Connection Formula for Heine's Hypergeometric Function with $|q| = 1$

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based on arXiv:1411.2307[math-ph], J. Phys. A in **48** (2015) 115204, with S. Odake

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Outline

1. Introduction
   - Exactly solvable Schrödinger equations
   - Exactly solvable difference Schrödinger equations
2. Strategy
3. “Solution”
4. Outlook

Target: Exactly solvable difference Schrödinger eq. with finitely many discrete eigenstates

Background

Exactly Solvable Quantum Mechanics $\Rightarrow$ orthogonal eigenfunctions

$\Downarrow$ provides

unified theory of classical orthogonal polynomials;
ordinary Schrödinger eq. Hermite, Laguerre, Jacobi,
difference Schrödinger eq. Wilson, Askey-Wilson, Racah, q-Racah etc, so-called Askey-scheme of hypergeometric orthogonal polynomials cf. S. Odake & RS: "Discrete Quantum Mechanics," J. Phys. **A44** (2011) 353001

Exactly solvable difference Schrödinger eq. with non-confining potentials $\Rightarrow$ non-complete set of eigenfunctions

$\Downarrow$

continuous spectrum $\Rightarrow$ scattering problem
$\Rightarrow$ connection formulas for (q-) hypergeometric functions, $|q| = 1$

Exactly solvable ordinary Schrödinger equations

- 1 degree of freedom, Schrödinger operator (Hamiltonian) $\mathcal{H} = -\frac{d^2}{dx^2} + U(x)$: all eigenvalues $\mathcal{E}_n$ and eigenfunctions $\phi_n(x)$ are exactly calculable

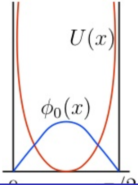
$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x), \quad n = 0, 1, \ldots,$$

- confining potentials, having ∞ discrete eigenlevels:
- Harmonic oscillator, Hermite polynomial $U(x) = x^2 - 1$, $-\infty < x < \infty$, $\mathcal{E}_n = 2n$, $\phi_n(x) = \phi_0(x)H_n(x)$, n=0,1..., $\phi_0(x) = e^{-x^2/2}$,
- Radial oscillator, Laguerre polynomial $U(x) = x^2 + g(g-1)/x^2 - (1+2g)$, $0 < x < \infty$, $\mathcal{E}_n = 4n$, $\phi_n(x) = \phi_0(x)L_n^{(\alpha)}(x^2)$, n=0,1..., $\alpha = g - 1/2$ $\phi_0(x) = e^{-x^2/2}x^g$,

Exactly solvable ordinary Schrödinger eqs 2

- confining potentials, having ∞ discrete eigenlevels: continued
- Pöschl-Teller, Jacobi polynomial
  $U(x) = \dfrac{g(g-1)}{\sin^2 x} + \dfrac{h(h-1)}{\cos^2 x} - (g+h)^2,\ 0 < x < \pi/2$
  $\mathcal{E}_n = 4n(n+g+h),\ \phi_n(x) = \phi_0(x) P_n^{(\alpha,\beta)}(\cos 2x)$, n=0,1. . . ,
  $\alpha = g - 1/2,\ \beta = h - 1/2,\ \phi_0(x) = (\sin x)^g (\cos x)^h$,
- ground state eigenfunction squared $\phi_0^2(x)$ provides the orthogonality weight function

no scattering problem!



Exactly solvable difference Schrödinger eqs

- confining potentials, having ∞ discrete eigenlevels: continued
- Wilson poly. $V(x) = \dfrac{\prod_{j=1}^{4}(a_j + ix)}{2ix(2ix+1)}$, $0 < x < \infty$, $Re(a_j) > 0$,

  singular at $x = 0$, $x = \infty$, $\mathcal{E}_n = n(n + b_1 - 1)$, $b_1 = \sum_{j=1}^{4} a_j$,
- Askey-Wilson poly. $V(x) = \dfrac{\prod_{j=1}^{4}(1 - a_j e^{ix})}{(1 - e^{2ix})(1 - qe^{2ix})}$, $0 < x < \pi$, $|a_j| < 1$, singular at $x = 0,\pi$, $\mathcal{E}_n = (q^{-n} - 1)(1 - b_4 q^{n-1})$, $b_4 = a_1 a_2 a_3 a_4$

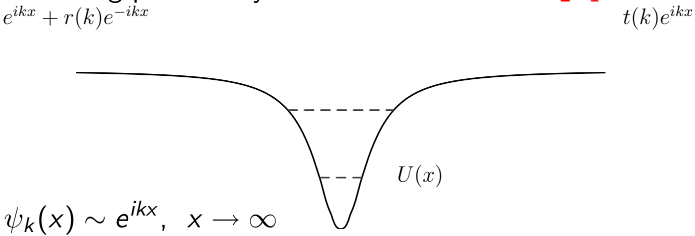
Exactly solvable ordinary Schrödinger eqs, non-confining

- $U(x) = \dfrac{g(g-1)}{\sin^2 x}$, confining Gegenbauer poly

  $\Downarrow\ x \longrightarrow \pi/2 - ix$
- $U(x) \propto -1/\cosh^2 x,\ \lim_{x\to\pm\infty} U(x) = 0$, non-confining, soliton potential, or so called 'soliton' potential finitely many discrete eigenstates

Exactly solvable ordinary Schrödinger eqs, non-confining 2

- non-confining potentials, having finite discrete eigenlevels: all eigenvalues $\mathcal{E}_n$ and eigenfunctions $\phi_n(x)$ and scattering amplitudes $t(k)$, $r(k)$ are exactly calculable
- scattering problem by connection formula for ${}_2F_1$



$\psi_k(x) \sim e^{ikx}, \quad x \to \infty$
analytically continue to $x \to -\infty$
$\psi_k(x) \sim A(k)e^{ikx} + B(k)e^{-ikx}$, $t(k) = 1/A(k)$,
$r(k) = B(k)/A(k)$, unitarity $|t(k)|^2 + |r(k)|^2 = 1$

Problem: Find the discrete analogues

- discrete analogues $x \to \pi/2 - ix$ in q-ultraspherical poly with $|q| = 1$, polynomials are OK
- discrete reflection potentials $\Rightarrow$ analogues of $1/\cosh^2 x$ potential
- for scattering problem connection formula for ${}_2\phi_1$ is needed
- ${}_2\phi_1$ with $|q| = 1$ unknown
- scattering amplitudes obtained from conjectured connection formula for ${}_2\phi_1$
- several supporting evidence presented

Solutions for $-h(h+1)/\cosh^2 x$ potential

- $U(x) = -h(h+1)/\cosh^2 x,\ -\infty < x < \infty,\ \mathcal{E}_n = -(h-n)^2,$
  $(n = 0, 1, \dots [h]')$, invariant $h \leftrightarrow -(h+1)$
  $\phi_n(x) = (2\cosh x)^{-h+n} \times P_n^{(h-n,h-n)}(\tanh x),$
  Gegenbauer or ultraspherical polynomial

$$t(k) = \frac{\Gamma(-h-ik)\Gamma(1+h-ik)}{\Gamma(-ik)\Gamma(1-ik)},$$

$$r(k) = \frac{\Gamma(ik)\Gamma(-h-ik)\Gamma(1+h-ik)}{\Gamma(-ik)\Gamma(-h)\Gamma(1+h)},\quad \text{invariant } h \leftrightarrow -(h+1)$$

- rewrite by Gauss hypergeometric function
  $$\phi_n(x) = (2\cosh x)^{-h+n} \times {}_2F_1\left(\begin{matrix} -n,\ -n+2h+1 \\ h-n+1 \end{matrix} \middle| \frac{1-\tanh x}{2}\right)$$
- introduce k by $n \to h + ik$:

$$(2\cosh x)^{ik}{}_2F_1\left(\begin{matrix} -h-ik,\ 1+h-ik \\ 1-ik \end{matrix} \middle| \frac{1-\tanh x}{2}\right) \to e^{ikx},\ x \to +\infty$$

Gaussian Connection formula ⇒ Scattering amplitudes

- analytically continue to $x \to -\infty$ by connection formula

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix}\middle| z\right)\\
&\quad= \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}\cdot {}_2F_1\left(\begin{matrix}\gamma-\alpha,\ \gamma-\beta\\ \gamma-\alpha-\beta+1\end{matrix}\middle| 1-z\right)\\
&\qquad+\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\cdot {}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \alpha+\beta-\gamma+1\end{matrix}\middle| 1-z\right),
\end{aligned}
$$

- positive integer $h \in \mathbb{N}$, $1/\Gamma(-h) = 0 \Rightarrow r(k) = 0$ reflectionless
- general reflectionless potential of Schrödinger eq. = profile of KdV soliton, last year's talk

general difference Schrödinger eqs

- Schrödinger operator (Hamiltonian):

$$\mathcal{H} = \sqrt{V(x)V^*(x-i\gamma)}\, e^{-i\gamma\partial} + \sqrt{V^*(x)V(x+i\gamma)}\, e^{+i\gamma\partial} - V(x) - V^*(x), \quad e^{\pm i\gamma\partial}\psi(x) = \psi(x \pm i\gamma),\ \gamma \in \mathbb{R},$$

- confining potentials, having ∞ discrete eigenlevels:
- $V(x) = \dfrac{\prod_{j=1}^4 (1 - a_j e^{ix})}{(1-e^{2ix})(1-q\, e^{2ix})}$, $|a_j| < 1$, $0 < x < \pi$,

  Askey-Wilson polynomial $\mathcal{E}_n = (q^{-n} - 1)(1 - b_4 q^{n-1})$,

  $$0 < q = e^{\gamma} < 1,\ b_4 = \prod_{j=1}^{4} a_j,\ \phi_n(x) = \phi_0(x) P_n(\cos x),$$

  $$P_n(\cos x) = p_n(\cos x\,; a_1, a_2, a_3, a_4 | q)$$

  $$\stackrel{\text{def}}{=} a_1^{-n} (a_1 a_2, a_1 a_3, a_1 a_4\,; q)_n\, {}_4\phi_3\left(\begin{matrix} q^{-n},\ b_4 q^{n-1},\ a_1 e^{ix},\ a_1 e^{-ix} \\ a_1 a_2,\ a_1 a_3,\ a_1 a_4 \end{matrix} \,\middle|\, q\,; q\right),$$

general difference Schrödinger eqs, 2

- squared ground state wave function ⇒ orthogonality weight function: $(a;q)_n \stackrel{\text{def}}{=} \prod_{j=1}^{n}(1-aq^{j-1})$,

$$\phi_0(x)^2=(e^{2ix};q)_\infty(e^{-2ix};q)_\infty\prod_{j=1}^{4}\left((a_je^{ix};q)_\infty(a_je^{-ix};q)_\infty\right)^{-1}$$

- non-confining potentials, having finite discrete eigenlevels: all eigenvalues $\mathcal{E}_n$ and eigenfunctions $\phi_n(x)$ are exactly calculable

Explicit form of discrete analogue of $1/\cosh^2 x$ potential

- $V(x;h) = e^{-i\gamma h}\dfrac{(1+e^{i\gamma h}e^{2x})(1+e^{i\gamma(h-1)}e^{2x})}{(1+e^{2x})(1+e^{-i\gamma}e^{2x})} \to 1,$
  $x \to \pm\infty,$
  $\mathcal{H}$: invariant $h \leftrightarrow -(h+1), \;\; -\infty < x < +\infty,$ $q = e^{-i\gamma}$,

$$\lim_{\gamma\to 0} \gamma^{-2}\mathcal{H} = -\frac{d^2}{dx^2} - \frac{h(h+1)}{\cosh^2 x},$$

Explicit form of discrete analogue of $1/\cosh^2 x$ potential 2

Discrete eigenstates

- $\mathcal{E}_n = -4\sin^2\frac{\gamma}{2}(h-n),\ n = 0, 1, \ldots, [h]'$.
  $\phi_n(x) = \phi_0(x)P_n(\eta),\ \eta = \sinh x$

  $$P_n(\eta) \propto p_n(i\sinh x; e^{i\frac{\gamma}{2}h}, e^{i\frac{\gamma}{2}(h-1)}, -e^{i\frac{\gamma}{2}h}, -e^{i\frac{\gamma}{2}(h-1)}|e^{-i\gamma}) : \text{AW}$$
  $$\propto C_n(i\sinh x; \beta|q),\ q = e^{-i\gamma}, |q| = 1,\ \beta \stackrel{\text{def}}{=} e^{i\gamma h} = q^{-h}$$
  $$= e^{nx}{}_2\phi_1\left(\begin{matrix} q^{-n}, \beta \\ \beta^{-1}q^{1-n}\end{matrix}\middle| q\,;\beta^{-1}qe^{-2x-i\pi}\right),$$

  q-ultraspherical polynomial well-defined for $|q| = 1$,
- Heine's q-hypergeometric functions

Discrete eigenstates, Quantum Dilog as weight function

- $(a;q)_\infty$ does Not converge for $|q|=1$,
- ground state eigenfunction by Quantum Dilog $\Phi_\gamma(x)$:

$$\phi_0(x)^2 = e^{2hx}(1+e^{2x})\frac{\Phi_{\frac{\gamma}{2}}\left(2x+i\gamma(h+\frac{1}{2})\right)}{\Phi_{\frac{\gamma}{2}}\left(2x-i\gamma(h+\frac{1}{2})\right)},$$

- quantum dilog, integral rep $\leftrightarrow$ q-Gamma function for $|q|=1$

$$\Phi_\gamma(z) = \exp\left(\int_{\mathbb{R}+i0} \frac{e^{-izt}}{4\sinh\gamma t\,\sinh\pi t}\frac{dt}{t}\right) \qquad (|\mathrm{Im}\,z| < \gamma+\pi),$$

- functional relations

$$\frac{\Phi_\gamma(z+i\gamma)}{\Phi_\gamma(z-i\gamma)} = \frac{1}{1+e^z}, \quad \frac{\Phi_\gamma(z+i\pi)}{\Phi_\gamma(z-i\pi)} = \frac{1}{1+e^{\frac{\pi}{\gamma}z}},$$

Reflectionless potentials in difference Schrödinger eqs

- Scattering problem, trivial potential $V(x) \equiv 1$,

$$p = -i\partial_x, \quad \mathcal{H} \to \mathcal{H}_0 = e^{\gamma p} + e^{-\gamma p} - 2,$$
$$\mathcal{H}_0 e^{\pm kx} = \tilde{\mathcal{E}}_k e^{\pm kx}, \; \tilde{\mathcal{E}}_k \stackrel{\text{def}}{=} -4\sin^2\frac{k\gamma}{2} < 0,$$
$$\mathcal{H}_0 e^{\pm ikx} = \mathcal{E}^{\mathrm{s}}_k e^{\pm ikx}, \; \mathcal{E}^{\mathrm{s}}_k \stackrel{\text{def}}{=} 4\sinh^2\frac{k\gamma}{2} > 0.$$

- $\gamma \to 0$ limit:

$$\lim_{\gamma\to 0}\gamma^{-2}\mathcal{H}_0 = p^2, \quad \lim_{\gamma\to 0}\gamma^{-2}\tilde{\mathcal{E}}_k = -k^2, \quad \lim_{\gamma\to 0}\gamma^{-2}\mathcal{E}^{\mathrm{s}}_k = k^2,$$

General discrete reflectionless potentials

- Darboux transformation with seed solutions

$$\psi_j(x) = e^{k_j x} + \tilde{c}_j e^{-k_j x} = \psi_j^*(x), \quad 0 < k_1 < k_2 < \cdots < k_N \le \tfrac{\pi}{\gamma},$$
$$\mathcal{H}_0 \psi_j(x) = \tilde{\mathcal{E}}_{k_j} \psi_j(x), \qquad (-1)^{j-1}\tilde{c}_j > 0,$$

- first step Darboux transformation

$$\mathcal{H}_0 = \hat{\mathcal{A}}_1^\dagger \hat{\mathcal{A}}_1 + \tilde{\mathcal{E}}_{k_1}, \quad \hat{\mathcal{A}}_1 \stackrel{\text{def}}{=} i\Big(e^{\frac{\gamma}{2}p}\sqrt{\hat{V}_1^*(x)} - e^{-\frac{\gamma}{2}p}\sqrt{\hat{V}_1(x)}\,\Big),$$
$$\hat{V}_1(x) \stackrel{\text{def}}{=} \frac{\psi_1(x - i\gamma)}{\psi_1(x)}.$$

- N-step Darboux transformation

$$\mathcal{H}^{[N]} = \mathcal{A}^{[N]\dagger}\mathcal{A}^{[N]} + \tilde{\mathcal{E}}_{k_N}, \mathcal{A}^{[N]} \stackrel{\text{def}}{=} i\Big(e^{\frac{\gamma}{2}p}\sqrt{V^{[N]*}(x)} - e^{-\frac{\gamma}{2}p}\sqrt{V^{[N]}(x)}\,\Big)$$
$$V^{[N]}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}_\gamma[\psi_1, \ldots, \psi_{N-1}](x - i\gamma)}{\mathrm{W}_\gamma[\psi_1, \ldots, \psi_{N-1}](x)} \frac{\mathrm{W}_\gamma[\psi_1, \ldots, \psi_N](x + i\frac{\gamma}{2})}{\mathrm{W}_\gamma[\psi_1, \ldots, \psi_N](x - i\frac{\gamma}{2})}$$

General discrete reflectionless potentials 2

- right moving wave $\Psi_k^{[N]}(x)$ $(k > 0)$

$$\mathcal{H}^{[N]}\Psi_k^{[N]}(x) = \mathcal{E}_k^s \Psi_k^{[N]}(x),$$

$$\Psi_k^{[N]}(x) = \frac{\mathrm{W}_\gamma[\psi_1, \ldots, \psi_N, e^{ikx}](x)}{\left(\mathrm{W}_\gamma[\psi_1, \ldots, \psi_N](x - i\frac{\gamma}{2})\mathrm{W}_\gamma[\psi_1, \ldots, \psi_N](x + i\frac{\gamma}{2})\right)^{\frac{1}{2}}}$$

- discrete eigenfunctions $\Phi_j^{[N]}(x)$ $(j = 1, 2, \ldots, N)$

$$\mathcal{H}^{[N]}\Phi_j^{[N]}(x) = \tilde{\mathcal{E}}_{k_j} \Phi_j^{[N]}(x),$$

$$\Phi_j^{[N]}(x) = \frac{\mathrm{W}_\gamma[\psi_1, \ldots, \breve{\psi}_j, \ldots, \psi_N](x)}{\left(\mathrm{W}_\gamma[\psi_1, \ldots, \psi_N](x - i\frac{\gamma}{2})\mathrm{W}_\gamma[\psi_1, \ldots, \psi_N](x + i\frac{\gamma}{2})\right)^{\frac{1}{2}}},$$

General discrete reflectionless potentials 3

- Casoratian

$$\mathrm{W}_\gamma[f_1,\ldots,f_n](x) \stackrel{\text{def}}{=} i^{\frac{1}{2}n(n-1)} \det\Big(f_k\big(x_j^{(n)}\big)\Big)_{1\le j,k\le n},$$

$$x_j^{(n)} \stackrel{\text{def}}{=} x + i(\tfrac{n+1}{2} - j)\gamma,$$

$$\lim_{\gamma\to 0} \gamma^{-\frac{1}{2}n(n-1)} \mathrm{W}_\gamma[f_1, f_2, \ldots, f_n](x) = \mathrm{W}[f_1, f_2, \ldots, f_n](x),$$

- Casoratian of exponentials

$$\mathrm{W}_\gamma[e^{k_1 x}, e^{k_2 x}, \ldots, e^{k_n x}](x) = \prod_{1\le i<j\le n} 2\sin\tfrac{\gamma}{2}(k_j - k_i)\cdot e^{\sum_{j=1}^n k_j x}$$

- transmission and reflection amplitudes:

$$t^{[N]}(k) = \prod_{j=1}^N \frac{\sin\frac{\gamma}{2}(ik - k_j)}{\sin\frac{\gamma}{2}(ik + k_j)} = \prod_{j=1}^N \frac{\sinh\frac{\gamma}{2}(k + ik_j)}{\sinh\frac{\gamma}{2}(k - ik_j)}, \quad r^{[N]}(k) = 0$$

Discrete analogue of $1/\cosh^2 x$ potentials 2

- choose integer wavenumbers $k_j = j, \quad \tilde{c}_j = (-1)^{j-1}$
  reflectionless potential

$$V^{[N]}(x) = e^{-i\gamma N}\frac{(1+e^{i\gamma N}e^{2x})(1+e^{i\gamma(N-1)}e^{2x})}{(1+e^{2x})(1+e^{-i\gamma}e^{2x})} \equiv V(x;N)$$

  ground state eigenfunction

$$\Phi_N^{[N]}(x) = \Big(\prod_{j=1}^{N-1} 2\sin\tfrac{\gamma}{2}j\Big)^{-1}\cdot\Big(\prod_{j=1}^{N} 4\cosh(x-i\tfrac{\gamma}{2}j)\cosh(x+i\tfrac{\gamma}{2}j)\Big)^{-\frac{1}{2}}$$

- quantum dialog degenerates to elementary functions for $h = N$.

Scattering Wavefunction

- scattering wave function is obtained by setting $n \to h + ik$ in eigenfunction:

$$\Psi_k(x) \overset{\text{def}}{=} e^{ikx} e^{2hx} \sqrt{1+e^{2x}} \left(\frac{\Phi_{\frac{\gamma}{2}}\left(2x + i\gamma(h+\frac{1}{2})\right)}{\Phi_{\frac{\gamma}{2}}\left(2x - i\gamma(h+\frac{1}{2})\right)} \right)^{\frac{1}{2}}$$

$$\times \, {}_2\phi_1\left(\begin{matrix} q^{-h-ik}, \, q^{-h} \\ q^{1-ik} \end{matrix} \middle| \, q\, ; q^{1+h} e^{-2x-i\pi} \right),$$

infinite series: convergence unclear for $|q| = 1$,

Connection Formula ${}_2\phi_1$, $0 < q < 1$

- connection formula for Heine's q-hypergeometric function known only for $0 < q < 1$ (Watson 1910)

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\middle| q\,;z\right) \\
&= \frac{(b, c/a; q)_\infty}{(c, b/a; q)_\infty}\frac{(az, q/(az); q)_\infty}{(z, q/z; q)_\infty}\cdot{}_2\phi_1\left(\begin{matrix} a,\ aq/c \\ aq/b \end{matrix}\middle| q\,;\frac{cq}{abz}\right) \\
&\quad + \frac{(a, c/b; q)_\infty}{(c, a/b; q)_\infty}\frac{(bz, q/(bz); q)_\infty}{(z, q/z; q)_\infty}\cdot{}_2\phi_1\left(\begin{matrix} b,\ bq/c \\ bq/a \end{matrix}\middle| q\,;\frac{cq}{abz}\right),
\end{aligned}
$$

- $(a; q)_\infty$ does Not converge for $|q| = 1$,

How to solve the scattering problem??

- starting from $\mathcal{H}_0$, construct general reflectionless potentials by Darboux transformations
- choose special integer wavenumbers to construct reflectionless analogue of difference $1/\cosh^2 x$ potential. Calculate the transmission amplitude $t(k)$.
- for the generic h, calculate the transmission and reflection amplitude based on conjectured connection formula of ${}_2\phi_1$
- check various properties: everything seems OK

Scattering amplitudes

- scattering wave function $n \longrightarrow h + ik$

$$\Psi_k(x) \stackrel{\text{def}}{=} e^{ikx} e^{2hx} \sqrt{1+e^{2x}} \left(\frac{\Phi_{\frac{\gamma}{2}}\left(2x + i\gamma(h+\frac{1}{2})\right)}{\Phi_{\frac{\gamma}{2}}\left(2x - i\gamma(h+\frac{1}{2})\right)} \right)^{\frac{1}{2}}$$

$$\times\, {}_2\phi_1\left(\begin{matrix} q^{-h-ik},\ q^{-h} \\ q^{1-ik} \end{matrix} \,\middle|\, q\,;\, q^{1+h} e^{-2x-i\pi} \right),$$

exclude q root of unity, assume $|q| = 1$ is approached from below $|q| \nearrow 1$

- $(a;q)_\infty$ replaced by quantum dilogarithm function

$$(e^z;q)_\infty \longrightarrow \text{const.}/\Phi^{(+)}_{\frac{\gamma}{2}}(z), \qquad \Phi^{(+)}_{\frac{\gamma}{2}}(z) \stackrel{\text{def}}{=} \Phi_{\frac{\gamma}{2}}\left(z + i\tfrac{\gamma}{2} + i\pi\right),$$

$$q = e^{-i\gamma}$$

conjectured connection formula for ${}_2\phi_1$

$$
{}_2\phi_1\left(\begin{matrix} e^{\lambda},\ e^{\mu} \\ e^{\nu} \end{matrix}\,\middle|\, q\,;e^{z}\right) =
$$

$$
\frac{\Phi^{(+)}_{\frac{\gamma}{2}}(\nu)\Phi^{(+)}_{\frac{\gamma}{2}}(\mu-\lambda)}{\Phi^{(+)}_{\frac{\gamma}{2}}(\mu)\Phi^{(+)}_{\frac{\gamma}{2}}(\nu-\lambda)}\ \frac{\Phi^{(+)}_{\frac{\gamma}{2}}(z)\Phi^{(+)}_{\frac{\gamma}{2}}(-i\gamma-z)}{\Phi^{(+)}_{\frac{\gamma}{2}}(\lambda+z)\Phi^{(+)}_{\frac{\gamma}{2}}(-i\gamma-\lambda-z)}
$$

$$
\times\ {}_2\phi_1\left(\begin{matrix} e^{\lambda},\ q\,e^{\lambda-\nu} \\ q\,e^{\lambda-\mu} \end{matrix}\,\middle|\, q\,;q\,e^{\nu-\lambda-\mu-z}\right)
$$

$$
+\ \frac{\Phi^{(+)}_{\frac{\gamma}{2}}(\nu)\Phi^{(+)}_{\frac{\gamma}{2}}(\lambda-\mu)}{\Phi^{(+)}_{\frac{\gamma}{2}}(\lambda)\Phi^{(+)}_{\frac{\gamma}{2}}(\nu-\mu)}\ \frac{\Phi^{(+)}_{\frac{\gamma}{2}}(z)\Phi^{(+)}_{\frac{\gamma}{2}}(-i\gamma-z)}{\Phi^{(+)}_{\frac{\gamma}{2}}(\mu+z)\Phi^{(+)}_{\frac{\gamma}{2}}(-i\gamma-\mu-z)}
$$

$$
\times\ {}_2\phi_1\left(\begin{matrix} e^{\mu},\ q\,e^{\mu-\nu} \\ q\,e^{\mu-\lambda} \end{matrix}\,\middle|\, q\,;q\,e^{\nu-\lambda-\mu-z}\right)
$$

Scattering amplitudes 2

$$\Psi_k(x) \to e^{ikx} e^{-i\frac{\gamma}{2}h(h+1)} \frac{\Phi^{(+)}_{\frac{\gamma}{2}}(-k\gamma - i\gamma)\Phi^{(+)}_{\frac{\gamma}{2}}(-k\gamma)}{\Phi^{(+)}_{\frac{\gamma}{2}}(-k\gamma + i\gamma h)\Phi^{(+)}_{\frac{\gamma}{2}}(-k\gamma - i\gamma(h+1))}$$

$$+ e^{-ikx} e^{-i\frac{\gamma}{2}h(h+1)+\frac{k\gamma}{2}(1-ik)} \frac{\Phi^{(+)}_{\frac{\gamma}{2}}(-k\gamma - i\gamma)\Phi^{(+)}_{\frac{\gamma}{2}}(k\gamma)}{\Phi^{(+)}_{\frac{\gamma}{2}}(i\gamma h)\Phi^{(+)}_{\frac{\gamma}{2}}(-i\gamma(h+1))}.$$

Scattering amplitudes

$$t(k) = e^{i\frac{\gamma}{2}h(h+1)} \times \frac{\Phi_{\frac{\gamma}{2}}(-k\gamma + i\gamma(h+\frac{1}{2}) + i\pi)\Phi_{\frac{\gamma}{2}}(-k\gamma - i\gamma(h+\frac{1}{2}) + i\pi)}{\Phi_{\frac{\gamma}{2}}(-k\gamma + i\frac{\gamma}{2} + i\pi)\Phi_{\frac{\gamma}{2}}(-k\gamma - i\frac{\gamma}{2} + i\pi)},$$

$$r(k) = e^{\frac{k\gamma}{2}(1-ik)} \times \frac{\Phi_{\frac{\gamma}{2}}(k\gamma + i\frac{\gamma}{2}+i\pi)\Phi_{\frac{\gamma}{2}}(-k\gamma + i\gamma(h+\frac{1}{2})+i\pi)\Phi_{\frac{\gamma}{2}}(-k\gamma - i\gamma(h+\frac{1}{2})+i\pi)}{\Phi_{\frac{\gamma}{2}}(-k\gamma + i\frac{\gamma}{2}+i\pi)\Phi_{\frac{\gamma}{2}}(i\gamma(h+\frac{1}{2}) + i\pi)\Phi_{\frac{\gamma}{2}}(-i\gamma(h+\frac{1}{2}) + i\pi)}$$

Scattering amplitude 4

- $t(k)$, $r(k)$: invariant under $h \leftrightarrow -(h+1)$
- unitarity satisfied $|t(k)|^2 + |r(k)|^2 = 1$
- h: integer: reproduces the reflectionless case

$$t(k) = \prod_{j=1}^{N} \frac{\sinh \frac{\gamma}{2}(k + i\,j)}{\sinh \frac{\gamma}{2}(k - i\,j)}, \quad r(k) = 0.$$

- some more supporting evidence for the conjectured connection formula
- **I do hope experts to provide an analytic proof of the connection formula.**

Further Challenge??

- There are other difference analogues of exactly solvable potentials, Morse potential, hyperbolic Pöschl-Teller potential, etc.
- Do they lead to new problems?? new challenges???

Thank you!!