{\mathcal{N}}(\eta(x))}{dx} \right)\Big|_{x=x_n}}, \quad n = 1, \ldots, \mathcal{N}, \quad m = 0, 1, \ldots, \mathcal{N} - 1.$$

Remarks

- **Remark 1** The matrix $\mathcal{M}$, constructed by the zeros of $P_{\mathcal{N}}(\eta(x))$ and the parameters of $\widetilde{\mathcal{H}}$ only, contains all the information of the values of lower degree polynomials at these zeros $\{P_m(\eta(x_n))\}$, $m = 0, 1, \ldots, \mathcal{N} - 1$ as eigenvectors.
- **Remark 2** The eigenvalues are $\mathrm{algebraic\ numbers}$ based on the basic parameters of $\widetilde{\mathcal{H}}$, $\mathcal{N}$ and the zeros, $\{\eta(x_n)\}$, $\{\dot{\eta}(x_n)\}$. That they are independent of the zeros means that the way the zeros enter the matrix elements $\mathcal{M}_{n\,m}$ is essential but not the explicit values of $\{\eta(x_n)\}$, $\{\dot{\eta}(x_n)\}$. The explicit values are indispensable for the exact values of the eigenvectors $\{\mathrm{v}_n^{(m)}\}$ to reproduce the lower degree polynomials $\{P_m(\eta)\}$. The equations satisfied by the zeros are essential. They are obtained by evaluating the polynomial equation (△) at the zeros: $0 = \widetilde{\mathcal{H}}_{\mathcal{N}} P_{\mathcal{N}}\left(\eta(x)\right)\Big|_{x=x_n}$.

Examples: Hermite polynomials

- basic data:

$$\widetilde{\mathcal{H}} = -\frac{d^2}{dx^2} + 2x\frac{d}{dx}, \ -\infty < x < \infty, \ \eta(x) = x, \ \mathcal{E}(n) = 2n,$$

$$\phi_0(x)^2 = e^{-x^2}, \quad P_n(\eta) = H_n(\eta), \quad \text{Hermite polynomial.}$$

$$\mathcal{M}_{n\,m} = 2\delta_{n\,m}\left(\mathcal{N} + {\sum_{j<k}^{\mathcal{N}}}' \frac{1}{x_n - x_j} \cdot \frac{1}{x_n - x_k} - x_n {\sum_{j=1}^{\mathcal{N}}}' \frac{1}{x_n - x_j}\right)$$

$$+ 2(1 - \delta_{n\,m})\frac{1}{x_n - x_m}\left({\sum_{j=1,\neq m}^{\mathcal{N}}}' \frac{1}{x_n - x_j} - x_n\right).$$

- eigenvalues are all **integers** $2(\mathcal{N} - m)$, $m = 0, 1, \ldots, \mathcal{N} - 1$, for arbitrary distinct $\{x_j\}$; $\sum'$: singular terms are omitted

Examples: Hermite polynomials 2

- algebraic equations among the zeros $\{x_n\}$: $\sum_{j=1}^{\mathcal{N}}{}' \frac{1}{x_n - x_j} = x_n$.
- $\mathcal{N} \times \mathcal{N}$ matrix A by Ahmed, Calogero, et al [1]:

$$A_{n\,m} = \delta_{n\,m} \sum_{j=1}^{\mathcal{N}}{}' \frac{1}{(x_n - x_j)^2} - (1 - \delta_{n\,m}) \frac{1}{(x_n - x_m)^2},$$

with eigenvector $v_n^{(m)} = \frac{H_m(x_n)}{H_{\mathcal{N}-1}(x_n)}$, $n = 1, \ldots, \mathcal{N}$, for eigenvalues $\mathcal{N} - m - 1$, $m = 0, 1, \ldots, \mathcal{N} - 1$.

- $\mathcal{M} = 2(A + 1)$ by $\sum_{j=1}^{\mathcal{N}}{}' \frac{1}{(x_n - x_j)^2} = \frac{2}{3}(\mathcal{N} - 1) - \frac{1}{3} x_n^2$.
- Corollary proved, since $H'_{\mathcal{N}}(x) = 2H_{\mathcal{N}-1}(x)$.

Examples: Laguerre polynomials

- basic data ($y_n \stackrel{\text{def}}{=} x_n^2$):

$$\widetilde{\mathcal{H}} = -\frac{d^2}{dx^2} + 2(x - \frac{g}{x})\frac{d}{dx}, \ 0 < x < \infty, \ \eta(x) = x^2, \ \mathcal{E}(n) = 4n,$$

$$\phi_0(x)^2 = e^{-x^2}(x^2)^g, \ P_n(\eta) = L_n^{(\alpha)}(\eta), \ \text{Laguerre}, \ \alpha \stackrel{\text{def}}{=} g - \tfrac{1}{2} > -1,$$

$$\mathcal{M}_{n\,m} = 4\delta_{n\,m}\left(\mathcal{N} + 2y_n \sum_{j<k}^{\mathcal{N}}{}' \frac{1}{y_n - y_j} \cdot \frac{1}{y_n - y_k} - (y_n - \alpha - 1)\sum_{j=1}^{\mathcal{N}}{}' \frac{1}{y_n - y_j}\right)$$

$$+ 4(1 - \delta_{n\,m})\frac{x_m}{x_n} \cdot \frac{1}{y_n - y_m}\left(2y_n \sum_{j=1,\neq m}^{\mathcal{N}}{}' \frac{1}{y_n - y_j} - (y_n - \alpha - 1)\right).$$

- eigenvalues are all **integers** $4(\mathcal{N} - m)$, $m = 0, 1, \ldots, \mathcal{N} - 1$, for arbitrary distinct $\{y_j\}$, $\{x_j\}$, except for 0.

Examples: Laguerre polynomials 2

- algebraic eqs among the zeros $\{y_n\}$:
$$y_n {\sum_{j=1}^{\mathcal{N}}}' \frac{1}{y_n - y_j} = \frac{1}{2}(y_n - (\alpha + 1))$$
- $\mathcal{N} \times \mathcal{N}$ matrix B by Ahmed, Calogero, et al [1]:

$$B_{n\,m} = \delta_{n\,m} {\sum_{j=1}^{\mathcal{N}}}' \frac{y_j}{(y_n - y_j)^2} - (1 - \delta_{n\,m}) \frac{y_m}{(y_n - y_m)^2},$$

with eigenvector $w_n^{(m)} = \dfrac{L_m^{(\alpha)}(y_n)}{L_{\mathcal{N}-1}^{(\alpha)}(y_n)}$, $n = 1, \ldots, \mathcal{N}$,
for eigenvalues $\frac{1}{2}(\mathcal{N} - m - 1)$, $m = 0, 1, \ldots, \mathcal{N} - 1$.

Examples: Laguerre polynomials 3

- $\mathcal{M} = 4\mathcal{D}(2B+1)\mathcal{D}^{-1}$, $\mathcal{D} \stackrel{\text{def}}{=} \text{diag}(x_1, x_2, \ldots, x_{\mathcal{N}})$, ($\square$)
  $$\Rightarrow \mathrm{v}_n^{(m)} = x_n \frac{L_m^{(\alpha)}(y_n)}{L_{\mathcal{N}-1}^{(\alpha)}(y_n)} \propto \frac{1}{x_n}\frac{L_m^{(\alpha)}(y_n)}{L_{\mathcal{N}}^{(\alpha)\prime}(y_n)}, \ m = 0, 1, \ldots, \mathcal{N}-1$$
- **Corollary** verified, since

$$\eta\frac{dL_n^{(\alpha)}(\eta)}{d\eta} = -\eta L_{n-1}^{(\alpha+1)}(\eta) = nL_n^{(\alpha)}(\eta) - (n+\alpha)L_{n-1}^{(\alpha)}(\eta).$$

- $\square$: by using another equation for $\{y_n\}$:

$$y_n^2 {\sum_{j=1}^{\mathcal{N}}}' \frac{1}{(y_n - y_j)^2} = -\frac{1}{12}\left((\alpha+1)(\alpha+5) - 2(2\mathcal{N}+\alpha+1)y_n + y_n^2\right).$$

Examples: Jocobi polynomials

- basic data

$$\widetilde{\mathcal{H}} = -\frac{d^2}{dx^2} - 2(g\cot x - h\tan x)\frac{d}{dx}, \quad 0 < x < \frac{\pi}{2},$$

$$\eta(x) = \cos 2x, \quad \dot{\eta}(x) = -2\sin 2x, \quad (\dot{\eta}(x))^2 = 4(1-\eta(x)^2),$$

$$\phi_0(x)^2 = (\sin^2 x)^g(\cos^2 x)^h,\ P_n(\eta) = P_n^{(\alpha,\beta)}(\eta),\ \text{Jacobi polynomial},$$

$$\mathcal{E}(n) = 4n(n+g+h) = 4n(n+\alpha+\beta+1),$$

$$\alpha \stackrel{\text{def}}{=} g - \tfrac{1}{2} > -1,\ \beta \stackrel{\text{def}}{=} h - \tfrac{1}{2} > -1.$$

Examples: Jocobi polynomials 2

$$\mathcal{M}_{n\,m} = 4\delta_{n\,m}\Bigg(\mathcal{N}(\mathcal{N}+\alpha+\beta+1)+2(1-y_n^2)\sum_{j<k}^{\mathcal{N}}{}'\frac{1}{y_n-y_j}\cdot\frac{1}{y_n-y_k}$$

$$-((\alpha+\beta)y_n+\alpha-\beta)\sum_{j=1}^{\mathcal{N}}{}'\frac{1}{y_n-y_j}\Bigg)$$

$$+4(1-\delta_{n\,m})\frac{\sin 2x_m}{\sin 2x_n}\cdot\frac{1}{y_n-y_m}\Bigg(2(1-y_n^2)\sum_{j=1,\neq m}^{\mathcal{N}}{}'\frac{1}{y_n-y_j}-((\alpha+\beta)y_n+\alpha-\beta)\Bigg)$$

- if $\alpha+\beta\in\mathbb{Z}$: $\mathcal{M}$ has a remarkable **Diophantine** property. $4(\mathcal{N}-m)(\mathcal{N}+m+\alpha+\beta+1),\quad m=0,1,\ldots,\mathcal{N}-1$, for **arbitrary distinct complex numbers** $\{y_n\}$ and $\{x_n\}$ except for 0 mod $\pi/2$.

Examples: Jocobi polynomials 3

- algebraic eqs among the zeros $\{y_n\}$:
$$(1-y_n^2)\sum_{j=1}^{\mathcal{N}}{}'\frac{1}{y_n-y_j}=\frac{1}{2}\big((\alpha+\beta+2)y_n+\alpha-\beta\big).$$
- $\mathcal{N}\times\mathcal{N}$ matrix C by Ahmed, Calogero, et al [1]:

$$C_{n\,m}=\delta_{n\,m}\sum_{j=1}^{\mathcal{N}}{}'\frac{(1-y_j^2)}{(y_n-y_j)^2}-(1-\delta_{n\,m})\frac{(1-y_m^2)}{(y_n-y_m)^2},$$

with eigenvector $w_n^{(m)}=\dfrac{P_m^{(\alpha,\beta)}(y_n)}{P_{\mathcal{N}-1}^{(\alpha,\beta)}(y_n)}$, $n=1,\ldots,\mathcal{N}$, for $\frac{1}{2}(\mathcal{N}-m-1)(\mathcal{N}+m+\alpha+\beta)$, $m=0,1,\ldots,\mathcal{N}-1$.

Examples: Jocobi polynomials 4

- $\mathcal{M} = 4\mathcal{D}(2C + 2\mathcal{N} + \alpha + \beta)\mathcal{D}^{-1}$, ($\square$)

$$\mathcal{D} \stackrel{\text{def}}{=} \text{diag}(\sin 2x_1, \sin 2x_2, \ldots, \sin 2x_{\mathcal{N}}),$$

$$\Rightarrow \mathrm{v}_n^{(m)} = \sin 2x_n \frac{P_m^{(\alpha,\beta)}(y_n)}{P_{\mathcal{N}-1}^{(\alpha,\beta)}(y_n)} \propto \frac{1}{\sin 2x_n} \frac{P_m^{(\alpha,\beta)}(y_n)}{P_{\mathcal{N}}^{(\alpha,\beta)\prime}(y_n)},$$

$$m = 0, 1, \ldots, \mathcal{N} - 1,$$

- **Corollary** verified, since (Szegö)

$$(2n + \alpha + \beta)(1 - \eta^2)dP_n^{(\alpha,\beta)}(\eta)/d\eta$$

$$= -n\left[(2n + \alpha + \beta)\eta + \beta - \alpha\right] P_n^{(\alpha,\beta)}(\eta) + 2(n + \alpha)(n + \beta)P_{n-1}^{(\alpha,\beta)}(\eta).$$

Examples: Jocobi polynomials 5

(□) is derived by using [1]

$$
\begin{aligned}
&(1-y_n^2)^2 \sum_{j=1}^{\mathcal{N}}{}' \frac{1}{(y_n-y_j)^2} \\
&= \frac{1}{3}(\mathcal{N}-1)(\mathcal{N}+\alpha+\beta+2) - \frac{1}{12}(\alpha-\beta)^2 - \frac{1}{6}(\alpha-\beta)(\alpha+\beta+6)y_n \\
&\quad - \frac{1}{12}\left[4\mathcal{N}(\mathcal{N}+\alpha+\beta+1)+(\alpha+\beta+2)(\alpha+\beta+6)\right] y_n^2 .
\end{aligned}
$$

General Features

- This group contains
  - $\eta(x) = x$: Meixner-Pollaczek, continuous Hahn polynomials
  - $\eta(x) = x^2$: continuous dual Hahn, Wilson polynomials
  - $\eta(x) = \cos x$: Askey-Wilson and its reduced form polynomials
- shift operators with pure imaginary shifts:
  $e^{\pm i\gamma\partial_x}\psi(x) = \psi(x \pm i\gamma)$, $\gamma \in \mathbb{R}$,
- general form of $\widetilde{\mathcal{H}}$ (second order difference eq):

$$\widetilde{\mathcal{H}} = V(x)(e^{-i\gamma\partial_x} - 1) + V^*(x)(e^{i\gamma\partial_x} - 1)$$

- $*$-operation
  If $f(x) = \sum_n a_n x^n$, $a_n \in \mathbb{C}$, $\Rightarrow f^*(x) \stackrel{\text{def}}{=} \sum_n a_n^* x^n$.

Polynomials having $\eta(x) = x$

$$\mathcal{M}_{n\,m} = \delta_{n\,m}\left(-\frac{V(x_n)\prod_{j\neq n}^{\mathcal{N}}(x_n - i - x_j) + V^*(x_n)\prod_{j\neq n}^{\mathcal{N}}(x_n + i - x_j)}{\prod_{j\neq n}^{\mathcal{N}}(x_n - x_j)} + \mathcal{E}(\mathcal{N}) + V(x_n) + V^*(x_n)\right)$$
$$+\, i(1-\delta_{n\,m})\frac{\left(V(x_n)\prod_{j\neq n,m}^{\mathcal{N}}(x_n - i - x_j) - V^*(x_n)\prod_{j\neq n,m}^{\mathcal{N}}(x_n + i - x_j)\right)}{\prod_{j\neq n}^{\mathcal{N}}(x_n - x_j)}.$$

- eigenvalues $\mathcal{E}(\mathcal{N}) - \mathcal{E}(m)$, $m = 0, 1, \ldots, \mathcal{N} - 1$, for arbitrary distinct complex values of $\{x_n\}$.

Polynomials having $\eta(x)=x$, 2

- Meixner-Pollaczek: $V(x)\stackrel{\text{def}}{=}e^{i(\frac{\pi}{2}-\phi)}(a+ix)$,$\mathcal{E}(n)=2n\sin\phi$, $a>0,\ 0<\phi<\pi,\ \phi_0(x)^2=e^{(2\phi-\pi)x}\Gamma(a+ix)\Gamma(a-ix)$,
$$P_n(x)=\frac{(2a)_n}{n!}\,e^{in\phi}{}_2F_1\Bigl(\begin{matrix}-n,\ a+ix\\ 2a\end{matrix}\Bigm|1-e^{-2i\phi}\Bigr)$$
- continuous Hahn: $V(x)=(a_1+ix)(a_2+ix)$,
$$\mathcal{E}(n)=n(n+b_1-1),\ b_1\stackrel{\text{def}}{=}\sum_{j=1}^{4}a_j,\ \{a_3,a_4\}=\{a_1^*,a_2^*\},\ \text{Re}a_j>0,$$
$$\phi_0(x)^2=\prod_{j=1}^{2}\Gamma(a_j+ix)\Gamma(a_j^*-ix),\ P_n(x)=$$
$$i^n\frac{(a_1+a_3)_n(a_1+a_4)_n}{n!}\,{}_3F_2\Bigl(\begin{matrix}-n,\ n+b_1-1,\ a_1+ix\\ a_1+a_3,\ a_1+a_4\end{matrix}\Bigm|1\Bigr).$$

Polynomials having $\eta(x) = x^2$

$$
\begin{aligned}
\mathcal{M}_{n\,m} = \delta_{n\,m}\Bigg(&-\frac{V(x_n)\prod_{j\neq n}^{\mathcal{N}}\big((x_n-i)^2-y_j\big)+V^*(x_n)\prod_{j\neq n}^{\mathcal{N}}\big((x_n+i)^2-y_j\big)}{\prod_{j\neq n}^{\mathcal{N}}(y_n-y_j)} \\
&+\mathcal{E}(\mathcal{N})+V(x_n)+V^*(x_n)\Bigg) \\
+(1-\delta_{n\,m})\frac{x_m}{x_n}\frac{1}{\prod_{j\neq n}^{\mathcal{N}}(y_n-y_j)}&\Bigg(V(x_n)(1+2ix_n)\prod_{j\neq n,m}^{\mathcal{N}}\big((x_n-i)^2-y_j\big) \\
&+V^*(x_n)(1-2ix_n)\prod_{j\neq n,m}^{\mathcal{N}}\big((x_n+i)^2-y_j\big)\Bigg)
\end{aligned}
$$

- eigenvalues $\mathcal{E}(\mathcal{N})-\mathcal{E}(m)$, $m=0,1,\ldots,\mathcal{N}-1$, for arbitrary distinct complex $\{x_n\}$ except for poles of V & V^*, $\{y_n = x_n^2\}$.

the Wilson polynomials

$$V(x) = \frac{(a_1 + ix)(a_2 + ix)(a_3 + ix)(a_4 + ix)}{2ix(2ix + 1)}, \quad V^*(x) = V(-x),$$

$$\phi_0(x)^2 = \frac{\prod_{j=1}^{4} \Gamma(a_j + ix)\Gamma(a_j - ix)}{\Gamma(2ix)\Gamma(-2ix)}, \quad \mathcal{E}(n) = n(n + b_1 - 1),$$

$$\{a_1^*, a_2^*, a_3^*, a_4^*\} = \{a_1, a_2, a_3, a_4\}, \quad \mathrm{Re}\, a_j > 0, \quad b_1 \stackrel{\text{def}}{=} \sum_{j=1}^{4} a_j,$$

$$P_n(\eta(x)) = (a_1 + a_2)_n (a_1 + a_3)_n (a_1 + a_4)_n \times {}_4F_3\Big(\begin{matrix} -n,\ n + b_1 - 1,\ a_1 + ix,\ a_1 - ix \\ a_1 + a_2,\ a_1 + a_3,\ a_1 + a_4 \end{matrix} \Big| \, 1\Big)$$

- integer eigenvalues if $b_1 \in \mathbb{Z}$,

the continuous dual Hahn polynomial

$$V(x) = \frac{(a_1 + ix)(a_2 + ix)(a_3 + ix)}{2ix(2ix + 1)}, \; V^*(x) = V(-x),$$

$$\{a_1^*, a_2^*, a_3^*\} = \{a_1, a_2, a_3\}, \; \mathrm{Re}\, a_j > 0,$$

$$\phi_0(x)^2 = \frac{\prod_{j=1}^{3} \Gamma(a_j + ix)\Gamma(a_j - ix)}{\Gamma(2ix)\Gamma(-2ix)}, \quad \mathcal{E}(n) = n,$$

$$P_n(\eta(x)) = (a_1 + a_2)_n (a_1 + a_3)_n \times {}_3F_2\Big(\begin{matrix} -n, , \; a_1 + ix, \; a_1 - ix \\ a_1 + a_2, \; a_1 + a_3 \end{matrix} \Big| \, 1 \Big).$$

- all integer eigenvalues

General Features

$$
\begin{aligned}
\mathcal{M}_{n\,m} = \delta_{n\,m}\Big\{&\mathcal{E}(\mathcal{N}) + V(x_n) + V^*(x_n) \\
&- \frac{1}{\prod_{j\neq n}^{\mathcal{N}}(y_n - y_j)}\Big(V(x_n)\prod_{j\neq n}^{\mathcal{N}}\big((qz_n + q^{-1}z_n^{-1})/2 - y_j\big) \\
&\qquad + V^*(x_n)\prod_{j\neq n}^{\mathcal{N}}\big((q^{-1}z_n + qz_n^{-1})/2 - y_j\big)\Big)\Big\} \\
+ (1-\delta_{n\,m})&\frac{\sin 2x_m}{\sin 2x_n}\,\frac{(q^{-1}-1)}{2\prod_{j\neq n}^{\mathcal{N}}(y_n - y_j)} \\
\times \Big(&V(x_n)z_n^{-1}(1 - qz_n^2)\prod_{j\neq n,m}^{\mathcal{N}}\big((qz_n + q^{-1}z_n^{-1})/2 - y_j\big) \\
&+ V^*(x_n)z_n(1 - qz_n^{-2})\prod_{j\neq n,m}^{\mathcal{N}}\big((q^{-1}z_n + qz_n^{-1})/2 - y_j\big)\Big)
\end{aligned}
$$

$$
z_n \stackrel{\text{def}}{=} e^{ix_n} = \cos x_n + i\sin x_n = y_n - i\dot{\eta}(x_n)
$$

General Features 2

$\mathcal{M}$ has eigenvalues $\mathcal{E}(\mathcal{N}) - \mathcal{E}(m)$, $m = 0, 1, \ldots, \mathcal{N} - 1$, for arbitrary distinct complex values of $\{x_n\}$ except for the poles of V and V^* and $\{y_n = \cos x_n\}$.

the Askey-Wilson polynomials

$$
\begin{aligned}
&V(x)=\frac{(1-a_1e^{ix})(1-a_2e^{ix})(1-a_3e^{ix})(1-a_4e^{ix})}{(1-e^{2ix})(1-qe^{2ix})},\\
&V^*(x)=V(-x),\quad \{a_1^*,a_2^*,a_3^*,a_4^*\}=\{a_1,a_2,a_3,a_4\},\ |a_j|<1,\\
&\phi_0(x)^2=\frac{(e^{2ix}\,;q)_\infty(e^{-2ix}\,;q)_\infty}{\prod_{j=1}^4(a_je^{ix}\,;q)_\infty(a_je^{-ix}\,;q)_\infty},\\
&\mathcal{E}(n)=(q^{-n}-1)(1-b_4q^{n-1}),\quad b_4\stackrel{\text{def}}{=}a_1a_2a_3a_4,\\
&P_n(\eta(x))=a_1^{-n}(a_1a_2,a_1a_3,a_1a_4\,;q)_n\\
&\qquad\times{}_4\phi_3\Bigl(\begin{matrix}q^{-n},\ b_4q^{n-1},\ a_1e^{ix},\ a_1e^{-ix}\\ a_1a_2,\ a_1a_3,\ a_1a_4\end{matrix}\Bigm|q\,;q\Bigr).
\end{aligned}
$$

The other polynomials

The other polynomials are obtained by restricting the parameters of the Askey-Wilson polynomial.

- the continuous dual q-Hahn polynomial: by $a_4 = 0$,
- the Al-Salam-Chihra polynomial: by $a_4 = a_3 = 0$,
- the continuous big q-Hermite polynomial: by $a_4 = a_3 = a_2 = 0$,
- the continuous q-Hermite polynomial: by $a_4 = a_3 = a_2 = a_1 = 0$,
- the continuous q-Jacobi polynomial: by $a_1 = q^{\frac{1}{2}(\alpha+\frac{1}{2})}$, $a_2 = q^{\frac{1}{2}(\alpha+\frac{3}{2})}$, $a_3 = q^{\frac{1}{2}(\beta+\frac{1}{2})}$, $a_4 = q^{\frac{1}{2}(\beta+\frac{3}{2})}$,
- the continuous q-Laguerre polynomial: by $a_1 = q^{\frac{1}{2}(\alpha+\frac{1}{2})}$, $a_2 = q^{\frac{1}{2}(\alpha+\frac{3}{2})}$, $a_3 = a_4 = 0$,

General Features

- This group contains
  - $\eta(x) = x$: Hahn, Krawtchouk, Meixner, Charlier polynomials,
  - $\eta(x)$ quadratic in x: dual Hahn, Racah polynomials,
  - $\eta(x)$ linear in $q^{\pm x}$: q-Hahn, quantum Krawtchouk, q-Krawtchouk, little q-Jacobi polynomials, etc
  - $\eta(x)$ bilinear in q^{-x} and q^{x}: dual q-Hahn, q-Racah polynomials,
- shift operators with real shifts: $e^{\pm\partial_x}\psi(x) = \psi(x \pm 1)$,
- general form of $\widetilde{\mathcal{H}}$ (second order difference eq):

$$\widetilde{\mathcal{H}} = B(x)(1 - e^{\partial}) + D(x)(1 - e^{-\partial}),\ \ B(x) \geq 0,\ \ D(x) \geq 0,$$

  acting on finite $[0, 1, \ldots, N]$ or infinite $[0, 1, \ldots, \infty)$ integer lattice
- boundary condition

$$D(0) = 0, \quad B(N) = 0, \quad \mathcal{N} < N,$$

Polynomials having $\eta(x) = x$

$$\mathcal{M}_{n\,m} = \delta_{n\,m}\Bigg(\frac{B(x_n)\prod_{j\neq n}^{\mathcal{N}}(x_n+1-x_j) + D(x_n)\prod_{j\neq n}^{\mathcal{N}}(x_n-1-x_j)}{\prod_{j\neq n}^{\mathcal{N}}(x_n-x_j)} + \mathcal{E}(\mathcal{N}) - B(x_n) - D(x_n)\Bigg)$$

$$+ (1-\delta_{n\,m})\frac{\Big(B(x_n)\prod_{j\neq n,m}^{\mathcal{N}}(x_n+1-x_j) - D(x_n)\prod_{j\neq n,m}^{\mathcal{N}}(x_n-1-x_j)\Big)}{\prod_{j\neq n}^{\mathcal{N}}(x_n-x_j)}$$

- eigenvalues $\mathcal{E}(\mathcal{N}) - \mathcal{E}(m)$, $m = 0, 1, \ldots, \mathcal{N}-1$, for arbitrary distinct complex values of $\{x_n\}$

Polynomials having $\eta(x) = x$, 2

- Hahn:

$$B(x) = (x+a)(N-x), \quad D(x) = x(b+N-x), \quad a>0,\ b>0,$$

$$\phi_0(x)^2 = \frac{N!}{x!\,(N-x)!}\,\frac{(a)_x\,(b)_{N-x}}{(b)_N}, \quad \mathcal{E}(n) = n(n+a+b-1),$$

$$P_n(x) = {}_3F_2\Big(\begin{matrix}-n,\ n+a+b-1,\ -x\\ a,\ -N\end{matrix}\ \Big|\ 1\Big)$$

- Krawtchouk:

$$B(x) = p(N-x), \quad D(x) = (1-p)x, \quad 0<p<1, \quad \mathcal{E}(n) = n,$$

$$\phi_0(x)^2 = \frac{N!}{x!\,(N-x)!}\Big(\frac{p}{1-p}\Big)^x, \quad P_n(x) = {}_2F_1\Big(\begin{matrix}-n,\ -x\\ -N\end{matrix}\ \Big|\ p^{-1}\Big)$$

Polynomials having $\eta(x) = x$, 3

- Meixner:

$$B(x) = \frac{c}{1-c}(x+\beta), \quad D(x) = \frac{1}{1-c}x, \qquad \beta > 0, \quad 0 < c < 1$$

$$\phi_0(x)^2 = \frac{(\beta)_x\, c^x}{x!}, \quad P_n(x) = {}_2F_1\Bigl(\begin{matrix}-n,\ -x\\ \beta\end{matrix}\Bigm| 1-c^{-1}\Bigr), \quad \mathcal{E}(n) = n$$

- Charlier:

$$B(x) = a, \quad D(x) = x, \quad a > 0, \quad \mathcal{E}(n) = n,$$

$$\phi_0(x)^2 = \frac{a^x}{x!}, \quad P_n(x) = {}_2F_0\Bigl(\begin{matrix}-n,\ -x\\ -\end{matrix}\Bigm| -a^{-1}\Bigr)$$

Polynomials having $\eta(x)$ quadratic in x

$$\mathcal{M}_{n\,m} = \delta_{n\,m}\Big(\frac{B(x_n)\prod_{j\neq n}^{\mathcal{N}}\big(\eta(x_n+1)-y_j\big)+D(x_n)\prod_{j\neq n}^{\mathcal{N}}\big(\eta(x_n-1)-y_j\big)}{\prod_{j\neq n}^{\mathcal{N}}(y_n-y_j)} + \mathcal{E}(\mathcal{N})-B(x_n)-D(x_n)\Big)$$

$$+(1-\delta_{n\,m})\frac{\dot{\eta}(x_m)}{\dot{\eta}(x_n)}\frac{1}{\prod_{j\neq n}^{\mathcal{N}}(y_n-y_j)}\Big(B(x_n)\prod_{j\neq m}^{\mathcal{N}}\big(\eta(x_n+1)-y_j\big) + D(x_n)\prod_{j\neq m}^{\mathcal{N}}\big(\eta(x_n-1)-y_j\big)\Big) \quad (\star)$$

- eigenvalues $\mathcal{E}(\mathcal{N})-\mathcal{E}(m)$, $m=0,1,\ldots,\mathcal{N}-1$, for arbitrary distinct complex values of $\{x_n\}$ except f poles of $B(x)$, $D(x)$.

Polynomials having $\eta(x)$ quadratic in x, 2

- Racah $(c = -N,\ d > 0,\ a > N + d,\ 0 < b < 1 + d)$:

$$B(x) = -\frac{(x+a)(x+b)(x+c)(x+d)}{(2x+d)(2x+1+d)},$$

$$D(x) = -\frac{(x+d-a)(x+d-b)(x+d-c)x}{(2x-1+d)(2x+d)},$$

$$\mathcal{E}(n) = n(n+\tilde{d}), \qquad \tilde{d} \stackrel{\text{def}}{=} a+b+c-d-1,$$

$$\eta(x) = x(x+d), \qquad \dot{\eta}(x) = 2x+d,$$

$$\phi_0(x)^2 = \frac{(a,b,c,d)_x}{(1+d-a,1+d-b,1+d-c,1)_x}\,\frac{2x+d}{d},$$

$$P_n(\eta(x)) = {}_4F_3\left(\begin{matrix}-n,\ n+\tilde{d},\ -x,\ x+d\\ a,\ b,\ c\end{matrix}\ \middle|\ 1\right)$$

Polynomials having $\eta(x)$ quadratic in x, 3

- dual Hahn:

$$B(x) = \frac{(x+a)(x+a+b-1)(N-x)}{(2x-1+a+b)(2x+a+b)}, \quad a>0, \quad b>0,$$

$$D(x) = \frac{x(x+b-1)(x+a+b+N-1)}{(2x-2+a+b)(2x-1+a+b)},$$

$$\mathcal{E}(n) = n, \quad \eta(x) = x(x+a+b-1), \quad \dot{\eta}(x) = 2x+a+b-1,$$

$$\phi_0(x)^2 = \frac{N!}{x!\,(N-x)!}\,\frac{(a)_x\,(2x+a+b-1)(a+b)_N}{(b)_x\,(x+a+b-1)_{N+1}},$$

$$P_n(\eta(x)) = {}_3F_2\Big(\begin{matrix}-n,\ x+a+b-1,\ -x\\ a,\ -N\end{matrix}\ \Big|\ 1\Big)$$

Polynomials having $\eta(x)$ linear in q^{-x}

- the rest of the polynomials has the $\mathcal{M}$ matrix of the same structure as that of the Racah $(\star)$.
- q-Hahn ($0 < a < 1$ and $0 < b < 1$):

$$B(x) = (1 - aq^x)(q^{x-N} - 1),\ D(x) = aq^{-1}(1 - q^x)(q^{x-N} - b),$$

$$\phi_0(x)^2 = \frac{(q;q)_N}{(q;q)_x(q;q)_{N-x}}\frac{(a;q)_x(b;q)_{N-x}}{(b;q)_N\, a^x}, \mathcal{E}(n) = (q^{-n} - 1)(1 - abq^{n-1}),$$

$$P_n(\eta(x)) = {}_3\phi_2\left(\begin{matrix} q^{-n},\ abq^{n-1},\ q^{-x} \\ a,\ q^{-N} \end{matrix} \;\middle|\; q\,;q\right)$$

- q-Charlier ($a > 0$):

$$B(x) = aq^x, \quad D(x) = 1 - q^x, \qquad \mathcal{E}(n) = 1 - q^n,$$

$$\phi_0(x)^2 = \frac{a^x q^{\frac{1}{2}x(x-1)}}{(q\,;q)_x},\ P_n(\eta(x)) = {}_2\phi_1\left(\begin{matrix} q^{-n},\ q^{-x} \\ 0 \end{matrix} \;\middle|\; q\,;-a^{-1}q^{n+1}\right)$$

Polynomials having $\eta(x)$ linear in q^x

- little q-Jacobi ($0 < a, b < q^{-1}$):

$$B(x)=a(q^{-x}-bq),\ D(x)=q^{-x}-1,\ \mathcal{E}(n)=(q^{-n}-1)(1-abq^{n+1}),$$

$$\phi_0(x)^2=\frac{(bq\,;q)_x}{(q\,;q)_x}(aq)^x,$$

$$P_n(\eta(x))=(-a)^{-n}q^{-\frac{1}{2}n(n+1)}\frac{(aq\,;q)_n}{(bq\,;q)_n}\,{}_2\phi_1\Bigl(\begin{matrix}q^{-n},\ abq^{n+1}\\ aq\end{matrix}\,\Bigm|\,q\,;q^{x+1}\Bigr)$$

- alternative q-Charlier ($a > 0$):

$$B(x)=a,\quad D(x)=q^{-x}-1,\quad \mathcal{E}(n)=(q^{-n}-1)(1+aq^n),$$

$$\phi_0(x)^2=\frac{a^x q^{\frac{1}{2}x(x+1)}}{(q;q)_x},\ \ P_n(\eta(x))=q^{nx}{}_2\phi_1\Bigl(\begin{matrix}q^{-n},q^{-x}\\ 0\end{matrix}\,\Bigm|\,q;-a^{-1}q^{-n+1}\Bigr)$$

Polynomials having $\eta(x)$ bilinear in q^{-x} & q^x

- q-Racah ($c = q^{-N}$, $0 < d < 1, 0 < a < q^N d$, $qd < b < 1$, $\tilde{d} \stackrel{\text{def}}{=} abcd^{-1}q^{-1}$):

$$B(x) = -\frac{(1-aq^x)(1-bq^x)(1-cq^x)(1-dq^x)}{(1-dq^{2x})(1-dq^{2x+1})},$$

$$D(x) = -\tilde{d}\,\frac{(1-a^{-1}dq^x)(1-b^{-1}dq^x)(1-c^{-1}dq^x)(1-q^x)}{(1-dq^{2x-1})(1-dq^{2x})},$$

$$\mathcal{E}(n) = (q^{-n}-1)(1-\tilde{d}q^n), \quad \eta(x) = (q^{-x}-1)(1-dq^x),$$

$$\phi_0(x)^2 = \frac{(a,b,c,d\,;q)_x}{(a^{-1}dq, b^{-1}dq, c^{-1}dq, q\,;q)_x\,\tilde{d}^x}\,\frac{1-dq^{2x}}{1-d},$$

$$P_n(\eta(x)) = {}_4\phi_3\left(\begin{matrix} q^{-n},\ \tilde{d}q^n,\ q^{-x},\ dq^x \\ a,\ b,\ c \end{matrix}\ \middle|\ q\,;q\right)$$

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6.7. Electrostatic interpretation of the zeros of the classical polynomials

Stieltjes gave (**4**, **5**; **6**, pp. 75–76; cf., also, Schur **1**) a very interesting derivation of the differential equations of the classical polynomials, which is closely connected with the calculation of the discriminant of these polynomials (cf. §6.71) and can be interpreted as a problem of electrostatic equilibrium.

(1) Problem. *Let p and q be two given positive numbers. If n unit "masses," $n \geqq 2$, at the variable points $x_1, x_2, x_3, \cdots, x_n$ in the interval $[-1, +1]$ and the fixed masses p and q at $+1$ and -1, respectively, are considered, for what position of the points $x_1, x_2, x_3, \cdots, x_n$ does the expression*

$$T(x_1, x_2, \cdots, x_n) = T(x) = \prod_{\kappa=1}^{n} (1 - x_\kappa)^p (1 + x_\kappa)^q \prod_{\substack{\nu,\mu=1,2,\cdots,n \\ \nu<\mu}} |x_\nu - x_\mu| \tag{6.7.1}$$

become a maximum?

Obviously, $\log (T^{-1})$ can be interpreted as the energy of the system of electrostatic masses just defined. They exert repulsive forces according to the law of logarithmic potential. The maximum position corresponds to the condition of electrostatic equilibrium. A maximum exists because T is a continuous function of $x_1, x_2, \cdots, x_n$ for $-1 \leqq x_\nu \leqq +1$, $\nu = 1, 2, \cdots, n$. It is clear that in the maximum position the x_ν are each different from ± 1 and from one another. In addition, this position is uniquely determined. To show this, let us suppose that (cf. Popoviciu **2**, p. 74)

$$\begin{aligned} &+1 > x_1 > x_2 > \cdots > x_n > -1, \\ &+1 > x_1' > x_2' > \cdots > x_n' > -1 \end{aligned} \tag{6.7.2}$$

are two positions of this kind; we write

$$y_\nu = (x_\nu + x_\nu')/2, \qquad \nu = 1, 2, \cdots, n. \tag{6.7.3}$$

Then

$$\begin{aligned} |y_\nu - y_\mu| &= \frac{|x_\nu - x_\mu| + |x_\nu' - x_\mu'|}{2} \geqq |x_\nu - x_\mu|^{\frac{1}{2}} |x_\nu' - x_\mu'|^{\frac{1}{2}}, \\ |1 \pm y_\nu| &\geqq |1 \pm x_\nu|^{\frac{1}{2}} |1 \pm x_\nu'|^{\frac{1}{2}}, \end{aligned} \tag{6.7.4}$$

so that $T(y) \geqq \{T(x)\}^{\frac{1}{2}} \{T(x')\}^{\frac{1}{2}}$, the equality sign being taken if and only if $x_\nu = x_\nu'$. This establishes the uniqueness.

Theorem 6.7.1. *Let $p > 0$, $q > 0$, and let $\{x_\nu\}$, $-1 \leqq x_\nu \leqq +1$, be a system of values for which the expression (6.7.1) becomes a maximum. Then the $\{x_\nu\}$ are the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, where $\alpha = 2p - 1$, $\beta = 2q - 1$.*