Exactly Solvable Quantum Mechanics and Infinite Families of Multi-indexed Orthogonal Polynomials

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Outline

Discovery of ∞ Multi-Indexed Orthogonal Polynomials

- Infinitely many *orthogonal polynomials satisfying second order differential equations*, after Hermite, Laguerre and Jacobi polynomials (Odake-Sasaki, 2009-11)
- called Infinite Families of Multi-Indexed orthogonal polynomials $P_{\mathcal{D},n}(x)$, $\mathcal{D} = \{d_1, \ldots, d_M\}$, $d_j \in \mathbb{N}$:

$$\int P_{\mathcal{D},n}(x) P_{\mathcal{D},m}(x) \mathcal{W}_{\mathcal{D}}(x) dx = h_{\mathcal{D},n} \delta_{n\,m}$$

- Solutions of *exactly solvable Schrödinger eq.*
- degree $\ell + n$ polynomial in x, but forming a complete set,
- No three term recurrence relations
- *global solutions* of Fuchsian differential equations with $3 + \ell$ *regular singularities*.

∞-many Exceptional Orthogonal Polynomials

- for $\mathcal{D} = \{\ell\}$, $\ell \geq 1 \Rightarrow$ Exceptional orthogonal (Jacobi & Laguerre, 2009) polynomials $\ell = 1, 2, \ldots,$, $P_{\ell,n}(x)$, $n = 0, 1, 2, \ldots,$, (n counts nodes in (x_1, x_2)):
$$\int_{x_1}^{x_2} P_{\ell,n}(x) P_{\ell,m}(x) \mathcal{W}_\ell(x) dx = h_{\ell,n} \delta_{n\,m}$$
- generalisation: *Exceptional (X_ℓ) Wilson, Askey-Wilson, Meixner-Pollaczek, continuous Hahn, Racah, q-Racah, dual Hahn, dual q-Hahn, little q-Jacobi, Meixner polynomials* are also discovered as solutions of *exactly solvable difference Schrödinger eq.* (discrete quantum mechanics) (Odake-Sasaki, '09, '10, '11)

Fuchsian Differential Equations 1: overview

$$y'' + f(x)y' + g(x)y = 0, \quad f(x) = \frac{\alpha}{x - x_0} + \sum_{n=0}^{\infty} f_n (x - x_0)^n,$$

$$g(x) = \frac{\beta}{(x - x_0)^2} + \frac{\gamma}{x - x_0} + \sum_{n=0}^{\infty} g_n (x - x_0)^n$$

x_0 : regular singularity

- singular solutions $y_j = (x - x_0)^{\rho_j}(1 + \sum_{n=1}^{\infty} a_n (x - x_0)^n)$
- ρ_1, ρ_2: *characteristic exponents* $\rho(\rho - 1) + \alpha\rho + \beta = 0$
- regular singularities only $\Rightarrow$ Fuchsian equation
- local theory only

Fuchsian Differential Equations 2: examples

- 3 regular singularities at 0, 1, ∞: Gauss hypergeometric eq.

$$x(1-x)y'' + (\gamma - (\alpha+\beta+1)x)y' - \alpha\beta y = 0$$

- solutions around $x = 0$: $\rho_1 = 0$, $\rho_2 = 1-\gamma$

$$y_1 = {}_2F_1(\alpha, \beta; \gamma | x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{x^n}{n!},$$
$$y_2 = x^{1-\gamma}{}_2F_1(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma | x)$$

  hypergeometric function $\Rightarrow$ globally continued

- 4 regular singularities 0, 1, ∞, t: Heun equation
- more than 4 regular singularities: global solution virtually unknown

Fuchsian Differential Equations 3

- if $\rho_2 - \rho_1 = n \in \mathbb{N}$: possible log terms (Frobenius)
- if $\rho_2 - \rho_1 = n \in \mathbb{N}$ and no log terms $\Rightarrow$ apparent singularity
- apparent singularity of Schrödinger eq. (at $x = 0$)

$$\mathcal{H} = -\frac{d^2}{dx^2} + \frac{\alpha}{x^2} + \frac{\beta}{x} + \text{ regular terms}$$

$$\rho = \frac{1 \pm \sqrt{1+4\alpha}}{2}$$

α	$\rho_2 - \rho_1$	
0	1	regular
3/4	2	Painlevé case
2	3	Darboux trans.
15/4	4	??
⋮	⋮	

- if all extra singularities are apparent $\Rightarrow$ global solutions possible

History

- $\ell = 1$ exceptional Laguerre and Jacobi polynomials, introduced by Gómez-Ullate et al (July '08) within Sturm-Liouville theory, by avoiding Bochner's theorem
- quantum mechanical reformulation with shape-invariant potentials by Quesne (July '08)
- $\forall \ell \geq 1$ exceptional Laguerre and Jacobi polynomials discovered by Odake-Sasaki (June '09)
- new (type II) $\ell = 2$ exceptional Laguerre & Jacobi introduced by Quesne (June '09)
- $\forall \ell \geq 2$ type II exceptional Laguerre & Jacobi introduced by Odake-Sasaki (Nov. '09)
- Gómez-Ullate et al discussed type II $\mathcal{D} = \{m_1, m_2\}$ (March '11)

General Structure of Factorised Hamiltonians

Problem: Find the complete set of eigenvalues and eigenfunctions

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x), \quad \int \phi_n^2(x)dx < \infty, \quad n = 0, 1, 2, \ldots,$$

by adjusting the const. of $\mathcal{H} \Rightarrow \mathcal{E}_0 = 0$
$\Rightarrow$ Positive Semi-Definite Hamiltonian $\mathcal{H}$ (Hermitian Matrix)

$$0 = \mathcal{E}_0 < \mathcal{E}_1 < \mathcal{E}_2 < \cdots, \quad \Rightarrow \quad \mathcal{H} = \mathcal{A}^\dagger \mathcal{A}$$

$$\mathcal{A} = d/dx - dw(x)/dx, \quad \mathcal{A}^\dagger = -d/dx - dw(x)/dx,$$

$\phi_0(x)$: groundstate wavefunction, no node, square integrable
$\phi_0(x) = e^{w(x)}$, $w(x) \in \mathbb{R}$: prepotential $\mathcal{A}\phi_0(x) = 0$

$$\mathcal{H} = -d^2/dx^2 + V(x), \qquad V(x) = (dw(x)/dx)^2 + d^2w(x)/dx^2$$

Orthogonality of Eigenfunctions

$$\mathcal{H}\phi_n = \mathcal{A}^\dagger\mathcal{A}\phi_n = \mathcal{E}_n\phi_n$$

$$0 = \mathcal{E}_n(\phi_m, \phi_n) = (\phi_m, \mathcal{A}^\dagger\mathcal{A}\phi_n) = (\mathcal{A}\phi_m, \mathcal{A}\phi_n), \qquad m \neq n$$

$\{\mathcal{A}\phi_n\}$, $n = 1, 2, \ldots$: a new orthogonal system

$\phi_n^{[1]}(x) \stackrel{\text{def}}{=} \mathcal{A}\phi_n(x)$ is the eigenvector of the first associated Hamiltonian $\mathcal{H}^{[1]} \stackrel{\text{def}}{=} \mathcal{A}\mathcal{A}^\dagger$

Structure of Associated Hamiltonians

associated Hamiltonian $\mathcal{H}^{[1]} \stackrel{\text{def}}{=} \mathcal{A}\mathcal{A}^\dagger$ intertwining relation

$$\mathcal{A}\mathcal{A}^\dagger\mathcal{A} = \mathcal{A}\mathcal{H} = \mathcal{H}^{[1]}\mathcal{A}, \qquad \mathcal{A}^\dagger\mathcal{A}\mathcal{A}^\dagger = \mathcal{A}^\dagger\mathcal{H}^{[1]} = \mathcal{H}\mathcal{A}^\dagger$$

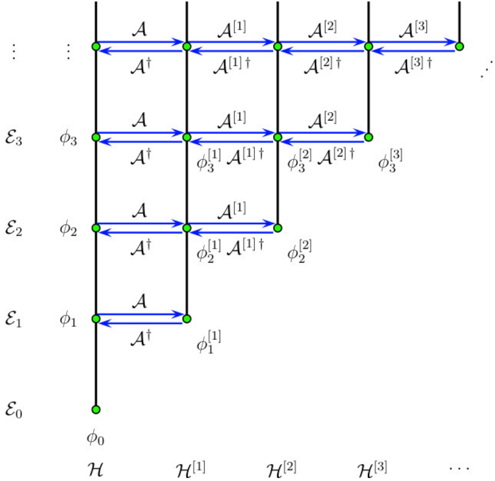
essentially iso-spectral

$$\phi_n^{[1]}(x) \stackrel{\text{def}}{=} \mathcal{A}\phi_n(x), \quad \phi_n(x) = \mathcal{A}^\dagger/\mathcal{E}_n\phi_n^{[1]}(x), \qquad n = 1, 2, \ldots,$$

$$\Rightarrow \mathcal{H}^{[1]}\phi_n^{[1]}(x) = \mathcal{E}_n\phi_n^{[1]}(x), \quad n = 1, 2, \ldots$$

removing the groundstate energy $\mathcal{E}_1$ from $\mathcal{H}^{[1]}$
$\Rightarrow$ positive semi-definite
$\Rightarrow$ factorisable $\mathcal{H}^{[1]} = \mathcal{A}^{[1]\dagger}\mathcal{A}^{[1]} + \mathcal{E}_1$ repeat!!

Schematic Picture of Crum's Theorem '55



deleting the eigenstates $\phi_0, \phi_1, \ldots, \phi_{\ldots}$ successively

Eigenfunctions etc for Crum's Theorem '55

in terms of Wronskian

$$\mathcal{H}^{[M]}\phi_n^{[M]}(x)=\mathcal{E}_n\phi_n^{[M]}(x)\quad(n=M,M+1,\ldots),$$

$$\phi_n^{[M]}(x)\stackrel{\text{def}}{=}\frac{\mathrm{W}[\phi_0,\phi_1,\ldots,\phi_{M-1},\phi_n](x)}{\mathrm{W}[\phi_0,\phi_1,\ldots,\phi_{M-1}](x)},$$

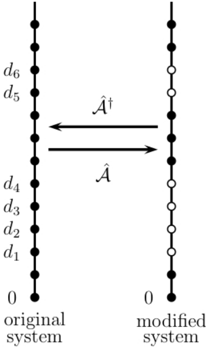
$$(\phi_m^{[M]},\phi_n^{[M]})=\prod_{j=0}^{M-1}(\mathcal{E}_n-\mathcal{E}_j)\cdot h_n\delta_{m\,n},$$

$$U^{[M]}(x)\stackrel{\text{def}}{=}U(x)-2\partial_x^2\log\bigl|(\mathrm{W}[\phi_0,\phi_1,\ldots,\phi_{M-1}](x)\bigr|,$$

also called M-step Darboux-Crum transformation

Schematic Picture of Modified Crum's Theorem

delete eigenstates $\mathcal{D} \stackrel{\text{def}}{=} \{d_1, d_2, \ldots, d_M\}$, $d_j \geq 0$, $\prod_{j=1}^{M}(m - d_j) \geq 0$, $\forall m \in \mathbb{Z}_{\geq 0}$



Eigenfunctions etc for Modified Crum's Theorem

by Krein-Adler ('57, '94)

$$\mathcal{H}^{[M]}\phi_n^{[M]}(x) = \mathcal{E}_n\phi_n^{[M]}(x) \quad (n \in \mathbb{Z}_{\geq 0}\backslash\mathcal{D}),$$

$$\phi_n^{[M]}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}, \phi_n](x)}{\mathrm{W}[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x)},$$

$$(\phi_m^{[M]}, \phi_n^{[M]}) = \prod_{j=1}^{M}(\mathcal{E}_n - \mathcal{E}_{d_j}) \cdot h_n\delta_{m\,n},$$

$$U^{[M]}(x) \stackrel{\text{def}}{=} U(x) - 2\partial_x^2 \log\left|\mathrm{W}[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x)\right|$$

exactly solvable $\Rightarrow$ exactly solvable

Darboux Transformation

underlying logic is the Darboux transformation

any Schrödinfer operator $\mathcal{H} = -\dfrac{d^2}{dx^2} + U(x)$, and any two solutions (not necessarily eigenstates) $\mathcal{H}\phi(x) = E\phi(x)$, $\mathcal{H}\tilde{\phi}(x) = \tilde{E}\tilde{\phi}(x)$,

$\Rightarrow \phi'(x) \stackrel{\text{def}}{=} \Big(d/dx - \partial_x\tilde{\phi}/\tilde{\phi}(x)\Big)\,\phi(x)$ is a solution of another

$\mathcal{H}' = -\dfrac{d^2}{dx^2} + U'(x)$, $U'(x) \stackrel{\text{def}}{=} U(x) - 2\partial_x^2 \log \tilde{\phi}(x)$, with the same energy $\mathcal{H}'\phi'(x) = E\phi'(x)$

In general $U'(x)$ has singularities at the zeros of $\tilde{\phi}(x)$:

$\tilde{\phi}(x) \sim \prod_j (x - x_j)$,

$$U'(x) = U(x) + \sum_j \frac{2}{(x - x_j)^2}, \qquad \text{regular singularity} \quad \rho_j = 2, -1$$

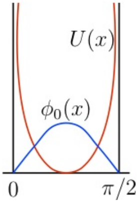
Crum, Krein-Adler give prescriptions of no-singularities in (x_1, x_2)

New Ingredient: Virtual State Solutions

- negative energy $\tilde{\mathcal{E}} < 0$: positivity of the norm guaranteed
- No Zeros in (x_1, x_2): $\Rightarrow$ No Singularity in the domain
- non-square-integrable due to the boundary condition at one boundary
- the above three properties of virtual state solutions are inherited at each step of Darboux transformation
- polynomial solutions $\Rightarrow$ Multi-indexed Orthogonal polynomials
- $\mathcal{D} = \{\ell\}$, $\ell \geq 1$: one-step Darboux transformation
  $\Rightarrow$ exceptional orthogonal polynomials

Example: Pöschl-Teller potential ⇒ Jacobi Polynomial

- $\mathcal{H} = -\frac{d^2}{dx^2} + U(x),\ U(x) = \frac{g(g-1)}{\sin^2 x} + \frac{h(h-1)}{\cos^2 x} - (g+h)^2,$ regular sing. $x = 0,\ g, 1-g,\ x = \pi/2,\ h, 1-h,\ \boldsymbol{\lambda} = \{g, h\},$
- groundstate wavefunct. $\phi_0(x) = (\sin x)^g (\cos x)^h, \quad g, h > 0,$ $w(x; \boldsymbol{\lambda}) = g \log \sin x + h \log \cos x, \quad 0 < x < \pi/2,$
- $\mathcal{E}_n(\boldsymbol{\lambda}) = 4n(n+g+h), \qquad \eta(x) \stackrel{\text{def}}{=} \cos 2x$
- $\phi_n(x; \boldsymbol{\lambda}) = \phi_0(x) P_n^{(g-1/2, h-1/2)}(\eta(x)), \quad P_n$: Jacobi polynomial



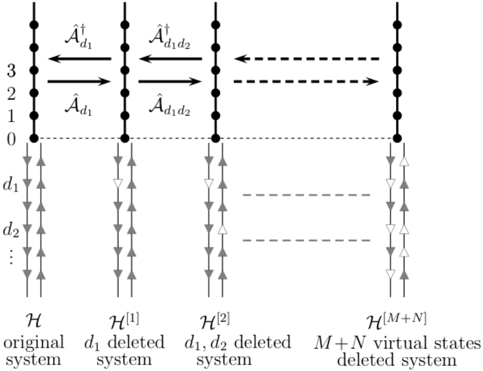
Multi-Indexed Orthogonal Polynomials 1

- Pöschl-Teller potential has virtual state solutions, type I and II, generated by the discrete symmetry of the potentail: $g \to 1-g$, and/or $h \to 1-h$
- negative energy and non-square integrable

$$\mathcal{H}\tilde{\phi}_{\rm v}(x) = \tilde{\mathcal{E}}_{\rm v}\tilde{\phi}_{\rm v}(x), \quad \tilde{\mathcal{E}}_{\rm v} < 0 \quad (\tilde{\phi}_{\rm v}, \tilde{\phi}_{\rm v}) = (1/\tilde{\phi}_{\rm v}, 1/\tilde{\phi}_{\rm v}) = \infty$$

- they have no zeros in $x \in (0, \pi/2)$
- delete virtual states à la Adler $\mathcal{D} \stackrel{\text{def}}{=} \{d_1^{\rm I}, \ldots, d_M^{\rm I}, d_1^{\rm II}, \ldots, d_N^{\rm II}\}$, $d_j^{\rm I,II} \geq 1$

Schematic Picture of Virtual States Deletion



Eigenfunctions etc after Virtual States Deletion

$$\mathcal{H}^{[M]}\phi_n^{[M]}(x)=\mathcal{E}_n\phi_n^{[M]}(x)\quad(n\in\mathbb{Z}_{\geq 0}),$$

$$\mathcal{H}^{[M]}\tilde{\phi}_{\mathrm{v}}^{[M]}(x)=\tilde{\mathcal{E}}_{\mathrm{v}}\tilde{\phi}_{\mathrm{v}}^{[M]}(x)\quad(\mathrm{v}\in\mathcal{V}\backslash\mathcal{D}),$$

$$\phi_n^{[M]}(x)\stackrel{\text{def}}{=}\frac{\mathrm{W}[\tilde{\phi}_{d_1},\tilde{\phi}_{d_2},\ldots,\tilde{\phi}_{d_M},\phi_n](x)}{\mathrm{W}[\tilde{\phi}_{d_1},\tilde{\phi}_{d_2},\ldots,\tilde{\phi}_{d_M}](x)},$$

$$(\phi_m^{[M]},\phi_n^{[M]})=\prod_{j=1}^{M}(\mathcal{E}_n-\tilde{\mathcal{E}}_{d_j})\cdot h_n\delta_{m\,n},$$

$$\tilde{\phi}_{\mathrm{v}}^{[M]}(x)\stackrel{\text{def}}{=}\frac{\mathrm{W}[\tilde{\phi}_{d_1},\tilde{\phi}_{d_2},\ldots,\tilde{\phi}_{d_M},\tilde{\phi}_{\mathrm{v}}](x)}{\mathrm{W}[\tilde{\phi}_{d_1},\tilde{\phi}_{d_2},\ldots,\tilde{\phi}_{d_M}](x)},$$

$$U^{[M]}(x)\stackrel{\text{def}}{=}U(x)-2\partial_x^2\log\left|\mathrm{W}[\tilde{\phi}_{d_1},\tilde{\phi}_{d_2},\ldots,\tilde{\phi}_{d_M}](x)\right|.$$

shape inv. exactly solvable $\Rightarrow$ shape inv. exactly solvable

Multi-Indexed Orthogonal Polynomials 2

- explicit forms of type I virtual states

$$\tilde{\phi}^{\mathrm{I}}_{\mathrm{v}}(x) \stackrel{\text{def}}{=} (\sin x)^{g}(\cos x)^{1-h}\xi^{\mathrm{I}}_{\mathrm{v}}(\eta(x);g,h),$$
$$\xi^{\mathrm{I}}_{\mathrm{v}}(\eta;g,h) \stackrel{\text{def}}{=} P_{\mathrm{v}}(\eta;g,1-h), \quad \mathrm{v}=0,1,\ldots,[h-\tfrac{1}{2}]',$$
$$\tilde{\mathcal{E}}^{\mathrm{I}}_{\mathrm{v}} \stackrel{\text{def}}{=} -4(g+\mathrm{v}+\tfrac{1}{2})(h-\mathrm{v}-\tfrac{1}{2}), \quad \tilde{\boldsymbol{\delta}}^{\mathrm{I}} \stackrel{\text{def}}{=} (-1,1)$$

- explicit forms of type II virtual states

$$\tilde{\phi}^{\mathrm{II}}_{\mathrm{v}}(x) \stackrel{\text{def}}{=} (\sin x)^{1-g}(\cos x)^{h}\xi^{\mathrm{II}}_{\mathrm{v}}(\eta(x);g,h),$$
$$\xi^{\mathrm{II}}_{\mathrm{v}}(\eta;g,h) \stackrel{\text{def}}{=} P_{\mathrm{v}}(\eta;1-g,h), \quad \mathrm{v}=0,1,\ldots,[g-\tfrac{1}{2}]',$$
$$\tilde{\mathcal{E}}^{\mathrm{II}}_{\mathrm{v}} \stackrel{\text{def}}{=} -4(g-\mathrm{v}-\tfrac{1}{2})(h+\mathrm{v}+\tfrac{1}{2}), \quad \tilde{\boldsymbol{\delta}}^{\mathrm{II}} \stackrel{\text{def}}{=} (1,-1)$$

Multi-Indexed Orthogonal Polynomials 3

- Multi-Indexed Orthogonal Polynomials $P_{\mathcal{D},n}(\eta)$:

$$\phi_n^{[M,N]}(x) \equiv \phi_{\mathcal{D},n}(x;\boldsymbol{\lambda}) = (-4)^{M+N}\psi_{\mathcal{D}}(x;\boldsymbol{\lambda})P_{\mathcal{D},n}(\eta(x);\boldsymbol{\lambda}),$$

$$\psi_{\mathcal{D}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\phi_0(x;\boldsymbol{\lambda}^{[M,N]})}{\Xi_{\mathcal{D}}(\eta(x);\boldsymbol{\lambda})},$$

- $\boldsymbol{\lambda}^{[M,N]} = (g+M-N, h-M+N)$,
  $\Xi_{\mathcal{D}}(\eta)$ has no node in $-1<\eta<1$;
- orthogonality

$$\int_{-1}^{1} d\eta\, \frac{W(\eta;\boldsymbol{\lambda}^{[M,N]})}{\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})^2} P_{\mathcal{D},m}(\eta;\boldsymbol{\lambda})P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda}) = h_{\mathcal{D},n}\delta_{nm}$$

- Pearson's equation satisfied

Multi-Indexed Orthogonal Polynomials 4

- Explicit Forms

$$P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathrm{W}[\mu_1,\ldots,\mu_M,\nu_1,\ldots,\nu_N,P_n](\eta) \times \Big(\frac{1-\eta}{2}\Big)^{(M+g+\frac{1}{2})N}\Big(\frac{1+\eta}{2}\Big)^{(N+h+\frac{1}{2})M}$$

$$\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathrm{W}[\mu_1,\ldots,\mu_M,\nu_1,\ldots,\nu_N](\eta) \times \Big(\frac{1-\eta}{2}\Big)^{(M+g-\frac{1}{2})N}\Big(\frac{1+\eta}{2}\Big)^{(N+h-\frac{1}{2})M}$$

$$\mu_j = \Big(\frac{1+\eta}{2}\Big)^{\frac{1}{2}-h}\xi^{\mathrm{I}}_{d^{\mathrm{I}}_j}(\eta;g,h), \quad \nu_j = \Big(\frac{1-\eta}{2}\Big)^{\frac{1}{2}-g}\xi^{\mathrm{II}}_{d^{\mathrm{II}}_j}(\eta;g,h)$$

- $P_{\mathcal{D},n}(\eta)$ degree $\ell+n$, $\Xi_{\mathcal{D}}(\eta)$ degree ℓ;

$$\ell = \sum_{j=1}^{M} d^{\mathrm{I}}_j + \sum_{j=1}^{N} d^{\mathrm{II}}_j - \frac{1}{2}M(M-1) - \frac{1}{2}N(N-1) + MN \geq 1$$

How it started: X_1 Jacobi polynomials

- X_1 Jacobi Hamiltonian (Gomez-Ullate et al, Quesne et al, '08)

$$\mathcal{H} = -\frac{d^2}{dx^2} + \frac{g(g+1)}{\sin^2 x} + \frac{h(h+1)}{\cos^2 x} - (2+g+h)^2 + \frac{8(g+h+1)}{1+g+h+(g-h)\cos 2x} - \frac{8(2g+1)(2h+1)}{(1+g+h+(g-h)\cos 2x)^2}$$

$$\phi_0(x) = (\sin x)^{g+1}(\cos x)^{h+1}\frac{3+g+h+(g-h)\cos 2x}{1+g+h+(g-h)\cos 2x}$$

- generalisation

$$\frac{P_1^{(g+2-3/2,-h-2-1/2)}(\cos 2x)}{P_1^{(g+1-3/2,-h-1-1/2)}(\cos 2x)}$$

$$w_\ell(x;\boldsymbol{\lambda}) = (g+\ell)\log\sin x + (h+\ell)\log\cos x + \log\frac{\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\xi_\ell(\eta;\boldsymbol{\lambda})}$$

$$\xi_\ell(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_\ell^{(g+\ell-3/2,-h-\ell-1/2)}(\eta), \quad \eta = \cos 2x$$

X_ℓ Jacobi Polynomials

- shape invariance can be verified directly

$$(\partial_x w_\ell(x;\boldsymbol{\lambda}))^2 - \partial_x^2 w_\ell(x;\boldsymbol{\lambda}) = (\partial_x w_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta}))^2 + \partial_x^2 w_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) + 4(g+h+2\ell+1)$$

- eigenvalues $\mathcal{E}_{\ell,n}(g,h) = \mathcal{E}_n(g+\ell, h+\ell) = 4n(n+g+h+2\ell)$
- eigenfunctions $\phi_{\ell,n}(x;\boldsymbol{\lambda}) = \psi_\ell(x;\boldsymbol{\lambda})P_{\ell,n}(\eta;\boldsymbol{\lambda})$

$$\psi_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{e^{w_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}}{\xi_\ell(\eta;\boldsymbol{\lambda})}, \quad c_n \stackrel{\text{def}}{=} n+h+1/2$$

$$P_{\ell,n}(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} c_n^{-1}\Big((h+\frac{1}{2})\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})P_n^{(g+\ell-3/2,h+\ell+1/2)}(\eta) + (1+\eta)\xi_\ell(\eta;\boldsymbol{\lambda})\partial_\eta P_n^{(g+\ell-3/2,h+\ell+1/2)}(\eta)\Big)$$

degree $\ell+n$ polynomial

X_ℓ Jacobi Polynomials 2: Fuchsian differential equation

- lowest degree $P_{\ell,0}(\eta;\boldsymbol{\lambda})=\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})$
- orthogonality

$$\int_0^{\pi/2}\psi_\ell(x;\boldsymbol{\lambda})^2P_{\ell,n}(\cos 2x;\boldsymbol{\lambda})P_{\ell,m}(\cos 2x;\boldsymbol{\lambda})dx=h_{\ell,n}(\boldsymbol{\lambda})\delta_{n\,m}$$

- Fuchsian differential eq.

$$\begin{aligned}&(1-\eta^2)\partial_\eta^2P_{\ell,n}(\eta;\boldsymbol{\lambda})\\&+\Bigl(h-g-(g+h+2\ell+1)\eta-2\frac{(1-\eta^2)\partial_\eta\xi_\ell(\eta;\boldsymbol{\lambda})}{\xi_\ell(\eta;\boldsymbol{\lambda})}\Bigr)\partial_\eta P_{\ell,n}(\eta;\boldsymbol{\lambda})\\&+\Bigl(-\frac{2(h+\frac12)(1-\eta)\partial_\eta\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\xi_\ell(\eta;\boldsymbol{\lambda})}+\ell(\ell+g-h-1)\\&\qquad\qquad+n(n+g+h+2\ell)\Bigr)P_{\ell,n}(\eta;\boldsymbol{\lambda})=0\end{aligned}$$

X_ℓ Jacobi Polynomials 3: Fuchsian differential equation 2

- regular singularities at ℓ roots of $\xi_\ell(\eta;\boldsymbol{\lambda})$, $\eta=\eta_j$:

$$\xi_\ell(\eta_j;\boldsymbol{\lambda})=0,\qquad j=1,2,\ldots,\ell$$

- in the neighbourhood of $\eta=\eta_j$:

$$\begin{aligned}&(1-\eta_j^2)y''-2\frac{(1-\eta_j^2)}{\eta-\eta_j}y'\\&-2(h+1/2)(1-\eta_j)\frac{\beta}{\eta-\eta_j}y+\text{regular terms}=0\end{aligned}$$

- characteristic eq.: same exponents everywhere

$$\rho(\rho-1)-2\rho=0\quad\Rightarrow\rho=0,3$$

$\rho=0$ corresponds to the polynomial solution

Other New Orthogonal Polynomial Solutions?

- Factorised Hamiltonian
- Crum-Krein-Adler Prescriptions, or Multi-step Darboux Transformations
- Virtual State Solutions

  known for most of the hypergeometric orthogonal polynomials of Askey-scheme, (q)-Racah, Wilson, Askey-Wilson polynomials, etc
- Discrete Quantum Mechanics with second order difference Schrödinger equation instead of differential eq.
- virtual state solutions 'known' since Exceptional (q)-Racah Wilson, Askey-Wilson, continuous Hahn, polynomials etc have been constructed (odake-Sasaki, '09 -'11)

General Setting of Discrete Quantum Mechanics

- Hamiltonian contains the exponentiated momentum operator differential equation $\Longrightarrow$ difference equation

$$p = -i\hbar\partial_x \qquad e^{\pm\beta p} = e^{\mp i\hbar\partial_x}$$
$$\hbar = 1 \qquad e^{\pm\beta p}\psi(x) = \psi(x \mp i\beta)$$

- β: real $\Longrightarrow$ pure imaginary shifts
  x: continuous $x \in (a, b)$; $(0, \pi)$, $(0, \infty)$, $(-\infty, +\infty)$
- β: pure imaginary $\Longrightarrow$ real shifts
  $\beta = \sqrt{-1}\gamma$ x: lattice with integral spacing $|\gamma| = 1$, finite $(0, 1, \ldots, N)$ or infinite $(0, 1, 2, \ldots, \ldots)$
- provides quantum mechanical treatments of various orthogonal polynomials
- developed mostly with Satoru Odake in about 30 papers

Real Shifts Discrete Quantum Mechanics

Eigenvalue Problem of
Finite or Infinite Tri-Diagonal Symmetric (Jacobi) Matrix

$$\mathcal{H} = \begin{pmatrix} B(0) & -\sqrt{B(0)D(1)} & 0 & \cdots & \cdots & 0 \\ -\sqrt{B(0)D(1)} & B(1)+D(1) & -\sqrt{B(1)D(2)} & 0 & \cdots & \vdots \\ 0 & -\sqrt{B(1)D(2)} & B(2)+D(2) & -\sqrt{B(2)D(3)} & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & -\sqrt{B(8)D(9)} & B(9)+D(9) & -\sqrt{B(9)D(10)} \\ 0 & \cdots & \cdots & 0 & -\sqrt{B(9)D(10)} & D(10) \end{pmatrix}$$

$N = 10$ Example

$\mathcal{H} = \mathcal{A}^\dagger\mathcal{A}$: $\mathcal{A}$: diagonal and super-diagonal
$\mathcal{A}^\dagger$: diagonal and sub-diagonal

- tri-diagonal $\Rightarrow$ Three term recurrence relation for the dual orthogonal polynimials

Explicit Form of $\mathcal{A}^\dagger$ and $\mathcal{A}$

$$\mathcal{A}^\dagger = \begin{pmatrix} \sqrt{B(0)} & 0 & 0 & \cdots & \cdots & 0 \\ -\sqrt{D(1)} & \sqrt{B(1)} & 0 & 0 & \cdots & \vdots \\ 0 & -\sqrt{D(2)} & \sqrt{B(2)} & 0 & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & -\sqrt{D(9)} & \sqrt{B(9)} & 0 \\ 0 & \cdots & \cdots & 0 & -\sqrt{D(10)} & 0 \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} \sqrt{B(0)} & -\sqrt{D(1)} & 0 & \cdots & \cdots & 0 \\ 0 & \sqrt{B(1)} & -\sqrt{D(2)} & 0 & \cdots & \vdots \\ 0 & 0 & \sqrt{B(2)} & -\sqrt{D(3)} & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & \sqrt{B(9)} & -\sqrt{D(10)} \\ 0 & \cdots & \cdots & 0 & 0 & 0 \end{pmatrix},$$

$N = 10$ Example

Factorised Hamiltonians 1: obviously self-adjoint

real shifts: discrete x

$$\mathcal{H} \stackrel{\text{def}}{=} -\sqrt{B(x)}\,e^{\partial}\sqrt{D(x)} - \sqrt{D(x)}\,e^{-\partial}\sqrt{B(x)} + B(x) + D(x),$$

$$= \mathcal{A}^{\dagger}\mathcal{A}, \quad \mathcal{A} = \sqrt{B(x)} - e^{\partial}\sqrt{D(x)}, \quad \mathcal{A}^{\dagger} = \sqrt{B(x)} - \sqrt{D(x)}\,e^{-\partial},$$

$$B(x) > 0, \quad D(x) > 0, \quad D(0) = 0\ ; \quad B(N) = 0 \ \text{ finite case},$$

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x), \qquad n = 0, 1, 2, \ldots, .$$

groundstate eigenfunction ϕ_0 is annihilated by $\mathcal{A}$:

$$\mathcal{A}\phi_0(x) = 0 \ \Rightarrow \mathcal{H}\phi_0(x) = 0, \quad \phi_0^2(x) = \prod_{y=0}^{x-1} B(y)/D(y+1), \ \ x = 1, 2, \ldots$$

excitedstate eigenfunction $\phi_n(x) = \phi_0(x)P_n(\eta(x))$,
($\eta(x)$:sinusoidal coordinate) satisfy simple difference equation

$$\widetilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0(x)^{-1} \circ \mathcal{H} \circ \phi_0(x) = \Big(B(x)(1 - e^{\partial}) + D(x)(1 - e^{-\partial})\Big),$$

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Factorised Hamiltonians 2: General Structure

real shifts: discrete x

- difference equation for the polynomials:

$$B(x)\left(P_n(\eta(x)) - P_n(\eta(x+1))\right) + D(x)\left(P_n(\eta(x)) - P_n(\eta(x-1))\right) = \mathcal{E}_n P_n(\eta(x))$$

- orthogonality measures are trivially given by the groundstate vector $\phi_0(x)^2$

$$(\phi_n, \phi_m) = \sum_{x=0}^{N} \phi_0^2(x) P_n(\eta(x)) P_m(\eta(x)) = \frac{1}{d_n^2}\delta_{n,m}$$

- duality $P_n(\eta(x)) = Q_x(\mathcal{E}(n))$, three term recurrence of $Q_x(\mathcal{E})$:

$$B(x)\left(Q_x(\mathcal{E}) - Q_{x+1}(\mathcal{E})\right) + D(x)\left(Q_x(\mathcal{E}) - Q_{x-1}(\mathcal{E})\right) = \mathcal{E} Q_x(\mathcal{E})$$

Explicit Examples, real shifts

real shifts: discrete x

- Hahn: $(0, 1, \ldots, N)$, $\eta(x) = x$,

$$B(x) = (x + a)(N - x), \quad D(x) = x(b + N - x), \quad a > 0, b > 0,$$

$$\mathcal{E}_n = n(n + a + b - 1), \quad \phi_0(x)^2 = \frac{N!}{x!\,(N - x)!}\,\frac{(a)_x\,(b)_{N-x}}{(b)_N},$$

$$P_n(\eta(x)) = {}_3F_2\Bigl(\begin{matrix}-n,\ n + a + b - 1,\ -x\\ a,\ -N\end{matrix}\Bigm| 1\Bigr).$$

Explicit Examples, real shifts cont'd

- Racah: $(0,1,\ldots,N)$, $\eta(x)=x(x+d)$, $\tilde{d}\stackrel{\text{def}}{=}a+b+c-d-1$,

$$B(x)=-\frac{(x+a)(x+b)(x+c)(x+d)}{(2x+d)(2x+1+d)},\quad \mathcal{E}_n=n(n+\tilde{d}),$$

$$D(x)=-\frac{(x+d-a)(x+d-b)(x+d-c)x}{(2x-1+d)(2x+d)},$$

$$a=-N,\quad 0<d<a+b,\quad 0<c<1+d,$$

$$\phi_0(x)^2=\frac{(a,b,c,d)_x}{(1+d-a,1+d-b,1+d-c,1)_x}\,\frac{2x+d}{d},$$

$$P_n(\eta(x))={}_4F_3\Big(\genfrac{}{}{0pt}{}{-n,\ n+\tilde{d},\ -x,\ x+d}{a,\ b,\ c}\Big|\,1\Big)$$

Explicit Examples, real shifts cont'd 2

- q-Racah: $(0,1,\ldots,N)$, $\eta(x)=(q^{-x}-1)(1-dq^{x})$

$$B(x)=-\frac{(1-aq^{x})(1-bq^{x})(1-cq^{x})(1-dq^{x})}{(1-dq^{2x})(1-dq^{2x+1})},$$

$$D(x)=-\tilde{d}\,\frac{(1-a^{-1}dq^{x})(1-b^{-1}dq^{x})(1-c^{-1}dq^{x})(1-q^{x})}{(1-dq^{2x-1})(1-dq^{2x})},$$

$$a=q^{-N},\quad a=q^{-N},\quad 0<ab<d<1,\quad qd<c<1,$$

$$\tilde{d}<q^{-1},\quad \tilde{d}\stackrel{\text{def}}{=}abcd^{-1}q^{-1},\quad \mathcal{E}_n=(q^{-n}-1)(1-\tilde{d}q^{n}),$$

$$\phi_0(x)^2=\frac{(a,b,c,d\,;q)_x}{(a^{-1}dq,b^{-1}dq,c^{-1}dq,q\,;q)_x\,\tilde{d}^{x}}\,\frac{1-dq^{2x}}{1-d},$$

$$P_n(\eta(x))={}_4\phi_3\Bigl(\begin{matrix}q^{-n},\ \tilde{d}q^{n},\ q^{-x},\ dq^{x}\\ a,\ b,\ c\end{matrix}\Bigm|q\,;q\Bigr).$$

Virtual state solutions: Type I

Racah case, $\mathfrak{t}$: twist operator, $\boldsymbol{\lambda} = (a, b, c, d)$, $\quad a = -N$

$$\mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (d - a + 1, d - b + 1, c, d), \quad \mathfrak{t}^2 = \text{id},$$

$$B'(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} B\big(x;\mathfrak{t}(\boldsymbol{\lambda})\big), \quad D'(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} D\big(x;\mathfrak{t}(\boldsymbol{\lambda})\big),$$

$$B'(x;\boldsymbol{\lambda}) = -\frac{(x+d-a+1)(x+d-b+1)(x+c)(x+d)}{(2x+d)(2x+1+d)},$$

$$D'(x;\boldsymbol{\lambda}) = -\frac{(x+a-1)(x+b-1)(x+d-c)x}{(2x-1+d)(2x+d)},$$

$$\Rightarrow B(x)D(x+1) = B'(x)D'(x+1), \quad \alpha' \stackrel{\text{def}}{=} -c(a+b+c-d-1) < 0,$$

$$B(x) + D(x) = B'(x) + D'(x) + \alpha',$$

$$B'(x) > 0 \ \ (x = 0, 1, \ldots, N+L-1), \quad D'(x) > 0 \ \ (x = 1, 2, \ldots, N),$$

$$D'(0) = D'(N+1) = 0,$$

Virtual state solutions: Type I, continued

$$\mathcal{H} = \mathcal{H}' + \alpha',$$

$$\mathcal{H}' \stackrel{\text{def}}{=} -\sqrt{B'(x)}\, e^{\partial} \sqrt{D'(x)} - \sqrt{D'(x)}\, e^{-\partial} \sqrt{B'(x)} + B'(x) + D'(x)$$

$$= \mathcal{A}'^{\dagger} \mathcal{A}', \qquad \text{positive definite} \Rightarrow \text{no zero-mode}$$

virtual solutions

$$\mathcal{H}'\tilde{\phi}_{\rm v}(x) = \mathcal{E}'_{\rm v}\tilde{\phi}_{\rm v}(x) \ \ (x = 0, 1, \ldots, N-1), \quad \mathcal{H}'\tilde{\phi}_{\rm v}(N) \neq \mathcal{E}'_{\rm v}\tilde{\phi}_{\rm v}(N),$$

$$\tilde{\phi}_{\rm v}(x) = \tilde{\phi}_0(x)\check{\xi}_{\rm v}(x), \quad \check{\xi}_{\rm v}(x) \stackrel{\text{def}}{=} \xi_{\rm v}(\eta(x)) \ \ (x = 0, 1, \ldots, N; {\rm v} \in \mathcal{V}),$$

$$\mathcal{A}'\tilde{\phi}_0(x) = 0 \quad (x = 0, 1, \ldots, N-1), \quad \mathcal{A}'\tilde{\phi}_0(N) \neq 0$$

$$\Rightarrow \tilde{\phi}_0(x) \stackrel{\text{def}}{=} \sqrt{\prod_{y=0}^{x-1} \frac{B'(y)}{D'(y+1)}} \ \ (x = 0, 1, \ldots, N),$$

$$B'(x)\big(\check{\xi}_{\rm v}(x) - \check{\xi}_{\rm v}(x+1)\big) + D'(x)\big(\check{\xi}_{\rm v}(x) - \check{\xi}_{\rm v}(x-1)\big) = \mathcal{E}'_{\rm v}\check{\xi}_{\rm v}(x).$$

Construction of Multi-Indexed (q)-Racah Polynomials

- discrete symmetries of (q)-Racah $a \leftrightarrow b \leftrightarrow c$
- virtual state solutions fail to satisfy the equation at one boundary

$$\begin{aligned}\text{type I} :& \mathcal{H}\tilde{\phi}_{\text{v}}(x) = \widetilde{E}_{\text{v}}\tilde{\phi}_{\text{v}}(x), \quad x = 0, 1, \ldots, N-1\\ & \mathcal{H}\tilde{\phi}_{\text{v}}(N) \neq \widetilde{E}_{\text{v}}\tilde{\phi}_{\text{v}}(N),\\ \text{type II} :& \mathcal{H}\tilde{\phi}_{\text{v}}(x) = \widetilde{E}_{\text{v}}\tilde{\phi}_{\text{v}}(x), \quad x = 1, \ldots, N\\ & \mathcal{H}\tilde{\phi}_{\text{v}}(0) \neq \widetilde{E}_{\text{v}}\tilde{\phi}_{\text{v}}(0),\end{aligned}$$

- M-step Darboux tr. with virtual solns. $\Rightarrow$ Multi-Indexed (q)-Racah will be published soon
- type I and II virtual solns cannot coexist
- $\mathcal{D} = \{\ell\} \Rightarrow$ exceptional (q)-Racah polynomials, Prog. Theor. Phys. **125** (2011) 851-870. `arXiv:1102.0812[math-ph]`.

Curious Identities of Laguerre polynomials

- definition $h_n(x;\alpha) \stackrel{\text{def}}{=} L_n^{(\alpha+n)}(-x)$
- trivial expansion: $h_n(x;\alpha+1) = \sum_{j=0}^{n} C_j h_{n-j}(x,\alpha)$
- C_j: independent of n and α
- C_j: all positive integer
- $C_j = \{1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16\,796, 58\,786, 208\,012, \ldots,\}$
- C_j: Catalan Numbers $C_j = \binom{2j}{j}/(j+1)$
- verified by Mathematica to $n = 50$, 19 seconds, no proof yet
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Summary and Outlook

- Infinitely many new orthogonal polynomials satisfying second order differential or difference equations are discovered recently.
  Hopefully they will find many interesting applications.
  At least, they give infinitely many examples of exactly solvable Birth and Death Processes.
- Various concepts and methods of QM have much wider currency and utility in the theory of ordinary differential and difference equations than is usually regarded.
- Various properties of the Askey-scheme of hypergeometric orthogonal polynomials can be understood in a unified fashion, both of a continuous and a discrete variable.
- Multi-variable Multi-Indexed Orthogonal polynomials are the next challenge

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