Another Face of Solitons

Ryu SASAKI

Department of Physics, Shinshu University

21st Numadu Meeting,
Numadu National College of Technology
Numadu, 7 March 2014

Outline

Overview:

- KdV equation: shallow water wave in a narrow canal

$$\partial_t U - 6U\partial_x U + \partial_x^3 U = 0, \quad U \equiv U(x;t)$$

- Schrödinger equation with potential U for $\psi(x;t)$

$$\mathcal{H}_U \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + U(x;t), \quad \mathcal{H}_U\psi(x;t) = \mathcal{E}(t)\psi(x;t), \quad (S)$$

- time evolution of $\psi(x;t)$

$$A_U \stackrel{\text{def}}{=} -4\frac{d^3}{dx^3} + 3U\frac{d}{dx} + 3\frac{d}{dx}U, \quad \partial_t\psi = A_U\psi, \quad (t)$$

$$[A_u, \mathcal{H}_U] = 6U\partial_x U - \partial_x^3 U,$$

- Compatibility of (S) & (t): $\Rightarrow \partial_t\mathcal{E} = 0$
- Lax Representation KdV eq $\Leftrightarrow \partial_t\mathcal{H}_U = [A_u, \mathcal{H}_U]$ *Lax '68*

Overview 2: Soliton = Reflectionless Potential

- *N-Soliton Solution of KdV eq.*

  $\equiv$ *Reflectionless Potential with N discrete eigenlevels* only

$$\mathcal{E}_j = -k_j^2, \quad j = 1, 2, \ldots, N, \quad 0 < k_1 < \cdots < k_N$$

  $\equiv$ *Solution of Gelfand-Levitan eq.* with *degenerate Kernel*,

$$K(x,t) + F(x+t) + \int_0^\infty F(x+t+s)K(x,s)ds = 0,$$

$$F(x) \stackrel{\text{def}}{=} 2\sum_{j=1}^{N} c_j e^{-2k_j x}, \quad c_j > 0, \quad U(x) = -2\frac{d}{dx}K(x,0)$$

$$U_N(x) = -2\partial_x^2 \log u_N(x), \quad u_N(x) \stackrel{\text{def}}{=} \det_{1\le j,k\le N}\left(\delta_{j\,l} + \frac{c_j e^{-(k_j+k_l)x}}{k_j + k_l}\right)$$

depends on 2N *positive parameters* $\{k_j, c_j\}$ *Kay-Moses, '56*

$c_j \to c_j exp[-8k_j^3 t]$: KdV soliton, *Hirota, '71*

$$c_j \to c_j exp[-8\sum_{n=0}^{\infty} k_j^{2n+1} t_n]: \text{ KdV hierarchy soliton}$$

Overview 3: Reflectionless Potential is Solvable

- eigenfunctions of the Reflection Potential

$$\mathcal{H}_U \phi_{N,j}(x) = -k_j^2 \phi_{N,j}(x), \quad \phi_{N,j}(x) \sim \begin{cases} e^{-k_j x} & x \to +\infty \\ con.e^{+k_j x} & x \to -\infty \end{cases}$$

  depending on 2N parameters

- *Naive expectation*
  Can new types of solitons be generated through deformations of the N-soliton solution by its eigenfunctions in items of Darboux and other transformations??
- *New*, if the obtained N-M solitons depend on all the original 2N parameters.
- Harsh reality *No new type of solitons*

Overview 4: Various Identities

- *Uniqueness of reflectionless potential* *Kay-Moses, '56*
- *No new Solitons*
  $\Rightarrow$ Various identities satisfied by the eigenfunctions of the reflection potential
- Similar identities have been obtained for the classical orthogonal polynomials.
  They represent the solvability of the underlying potentials.
- Wronskian identities satisfied by Hermite, Laguerre and Jacobi polynomials, Odake-Sasaki 2013
- Casoratian identities satisfied by Wilson, Askey-Wilson and their reduced type polynomials, Odake-Sasaki 2014

Darboux Transformations 1

Starting point:

$$\mathcal{H} = -\frac{d^2}{dx^2} + U(x), \quad \mathcal{H}\psi(x) = \mathcal{E}\psi(x) \quad (\mathcal{E}, U(x) \in \mathbb{C}),$$

seed solutions $\mathcal{H}\varphi_j(x) = \tilde{\mathcal{E}}_j\varphi_j(x) \quad (\tilde{\mathcal{E}}_j \in \mathbb{C}\ ;\ j = 1, 2, \ldots, M),$

Darboux transformation in terms of φ_1:

$$\mathcal{H}^{(1)} \stackrel{\text{def}}{=} \mathcal{H} - 2\partial_x^2 \log\bigl|\varphi_1(x)\bigr|,$$

$$\psi^{(1)}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}[\varphi_1, \psi](x)}{\varphi_1(x)} = \frac{\varphi_1\partial_x\psi - \partial_x\varphi_1\psi}{\varphi_1(x)}, \quad \varphi_{1,k}^{(1)}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}[\varphi_1, \varphi_k](x)}{\varphi_1(x)},$$

$$\mathcal{H}^{(1)}\psi^{(1)}(x) = \mathcal{E}\psi^{[1]}(x), \quad \mathcal{H}^{(1)}\varphi_1^{-1}(x) = \tilde{\mathcal{E}}_1\varphi_1^{-1}(x),$$

$$\mathrm{W}[f_1, \cdots, f_n](x) \stackrel{\text{def}}{=} \det\bigl(\partial_x^{j-1} f_k(x)\bigr)_{1\le j,k\le n}$$

Darboux Transformations 2

Repeating M times:

$$\mathcal{H}^{(M)} \stackrel{\text{def}}{=} \mathcal{H} - 2\partial_x^2 \log\big|\mathrm{W}[\varphi_1, \varphi_2, \ldots, \varphi_M](x)\big|,$$

$$\psi^{(M)}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}[\varphi_1, \varphi_2, \ldots, \varphi_M, \psi](x)}{\mathrm{W}[\varphi_1, \varphi_2, \ldots, \varphi_M](x)},$$

$$\breve{\varphi}_j^{(M)}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}[\varphi_1, \varphi_2, \ldots, \breve{\varphi}_j, \ldots, \varphi_M](x)}{\mathrm{W}[\varphi_1, \varphi_2, \ldots, \varphi_M](x)} \quad (j = 1, 2, \ldots, M),$$

$$\mathcal{H}^{(M)}\psi^{(M)}(x) = \mathcal{E}\psi^{(M)}(x),\ \mathcal{H}^{(M)}\breve{\varphi}_j^{(M)}(x) = \tilde{\mathcal{E}}_j\breve{\varphi}_j^{(M)}(x) \quad (j = 1, 2, \ldots, M)$$

$\breve{\varphi}_j$ means that φ_j is removed from the Wronskian

Abraham-Moses Transformations '80

notation:

$$\langle f,g\rangle(x)\stackrel{\text{def}}{=}\int_{-\infty}^{x}dyf(y)g(y)=\langle g,f\rangle(x),\ \frac{d}{dx}\langle f,g\rangle(x)=f(x)g(x),$$

$$\langle f,g\rangle(-\infty)=0,\quad \langle f,g\rangle(+\infty)=(f,g)\stackrel{\text{def}}{=}\int_{-\infty}^{\infty}f(x)g(x)dx.$$

use φ_1 for adding/deleting one bound state

$$\mathcal{H}\to\mathcal{H}^{(1)}(x)\stackrel{\text{def}}{=}\mathcal{H}-2\partial_x^2\log\big(e_1\pm\langle\varphi_1,\varphi_1\rangle\big),\quad e_1>0,$$

$$\varphi_1\to\varphi_1^{(1)}\stackrel{\text{def}}{=}\frac{\varphi_1}{e_1\pm\langle\varphi_1,\varphi_1\rangle},\qquad \mathcal{H}^{(1)}\varphi_1^{(1)}=\tilde{\mathcal{E}}_1\varphi_1^{(1)},$$

$$\psi\to\psi^{(1)}\stackrel{\text{def}}{=}\psi\mp\varphi_1^{(1)}\langle\varphi_1,\psi\rangle,\quad \mathcal{H}^{(1)}\psi^{(1)}=\mathcal{E}\psi^{(1)}.$$

Abraham-Moses Transformations 2

Repeating M times:

$$\mathcal{H}^{(M)}(x) = \mathcal{H} - 2\partial_x^2 \log\det\big(\mathcal{F}_M\big),$$

$$\psi^{(M)} = \psi \mp \sum_{j,l=1}^{M} \varphi_j\big(\mathcal{F}_M^{-1}\big)_{j\,l}\langle\varphi_l, \psi\rangle, \quad \mathcal{H}^{(M)}\psi^{(M)} = \mathcal{E}\psi^{(M)},$$

$$\varphi_j^{(M)} = \sum_{l=1}^{M}\big(\mathcal{F}_M^{-1}\big)_{j\,l}\varphi_l, \quad \mathcal{H}^{(M)}\varphi_j^{(M)} = \tilde{\mathcal{E}}_j\varphi_j^{(M)}, \quad (j, l = 1, \ldots, M),$$

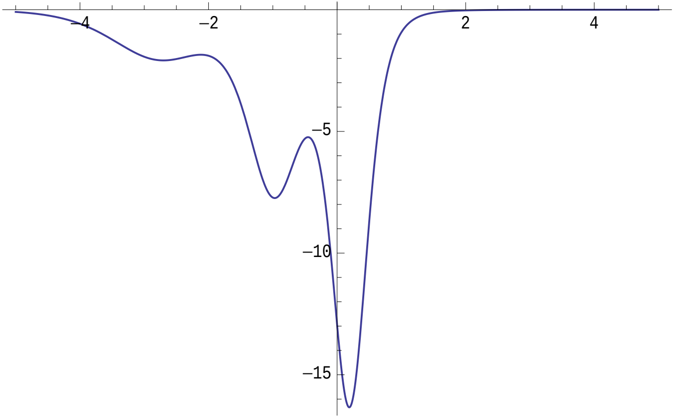
$\mathcal{F}_M$: $M \times M$ symmetric & positive definite matrix defined by:

$$(\mathcal{F}_M)_{j\,l} \stackrel{\text{def}}{=} e_j\delta_{j\,l} \pm \langle\varphi_j, \varphi_l\rangle, \; e_j \begin{cases} > 0 \text{ arbitrary} & \text{add} \\ \stackrel{\text{def}}{=} (\varphi_j, \varphi_j) & \text{delete} \end{cases}, \; (j, l = 1, \ldots, M)$$

Properties of the Reflectionless Potential

$$U_N(x) = -2\partial_x^2 \log u_N(x) < 0,$$ always negative

$$u_N(x) \stackrel{\text{def}}{=} \det_{1\le j,k\le N}\left(\delta_{j\,l} + \frac{c_j e^{-(k_j+k_l)x}}{k_j+k_l}\right),$$ *velocity* $\propto$ *depth* $\propto k_j^2$



Properties of the Reflectionless Potential 2

at a complex simple zero of $u_N(x)$, $u_N(x) = (x - x_0)r(x)$, $r(x_0) \neq 0$:

$$U(x) = \frac{2}{(x - x_0)^2} + \text{regular terms},$$

regular singularity with characteristic exponents 2,-1

$\Rightarrow$ monodromy free apparent singularities

generic solutions of Schrödinger equations with potential $U(x)$ are global solutions (everywhere monodromy free)

Scattering Problem

$U(x) \to 0$ at $x \to \pm\infty$,

$$\mathcal{H}\psi_k(x) = k^2\psi_k(x),\ k \in \mathbb{R}, \quad \psi_k(x) \to e^{\pm ikx},\ x \to \pm\infty,$$

normalise at $x \to +\infty$:

$$\psi_k(x) \to e^{ikx}, \quad x \to +\infty,$$

then

$$\psi_k(x) \to A(k)e^{ikx} + B(k)e^{-ikx}, \quad x \to -\infty,$$

$B(k)/A(k)$: Reflection amplitude, $1/A(k)$: Transmission amplitude
If $B(k)/A(k) \equiv 0$: Reflectionless

Another Derivation of Reflectionless Potential

- start from trivial potential $U \equiv 0$: $0 < k_1 < \cdots < k_N$

$$\mathcal{H}\psi_j(x) = -k_j^2\psi_j(x), \quad \psi_j(x) \stackrel{\text{def}}{=} e^{k_j x} + c_j' e^{-k_j x}, \quad (-1)^{j-1} c_j' > 0,$$

- multiple Darboux tr. on terms of $\{\psi_j(x)\}$

$$U_N(x) = -2\partial_x^2 \log \mathrm{W}[\psi_1, \ldots. \psi_N](x),$$

$$\mathrm{W}[e^{\alpha_1 x}, \ldots, e^{\alpha_N x}](x) = \prod_{j>l}^{N} (\alpha_j - \alpha_l) \cdot e^{\sum_{j=1}^{N} \alpha_j x}$$

$\mathrm{W}[\psi_1, \ldots. \psi_N](x) > 0$:

- all positive $k_j \Rightarrow \prod_{j>l}^{N} (k_j - k_l) > 0$,
- all but one k_j are positive, $c_j' \prod_{j>l} (-k_j - k_l) > 0$,
- all but two k_{j_1}, k_{j_2} are positive, ..., • ...

Another Derivation of Reflectionless Potential 2

- reflectionless

$$e^{ikx} \to \frac{\mathrm{W}[\psi_1, \cdots, \psi_N, e^{ikx}](x)}{\mathrm{W}[\psi_1, \cdots, \psi_N](x)} \sim \begin{cases} \prod_{j=1}^N (ik - k_j) \cdot e^{ikx} & x \to +\infty \\ \prod_{j=1}^N (ik + k_j) \cdot e^{ikx} & x \to -\infty \end{cases}$$

- N eigenfunctions

$$\phi_{N,j}(x) \propto \frac{\mathrm{W}[\psi_1, \cdots, \breve{\psi}_j, \cdots, \psi_N](x)}{\mathrm{W}[\psi_1, \cdots, \psi_N](x)}, \quad j = 1, \ldots, N,$$

- similar derivation by Matveev-Salle "Darboux transformations and solitons," without the eigenfunctions
  $\psi_1(x) = \cosh k_1(x - x_1)$, $\psi_2(x) = \sinh k_2(x - x_2)$,
  $\psi_3(x) = \cosh k_3(x - x_3)$, $\psi_4(x) = \sinh k_4(x - x_4)$...,

N-Eigenfunctions: Kay-Moses '56

$$\phi_{N,j}(x) \stackrel{\text{def}}{=} \frac{\tilde{u}_{N,j}(x)}{u_N(x)} e^{-k_j x}, \quad \tilde{u}_{N,j}(x) \stackrel{\text{def}}{=} \det \widetilde{A}_{N,j}(x), \quad j = 1, \ldots, N,$$

$$(\widetilde{A}_{N,j}(x))_{m\,n} \stackrel{\text{def}}{=} \delta_{m\,n} + \frac{k_j - k_m}{k_j + k_m} \frac{c_m e^{-(k_m+k_n)x}}{k_m + k_n}, \qquad m, n = 1, \ldots, N.$$

$\tilde{u}_{N,j}(x)$ is obtained from $u_N(x)$ by the replacement

$$\tilde{u}_{N,j}(x): \; c_m \to c_m \times (k_j - k_m)/(k_j + k_m), \quad m = 1, \ldots, N.$$

- asymptotic behaviours $x \to +\infty$ $u_N(x) \to 1$, $\tilde{u}_{N,j}(x) \to 1$
$x \to -\infty$ $u_N(x) \to\propto e^{-2\sum_{l=1}^{N} k_l x}$, $\tilde{u}_{N,j}(x) \to\propto e^{-2\sum_{l=1}^{N} k_l x + 2k_j x}$,
$x \to +\infty$ $\phi_{N,j}(x) \to e^{-k_j x}$, $\quad x \to -\infty$ $\phi_{N,j}(x) \to e^{+k_j x}$,
- $\phi_{N,j}(x)$ has $N - j$ zeros, $\phi_{N,N}$: ground state without zero

N-soliton to N-1 soliton

- use $\phi_{N,N}(x)$ in Darboux transformation

$$\begin{aligned} U_N^{(1)}(x) &= U_N(x) - 2\partial_x^2 \log \phi_{N,N}(x) \\ &= -2\partial_x^2 \log u_N(x) - 2\partial_x^2 \log(\tilde{u}_{N,N}(x) e^{-k_j x}/u_N(x)) \\ &= -2\partial_x^2 \log \tilde{u}_{N,N}(x) \end{aligned}$$

$$c_m \to c_m^{(1)} \stackrel{\text{def}}{=} c_m \times (k_N - k_m)/(k_N + k_m), \qquad m = 1, \ldots, N-1.$$

- c_N dependence wiped out
- use $\phi_{N,j}(x)$, $j < N$ in Darboux transformation

$$\begin{aligned} U_N^{(1)}(x) &= U_N(x) - 2\partial_x^2 \log \phi_{N,j}(x) \\ &= -2\partial_x^2 \log \tilde{u}_{N,j}(x), \quad \text{singular potential} \end{aligned}$$

$$c_m \to c_m^{(1)} \stackrel{\text{def}}{=} c_m \times (k_j - k_m)/(k_j + k_m), \qquad m = 1, \ldots, N$$

- c_j dependence wiped out

N-soliton to N-M soliton

- use $\{\phi_{N,d_1},\ldots,\phi_{N,d_M}\}$ in multiple Darboux transformations,

$$U_N^{(M)}(x) = -2\partial_x^2 \log \tilde{u}_{N,\mathcal{D}}(x), \quad \mathcal{D} \stackrel{\text{def}}{=} \{d_1,\ldots,d_M\}$$

$$\tilde{u}_{N,\mathcal{D}}(x): \qquad c_m^{(M)} \stackrel{\text{def}}{=} c_m \times \prod_{j=1}^{M} (k_{d_j} - k_m)/(k_{d_j} + k_m),$$

Wronskian identity:

$$\mathrm{W}[\phi_{N,d_1},\ldots,\phi_{N,d_M}](x) \propto \tilde{u}_{N,\mathcal{D}}(x) e^{-\sum_{j=1}^{M} k_{d_j} x} / u_N(x),$$

- rather non-trivial since

$\mathrm{W}[af_1,\ldots,af_M](x) = a^M \mathrm{W}[f_1,\ldots,f_M](x)$, meaning

$$\mathrm{W}[\tilde{u}_{N,d_1} e^{-k_{d_1}x},\ldots,\tilde{u}_{N,d_M} e^{-k_{d_M}x}](x) \propto u_N(x)^{M-1} \tilde{u}_{N,\mathcal{D}}(x) e^{-\sum_{j=1}^{M} k_{d_j} x}$$

- positivity of $\tilde{u}_{N,\mathcal{D}}(x)$ guaranteed if $\mathcal{D}$ satisfies

$$\prod_{j=1}^{M} (d_j - m) \geq 0, \quad m = 1,\ldots,N$$

Multiple Abraham-Moses Transformations

$$(\mathcal{F}_M)_{j\,l} = e_j\delta_{j\,l} - \langle\varphi_j,\varphi_l\rangle = \left(\int_{-\infty}^{\infty} - \int_{-\infty}^{x}\right)\varphi_j(y)\varphi_l(y)dy = \int_x^{\infty}\varphi_j(y)\varphi_l(y)dy$$

$$\phi_{N,j}^2(x) = -\partial_x\left(\frac{\tilde{w}_{N,j}(x)}{u_N(x)}\cdot\frac{e^{-2k_jx}}{2k_j}\right),$$

$$\phi_{N,j}(x)\phi_{N,l}(x) = -\partial_x\left(\frac{\tilde{v}_{N;j,l}(x)}{u_N(x)}\cdot\frac{e^{-(k_j+k_l)x}}{k_j+k_l}\right), \quad \tilde{v}_{N;j,j}(x) \equiv \tilde{w}_{N,j}(x),$$

$$\tilde{w}_{N,j}(x) : c_m \to c_m \times \frac{(k_j-k_m)^2}{(k_j+k_m)^2}, \;\; \tilde{v}_{N;j,l}(x) : c_m \to c_m \times \frac{k_j-k_m}{k_j+k_m}\cdot\frac{k_l-k_m}{k_l+k_m}$$

N→N-1 soliton by Abraham-Moses Transformations

$$U_N(x) \to U_N^{(1)}(x) = -2\partial_x^2 \log u_N(x) - 2\partial_x^2 \log \int_x^\infty \phi_{N,j}^2(y)dy$$
$$= -2\partial_x^2 \log \tilde{w}_{N,j}(x).$$

- N→N-M soliton by Multiple Abraham-Moses Transformations

$$U_N(x) \to U_N^{(M)}(x) = -2\partial_x^2 \log \tilde{w}_{N,\mathcal{D}}(x),$$
$$\tilde{w}_{N,\mathcal{D}}(x): \ c_m \to c_m \times \prod_{j=1}^{M} (k_{d_j} - k_m)^2/(k_{d_j} + k_m)^2.$$

Determinant identity:

$$\det\left(\int_x^\infty \phi_{N,d_j}(y)\phi_{N,d_l}(y)dy\right)_{1\le j,l\le M} \propto \frac{\tilde{w}_{N,\mathcal{D}}(x)}{u_N(x)} e^{-2\sum_{j=1}^{M} k_{d_j} x}$$

Main References

• present work based on
Ryu Sasaki, "Exactly solvable potentials with finitely many discrete eigenvalues of arbitrary choice," arXiv:1402.5474[math-ph]

• Wronskian Identities for the Hermite, Laguerre and Jacobi polynomials, Satoru Odake and Ryu Sasaki, "Krein-Adler transformations for shape-invariant potentials and pseudo virtual states," J. Phys. **A46** (2013) 245201 (24pp), arXiv:1212.6595[math-ph]

• Casoratian Identities for the Wilson, Askey-Wilson polynomials and their reduced form polynomials, "Casoratian Identities for the Wilson and Askey-Wilson Polynomials," to be published in J. Approximation Theory, Richard Askey 80-th birthday special issue, arXiv:1308.4240[math-ph]