コンパクト等質空間上の不変なアインシュタイン計量について

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outline

On invariant Einstein metrics of compact homogeneous spaces

based on joint works with
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- Short history on invariant Einstein metrics on Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-k)$
- main results
- Ricci tensor of a compact homogeneous space
- The Stiefel manifolds $V_{2p}\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-2p)$

(M, g): Riemannian manifold

- (M, g) is called Einstein if the Ricci tensor $\mathrm{r}(g)$ of the metirc g satisfies $\mathrm{r}(g) = cg$ for some constant c.

  We consider G-invariant Einstein metrics on a homogeneous space G/H.

- General Problem: Find G-invariant Einstein metrics on a homogeneous space G/H and classify them if it is not unique.

- Einstein homogeneous spaces can be divided into three cases depending on Einstein constant c.
  Here we consider the case $c > 0$.

Examples of the case $c > 0$
(G/H is compact and $\pi_1(G/H)$ is finite).

- Sphere $(S^n = SO(n+1)/SO(n), g_0)$,
  Complex Projective space $(\mathbb{C}P^n = SU(n+1)/(S(U(1) \times U(n)))$,
  Symmetric spaces of compact type,
  isotropy irreducible spaces (in these cases G-invariant Einstein metrics is unique)
- Generalized flag manifolds (Kähler C-spaces) (if we fix a complex structure, it admits a unique Kähler-Einstein metric, but complex structure may not be unique)

- A short history on homogeneous Einstein metrics on Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$:

  In 1963, S. Kobayashi proved
  existence of a homogeneous Einstein metric on the unit tangent bundle $T_1S^n = \mathrm{SO}(n)/\mathrm{SO}(n-2) = V_2\mathbb{R}^n$ (as S^1-bundle over a Kähler manifold $\mathrm{SO}(n)/(\mathrm{SO}(n-2)\times\mathrm{SO}(2))$).

  In 1970, A. Sagle proved that for $k \geq 3$
  the Stiefel manifold $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$ admits a homogeneous Einstein metric.

  In 1973, G. Jensen proved that
  the Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$ for $k \geq 3$ admits at least two homogeneous Einstein metric.

Introduction

- $V_2\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-2)$
  By a result of A. Back and W.Y. Hsiang (1987) and M. Kerr (1998), for $n \geq 5$ it is known that $\mathrm{SO}(n)/\,\mathrm{SO}(n-2)$ admits exactly one homogeneous Einstein metric.
  (the diagonal metrics are only homogeneous Einstein metrics)

- $V_2\mathbb{R}^4 = \mathrm{SO}(4)/\,\mathrm{SO}(2)$
  By a result of D. V. Alekseevsky, I. Dotti and C. Ferraris(1996), $\mathrm{SO}(4)/\,\mathrm{SO}(2)$ admits exactly two invariant Einstein metrics (one is diagonal metric and the other is non-diagonal metric).
  Jensen's metric is a diagonal metric. Note that $\mathrm{SO}(4)/\,\mathrm{SO}(2)$ is diffeomorphic to $S^3 \times S^2$. The non-diagonal Einstein metric comes from the product metric on $S^3 \times S^2$.

- For the tangent space $\mathfrak{p}$ of Stiefel manifold $SO(n)/SO(n-4)$

$$\mathfrak{p} = \left(\begin{array}{cccc|c} 0 & a_{12} & a_{13} & a_{14} & A_{15} \\ -a_{12} & 0 & a_{23} & a_{24} & A_{25} \\ -a_{13} & -a_{23} & 0 & a_{34} & A_{35} \\ -a_{14} & -a_{24} & a_{34} & 0 & A_{45} \\ \hline -{}^tA_{15} & -{}^tA_{25} & -{}^tA_{35} & -{}^tA_{45} & * \end{array}\right),$$

  we have a decomposition $\mathfrak{p} = \sum_{i<j} \mathfrak{p}_{ij} \oplus \sum_{k<5} \mathfrak{p}_{k5}$ as $\mathrm{Ad}(SO(n-4))$-submodules, where $\dim \mathfrak{p}_{ij} = 1$ and $\dim \mathfrak{p}_{k5} = n-4$. Note that submodules $\mathfrak{p}_{ij}$ are equivalent each other and submodules $\mathfrak{p}_{k5}$ are equivalent each other.
- For general invariant metrics on the Stiefel manifolds $V_k\mathbb{R}^n = SO(n)/SO(n-k)$, it would be difficult to find all homogeneous Einstein metrics.

The Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$

- Open problem : How many homogeneous Einstein metrics are there on Stiefel manifold $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$?

  Are there homogeneous Einstein metrics on Stiefel manifold $\mathrm{SO}(n)/\mathrm{SO}(n-k)$ other than Jensen's Einstein metrics ?

Related known results

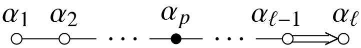
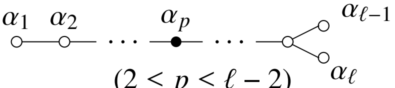
- A. Arvanitoyeorgos, V.V. Dzhepko and Yu. G. Nikonorov (2009) proved that for $s > 1$ and $\ell > k \geq 3$, the Stiefel manifold $\mathrm{SO}(sk + \ell)/\mathrm{SO}(\ell)$ admits at least four $\mathrm{SO}(sk + \ell)$-invariant Einstein metrics that are also $\mathrm{Ad}(\mathrm{SO}(k)^s \times \mathrm{SO}(\ell))$-invariant and two of which are Jensen's metrics.

  In 2007, they have treated $\mathrm{SO}(k + k + \ell)/\mathrm{SO}(\ell)$ ($\ell > k \geq 3$) as the special case of above and proved that the Stiefel manifold $\mathrm{SO}(k + k + \ell)/\mathrm{SO}(\ell)$ admits at least four $\mathrm{SO}(k + k + \ell)$-invariant Einstein metrics that are also $\mathrm{Ad}(\mathrm{SO}(k) \times \mathrm{SO}(k) \times \mathrm{SO}(\ell))$-invariant.

Related known results

- A. Arvanitoyeorgos, Y. S. and M. Statha (2014) proved that the Stiefel manifold $\mathrm{SO}(n)/\mathrm{SO}(n-4)$ $(n \geq 6)$ admits two new Einstein metrics, except Jensen's metrics, that are $\mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(n-4))$-invariant (solutions of polynomial of degree 10),
  the Stiefel manifold $\mathrm{SO}(7)/\mathrm{SO}(2)$ admits two new Einstein metrics, except Jensen's metrics, that are $\mathrm{Ad}(\mathrm{SO}(2) \times \mathrm{SO}(3) \times \mathrm{SO}(2))$-invariant (solutions of polynomial of degree 22)
  and
  the Stiefel manifold $\mathrm{SO}(7)/\mathrm{SO}(2)$ admits two new Einstein metrics, except Jensen's metrics, that are $\mathrm{Ad}(\mathrm{SO}(4) \times \mathrm{SO}(2))$-invariant (solutions of polynomial of degree 10).

We consider the generalized flag manifolds with two isotropy summands $B(\ell, p) = \mathrm{SO}(2\ell+1)/\,\mathrm{U}(p) \times \mathrm{SO}(2\ell+1-2p)$ and $D(\ell, p) = \mathrm{SO}(2\ell)/\,\mathrm{U}(p) \times \mathrm{SO}(2\ell-2p)$. From these spaces we can study the Stiefel manifolds $V_{2p}\mathbb{R}^{2\ell+1}$ and $V_{2p}\mathbb{R}^{2\ell}$.

G	(Π, Π_K)	K
B_ℓ	 $(2 \leq p \leq \ell - 1)$	$\mathrm{U}(p) \times \mathrm{SO}(2(\ell-p)+1)$
D_ℓ	 $(2 \leq p \leq \ell - 2)$	$\mathrm{U}(p) \times \mathrm{SO}(2(\ell-p))$

The tangent space of generalized flag manifold can be decomposed into $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ as irreducible Ad(K)-modules.

- Now we state our main results:

Theorem

For $2 \leq p \leq \frac{2}{5}n - 1$ the Stiefel manifold $V_{2p}\mathbb{R}^n$ admits at least four invariant Einstein. Two of the metrics are Jensen's metrics and the other two are different from Jensen's metrics.

In fact, we consider $\mathrm{Ad}(\mathrm{U}(p) \times \mathrm{SO}(n-2p))$-invariant metrics on Stiefel manifolds $V_{2p}\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-2p)$.

Ricci tensor of a compact homogeneous space G/H

- Let G be a compact semi-simple Lie group and H a connected closed subgroup of G.

  We assume that there exists a connected closed subgroup K of G such that H is a normal subgroup of K and $K = HL$ for a closed subgroup L of K.

  Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to B $(= -$ Killing form of $\mathfrak{g}$). Then we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $[\,\mathfrak{h}, \mathfrak{m}\,] \subset \mathfrak{m}$.

- We assume that $\mathfrak{m}$ is decomposed into irreducible Ad(K)-modules:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q.$$

where Ad(K)-modules $\mathfrak{m}_j$ $(j = 1, \cdots, q)$ are mutually non-equivalent.
Then G-invariant metrics on G/H which are also Ad(K)-invariant can be written as

$$< \, , \, >= x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q}, \quad (1)$$

for positive real numbers $x_1, \cdots, x_q$.

Ricci tensor of a compact homogeneous space G/H

- Note that G-invariant symmetric covariant 2-tensors on G/H which are also Ad(K)-invariant are the same form as the metrics.
  In particular, the Ricci tensor r of a G-invariant Riemannian metric on G/H which are also Ad(K)-invariant is of the same form as (1).

- Let $\{e_\alpha\}$ be a B-orthonormal basis adapted to the decomposition of $\mathfrak{m}$, i.e., $e_\alpha \in \mathfrak{m}_i$ for some i, and $\alpha < \beta$ if $i < j$ (with $e_\alpha \in \mathfrak{m}_i$ and $e_\beta \in \mathfrak{m}_j$).

- We put $A^\gamma_{\alpha\beta} = B\left(\left[e_\alpha, e_\beta\right], e_\gamma\right)$, so that $\left[e_\alpha, e_\beta\right] = \sum_\gamma A^\gamma_{\alpha\beta} e_\gamma$, and set $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A^\gamma_{\alpha\beta})^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i,\ e_\beta \in \mathfrak{m}_j,\ e_\gamma \in \mathfrak{m}_k$.

- Notations $\begin{bmatrix} k \\ ij \end{bmatrix}$ are introduced by Wang and Ziller in 1986.
- Then, the non-negative number $\begin{bmatrix} k \\ ij \end{bmatrix}$ is independent of the B-orthonormal bases chosen for $\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k$, and

$$\begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}. \tag{2}$$

- For simplicity, we write $A_{ijk} = \begin{bmatrix} k \\ ij \end{bmatrix}$.
  Let $d_k = \dim \mathfrak{m}_k$. Then we have (cf. Park - S. (1997))

Lemma

The components $r_1, \cdots, r_q$ of Ricci tensor r of the metric $< \, , \, >= x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q}$ on G/H which are also Ad(K)-invariant are given by

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} A_{jik} - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} A_{kij} \quad (k = 1, \cdots, q) \qquad (3)$$

where the sum is taken over $i, j = 1, \cdots, q$.

- We consider the homogeneous space $G/K = \mathrm{SO}(n)/\mathrm{U}(p) \times \mathrm{SO}(n-2p)$, where the embedding of K in G is diagonal. The tangent space $\mathfrak{p}$ of G/K decomposes into two $\mathrm{Ad}(K)$-submodules

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2,$$

with

$$\mathfrak{p}_1 = \left\{\begin{pmatrix} 0 & A_{12} \\ -{}^tA_{12} & 0 \end{pmatrix}\right\} \text{ and } \mathfrak{p}_2 \subset \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

where $A_{12} \in M(n-2p, 2p)$ ($M(p,q)$ the set of all $p \times q$ matrices) and $A_{11} \in \mathfrak{so}(2p)$. Note that the irreducible $\mathrm{Ad}(H)$-submodules $\mathfrak{p}_1$, $\mathfrak{p}_2$ are mutually non-equivalent.

- The tangent space $\mathfrak{m}$ of the Stiefel manifold can be written as follows

$$\mathfrak{m} = \mathfrak{h}_0 \oplus \mathfrak{su}(p) \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \tag{4}$$

where $\mathfrak{h}_0$ is 1 dimensional center of $\mathfrak{su}(p)$ and $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ is the tangent space of generalized flag manifolds $B(\ell, p)$ or $D(\ell, p)$. Note that $\mathrm{Ad}(\mathrm{U}(p) \times \mathrm{SO}(n-2p))$-modules in the decomposition (4) are mutually non-equivalent.
For simplicity, we rewrite the decomposition (4) as

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \tag{5}$$

- We write the invariant metrics on Stiefel manifold $\mathrm{SO}(n)/\mathrm{SO}(n-2p)$ defined by the $\mathrm{Ad}(\mathrm{U}(p)\times\mathrm{SO}(n-2p))$-invariant inner products on $\mathfrak{m}$ as

$$\langle\cdot,\cdot\rangle = u_0(-B)|_{\mathfrak{m}_0} + u_1(-B)|_{\mathfrak{m}_1} + u_2(-B)|_{\mathfrak{m}_2} + u_3(-B)|_{\mathfrak{m}_3} \qquad (6)$$

where $u_i \in \mathbb{R}_+$ $(i = 0, 1, 2, 3)$.

We set $d_1 = \dim(\mathfrak{m}_1)$, $d_2 = \dim(\mathfrak{m}_2)$ and $d_3 = \dim(\mathfrak{m}_3)$ (and $d_0 = \dim(\mathfrak{m}_0) = 1$.) It easy to see that the following relations hold:

$$[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_3,$$

$$[\mathfrak{m}_3, \mathfrak{m}_3] \subset \mathfrak{m}_0 \oplus \mathfrak{m}_1, \quad [\mathfrak{m}_2, \mathfrak{m}_3] \subset \mathfrak{m}_2.$$

- We see that the only non zero numbers triples (up to permutation of indices) for the metric corresponding to (6) are

$$A_{220}, \quad A_{330}, \quad A_{111}, \quad A_{122}, \quad A_{133}, \quad A_{322}.$$

From A. Arvanitoyeorgos, K. Mori and Y. Sakane, we have the following:

Lemma

The triples A_{ijk} are given as follows:

$$A_{220} = \frac{d_2}{(d_2+4d_3)}, \quad A_{330} = \frac{4d_3}{(d_2+4d_3)} \quad A_{111} = \frac{2d_3(2d_1+2-d_3)}{(d_2+4d_3)}$$
$$A_{122} = \frac{d_1 d_2}{(d_2+4d_3)} \quad A_{133} = \frac{2d_3(d_3-2)}{(d_2+4d_3)} \quad A_{322} = \frac{d_2 d_3}{(d_2+4d_3)}.$$

Proposition

The components of the Ricci tensor r *for the invariant metric* $\langle \cdot, \cdot \rangle$ *on Stiefel manifold* $G/H = \mathrm{SO}(n)/\mathrm{SO}(n-2p)$ *defined by* (6) *are given as follows:*

$$
\begin{aligned}
r_0 &= \frac{u_0}{4u_2^2}\frac{d_2}{(d_2+4d_3)} + \frac{u_0}{4u_3^2}\frac{4d_3}{(d_2+4d_3)} \\
r_1 &= \frac{1}{4d_1u_1}\frac{2d_3(2d_1+2-d_3)}{(d_2+4d_3)} + \frac{u_1}{4u_2^2}\frac{d_2}{(d_2+4d_3)} + \frac{u_1}{2d_1u_3^2}\frac{d_3(d_3-2)}{(d_2+4d_3)} \\
r_2 &= \frac{1}{2u_2} - \frac{u_3}{2u_2^2}\frac{d_3}{(d_2+4d_3)} - \frac{1}{2u_2^2}\left(u_0\frac{1}{(d_2+4d_3)} + u_1\frac{d_1}{(d_2+4d_3)}\right) \\
r_3 &= \frac{1}{u_3}\left(\frac{1}{2} - \frac{1}{2}\frac{d_2}{(d_2+4d_3)}\right) + \frac{u_3}{4u_2^2}\frac{d_2}{(d_2+4d_3)} \\
&\quad -\frac{1}{u_3^2}\left(u_0\frac{2}{(d_2+4d_3)} + u_1\frac{d_3-2}{(d_2+4d_3)}\right)
\end{aligned}
$$

- The metric of the form (6) is Einstein if and only if the system of equations

$$r_0 = r_1, \quad r_1 = r_2, \quad r_2 = r_3, \tag{7}$$

has positive solutions.

- After normalizing with $u_3 = 1$,
  the system of equations (7) becomes

$$
\begin{aligned}
f_1 &= u_0u_1(n-2p) - {u_1}^2(n-2p) + 2(p-1)u_0u_1{u_2}^2 \\
&\quad -(p-2){u_1}^2{u_2}^2 - p{u_2}^2 = 0, \\
f_2 &= {u_1}^2\left(np - p^2 - 1\right) - 2(n-2)pu_1u_2 + p^2{u_2}^2 \\
&\quad +(p-2)p{u_1}^2{u_2}^2 + (p-1)pu_1 + u_0u_1 = 0, \\
f_3 &= 2(n-2)pu_2 + p(-n+p+1) + 2(p-2)(p+1)u_1{u_2}^2 \\
&\quad -(p-1)(p+1)u_1 - 4(p-1)p{u_2}^2 + 4u_0{u_2}^2 - u_0 = 0.
\end{aligned}
\tag{8}
$$

- We consider a polynomial ring $R = \mathbb{Q}[n, p][z, u_0, u_1, u_2]$ and an ideal J generated by $\{f_1, f_2, f_3, z\,u_0\,u_1\,u_2 - 1\}$
  to find non zero solutions of equation (8).

  We take the lexicographic order $>$ with $z > u_0 > u_1 > u_2$ for a monomial ordering on R.

  Then, by an aid of computer, we see that a Gröbner basis for the ideal J contains the polynomial
  $(2(p-1){u_2}^2 - 2(n-2)u_2 + n - 1)\, G_{n,p}(u_2)$,
  where

$$
\begin{aligned}
G_{n,p}(u_2) &= 8\left(5(p-2)^3+22(p-2)^2+29(p-2)+8\right)(p-2)(p-1){u_2}^8\\
&-8(n-2)(p-2)(3p-1)\left(p^2-p+2\right){u_2}^7+((68(p-2)^4+312(p-2)^3\\
&+484(p-2)^2+288(p-2)+64)(n-2p)+16(p-2)^5+100(p-2)^4\\
&+296(p-2)^3+452(p-2)^2+320(p-2)+96){u_2}^6\\
&-8(n-2)(4(p-2)^3+15(p-2)^2+21(p-2)+8)(n-2p){u_2}^5\\
&+2(n-2p)(p-1)(p(21p-26)(n-2p)+10(p-2)^3+42(p-2)^2\\
&+80(p-2)+48){u_2}^4-2(n-2)\left(7p^2-8p+4\right)(n-2p)^2{u_2}^3\\
&+\Big(p(11p-12)(n-2p)+8(p-2)^3+33(p-2)^2+56(p-2)\\
&+30\Big)(n-2p)^2{u_2}^2-(2p(n-2p)+4(p-1)p)\,(n-2p)^3u_2\\
&+p(n-2p)^4+\left(p^2-p+1\right)(n-2p)^3.
\end{aligned}
$$

(coefficients of even degree are positive and coefficients of odd degree are negative)

- Thus we see that, if the equation $G_{n,p}(u_2) = 0$ has real solutions, then these are positive for $2 \leq p \leq n/2$.
- Now we take the lexicographic order $>$ with $z > u_1 > u_2 > u_0$ for a monomial ordering on R. Then, by the aid of computer, we see that a Gröbner basis for the ideal J contains the polynomial $(u_0 - 1)H_{n,p}(u_0)$ such that

$$H_{n,p}(u_0) = \sum_{k=0}^{8} a_k(n,p){u_0}^k,$$

where $a_k(n,p)$ ($k = 0, 1, \cdots, 8$) are polynomials of n, p with property that $a_k(n,p)$ are positive for k is even and negative for k is odd, if $n - 5(p+1)/2 \geq 0$, that is, $p \leq 2n/5 - 1$.

for example,

$$
\begin{aligned}
a_0(n,p) &= 64(p-2)p^3\Big(n-\frac{5(p+1)}{2}\Big)^9 + 96(p-2)p^3(11p+7)\Big(n-\frac{5(p+1)}{2}\Big)^8 \\
&+576(p-2)p^3(p+1)(13p+5)\Big(n-\frac{5(p+1)}{2}\Big)^7 + (30004(p-2)^7 + 427488(p-2)^6 \\
&+2515972(p-2)^5 + 7835288(p-2)^4 + 13626592(p-2)^3 + 12555872(p-2)^2 + 4791744(p-2) \\
&+512)\Big(n-\frac{5(p+1)}{2}\Big)^6 + 4(18789(p-2)^6 + 248313(p-2)^5 + 1295975(p-2)^4 \\
&+3340687(p-2)^3 + 4256148(p-2)^2 + 2146136(p-2) + 928)p^2\Big(n-\frac{5(p+1)}{2}\Big)^5 \\
&+(122679(p-2)^7 + 1995834(p-2)^6 + 13365614(p-2)^5 + 47170188(p-2)^4 \\
&+92568607(p-2)^3 + 95837594(p-2)^2 + 40950780(p-2) + 44512)p^2\Big(n-\frac{5(p+1)}{2}\Big)^4 \\
&+2(65563(p-2)^9 + 1402351(p-2)^8 + 12986098(p-2)^7 + 68013362(p-2)^6 \\
&+220420041(p-2)^5 + 452819519(p-2)^4 + 576171516(p-2)^3 + 415509440(p-2)^2 \\
&+130343854(p-2) + 282848)p\Big(n-\frac{5(p+1)}{2}\Big)^3
\end{aligned}
$$

$$+\frac{1}{4}(354867(p-2)^{10} + 8723046(p-2)^{9}$$
$$+94326525(p-2)^{8} + 589028208(p-2)^{7} + 2341360757(p-2)^{6} + 6145435310(p-2)^{5}$$
$$+10655244691(p-2)^{4} + 11775095484(p-2)^{3} + 7534446728(p-2)^{2} + 2135654080(p-2)$$
$$+8047168)p\Big(n-\frac{5(p+1)}{2}\Big)^{2} + \frac{1}{4}(3936(p-2)^{8} + 75201(p-2)^{7} + 603308(p-2)^{6}$$
$$+2638758(p-2)^{5} + 6806032(p-2)^{4} + 10370489(p-2)^{3} + 8665044(p-2)^{2} + 3082288(p-2)$$
$$+18432)p(p+1)(5p+1)(7p+11)\Big(n-\frac{5(p+1)}{2}\Big) + \frac{1}{16}(92455(p-2)^{11} + 2535522(p-2)^{10}$$
$$+30740389(p-2)^{9} + 217376892(p-2)^{8} + 994341005(p-2)^{7} + 3078217258(p-2)^{6}$$
$$+6539013051(p-2)^{5} + 9423500944(p-2)^{4} + 8830656620(p-2)^{3} + 4872616168(p-2)^{2}$$
$$+1215054848(p-2) + 10612736)(p+1)^{2}.$$

- Thus we see that, if the equation $H_{n,p}(u_0) = 0$ has real solutions u_0, then these are positive for if $n - 5(p+1)/2 \geq 0$, that is, $p \leq 2n/5 - 1$.

- Now we take the lexicographic order $>$ with $z > u_0 > u_2 > u_1$ for a monomial ordering on R. Then, by the aid of computer, we see that a Gröbner basis for the ideal J contains the polynomial $(u_1 - 1)F_{n,p}(u_1)$.

- Moreover, the Gröbner basis contains polynomials of the form such that $b(n,p)u_2 - X(u_1)$ and $c(n,p)u_0 - Y(u_1)$, where $X(u_1)$ and $Y(u_1)$ are polynomials of degree 7 with coefficients in $\mathbb{Q}[n,p]$, and $b(n,p)$ and $c(n,p)$ are in $\mathbb{Q}[n,p]$.
- We also can show that $b(n,p)$ and $c(n,p)$ are non zero for $n-2p \geq 0$ and $p \geq 2$. In particular, if u_1 is reals, then the solutions u_2 and u_0 are real for $b(n,p)u_2 = X(u_1)$ and $c(n,p)u_0 = Y(u_1)$.
- In the above we have seen that, if u_2 and u_0 are real solutions, these are positive.

$$
\begin{aligned}
&F_{n,p}(u_1) = (2n-p-1)^4(p-2)^2(p+1)^2{u_1}^8 \\
&-2(2n-p-1)^3(p-2)(p+1)\left(-10p^3+np^2+25p^2-20p+4n\right){u_1}^7 \\
&-2(2n-p-1)(-182p^7-83np^6+1407p^6+505n^2p^5-1587np^5-2570p^5-284n^3p^4 \\
&-603n^2p^4+5381np^4+504p^4+36n^4p^3+710n^3p^3-1440n^2p^3-4595np^3+1377p^3-64n^4p^2 \\
&-616n^3p^2+2768n^2p^2-250np^2+202p^2+32n^4p+48n^3p-688n^2p+344np-480p+32n^4 \\
&-112n^3+112n^2+64n+32){u_1}^6+(2n-p-1)^2(140p^6-10np^5-726p^5-99n^2p^4+414np^4 \\
&+1057p^4+44n^3p^3+68n^2p^3-930np^3-178p^3-112n^3p^2+312n^2p^2+420np^2-596p^2 \\
&+80n^3p-208n^2p-280np+816p-64n^3+272n^2-304n-64){u_1}^5+(46p^8+1262np^7 \\
&-2706p^7-2685n^2p^6+2570np^6+7173p^6+828n^3p^5+10434n^2p^5-31616np^5+5706p^5 \\
&+1996n^4p^4-20068n^3p^4+34511n^2p^4+15458np^4-15743p^4-1728n^5p^3+9248n^4p^3 \\
&+7624n^3p^3-75052n^2p^3+63050np^3-14418p^3+384n^6p^2+864n^5p^2-22832n^4p^2 \\
&+65888n^3p^2-54384n^2p^2+16444np^2-1358p^2-896n^6p+6336n^5p-12288n^4p+2384n^3p \\
&+7616n^2p-6256np+1784p+256n^6-2048n^5+5632n^4-6624n^3+4336n^2-1632n+248){u_1}^4
\end{aligned}
$$

$$
\begin{aligned}
&-2(-330p^8+137np^7+2129p^7+852n^2p^6-4328np^6-2144p^6-1120n^3p^5+2522n^2p^5\\
&+6676np^5-2146p^5+704n^4p^4-1288n^3p^4-2508n^2p^4-4002np^4+5788p^4-304n^5p^3\\
&+1096n^4p^3-2600n^3p^3+10706n^2p^3-14817np^3+3701p^3+64n^6p^2-224n^5p^2+1104n^4p^2\\
&-6616n^3p^2+13188n^2p^2-6654np^2+1114p^2-64n^6p-128n^5p+3008n^4p-8600n^3p\\
&+8552n^2p-4428np+1000p+128n^6-1024n^5+2944n^4-3952n^3+3024n^2-1288n+256){u_1}^3\\
&+(444p^8-782np^7-1226p^7+363n^2p^6+2566np^6-121p^6+232n^3p^5-2758n^2p^5+708np^5\\
&+1854p^5-292n^4p^4+1524n^3p^4+447n^2p^4-5066np^4+439p^4+80n^5p^3-160n^4p^3-1948n^3p^3\\
&+5916n^2p^3-1638np^3-646p^3-96n^5p^2+896n^4p^2-1760n^3p^2-1512n^2p^2+4028np^2\\
&-1020p^2-320n^4p+1792n^3p-2768n^2p+600np+80p-224n^3+896n^2-848n+320){u_1}^2\\
&+2(3p^2-2np+p-2)(18p^6-35np^5-5p^5+31n^2p^4-17np^4-10p^4-16n^3p^3+37n^2p^3\\
&-25np^3+62p^3+4n^4p^2-18n^3p^2+26n^2p^2-53np^2+9p^2+4n^3p-16n^2p+44np-56p+16)u_1\\
&+(p^2-np+p-1)^2\left(3p^2-2np+p-2\right)^2.
\end{aligned}
$$

We see that,

$$F_{n,p}(1) > 0 \ \text{ for } \ 3 \leq p \leq \frac{2}{5}n - 1 \text{ and } F_{n,2}(2n) > 0 \ \text{ for } \ p = 2,$$

and $F_{n,p}(0) = (p^2 - np + p - 1)^2 \left(3p^2 - 2np + p - 2\right)^2 > 0.$
We also see that

$$F_{n,p}\Big(\frac{1}{4}\Big) < 0 \ \text{ for } \ 2 \leq p \leq \frac{2}{5}n - 1.$$

Thus we obtain at least two positive solutions for the equation $F_{n,p}(u_1) = 0$ for $2 \leq p \leq \dfrac{2}{5}n - 1$. Hence, we obtain two $\mathrm{Ad}(\mathrm{U}(p) \times \mathrm{SO}(n-2p))$-invariant Einstein metrics on the Stiefel manifolds $V_{2p}\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-2p)$ which are not Jensen's Einstein metrics.

The Stiefel manifolds $V_{2p}\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-2p)$

For $n = 31$, we have $2 \le p \le 2n/5 - 1 = 62/5 - 1 = 11.4$, but for $p = 12, 13$, we see the following:

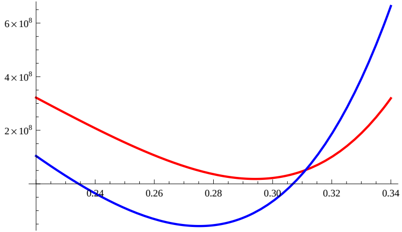
Except Jensen's Einstein metrics,

$V_{26}\mathbb{R}^{31} = \mathrm{SO}(31)/\mathrm{SO}(5)$, there are no $\mathrm{Ad}(\mathrm{U}(13) \times \mathrm{SO}(5)$-invariant Einstein metrics,

$V_{24}\mathbb{R}^{31} = \mathrm{SO}(31)/\mathrm{SO}(7)$, there are two more $\mathrm{Ad}(\mathrm{U}(12) \times \mathrm{SO}(7)$-invariant Einstein metrics.

In[5]:=
```
Plot[{FU1[u1, 31, 13], FU1[u1, 31, 12]},
 {u1, 0.22, 0.34}, PlotStyle → {{Red, Thick}, {Blue, Thick}}]
```

Out[5]=



$FU1(u1, 31, 13) = 491774976{u_1}^8 + 1682093952{u_1}^7 + 4011833808{u_1}^6 + 2082493764{u_1}^5 + 1342556360{u_1}^4 - 2795832361{u_1}^3 + 1093464243{u_1}^2 - 193555008u_1 + 15968016,$

$FU1(u1, 31, 12) = 24356284225{u_1}^8 + 71363530420{u_1}^7 + 235478881736{u_1}^6 + 125628595904{u_1}^5 + 221500487082{u_1}^4 - 235075487612{u_1}^3 + 75786327156{u_1}^2 - 12840182320u_1 + 1073676289.$