Invariant Einstein metrics on Stiefel manifolds

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Invariant Einstein metrics
on
Stiefel manifolds

based on joint works with
Andreas Arvanitoyeorgos and Marina Statha

- introduction
  left-invariant Einstein metrics on compact Lie groups
  homogeneous Einstein metrics on Stiefel manifolds
  $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$
- main results
- Ricci tensor of a compact homogeneous space
- The Stiefel manifolds $V_{k_1+k_2}\mathbb{R}^{k_1+k_2+k_3} = \mathrm{SO}(k_1+k_2+k_3)/\mathrm{SO}(k_3)$
- The compact Lie groups $\mathrm{SO}(n)$ $(n \geq 7)$

(M, g): Riemannian manifold

- (M, g) is called Einstein if the Ricci tensor $\mathrm{r}(g)$ of the metirc g satisfies $\mathrm{r}(g) = cg$ for some constant c.

  We consider G-invariant Einstein metrics on a homogeneous space G/K.

- General Problem: Find G-invariant Einstein metrics on a homogeneous space G/K and classify them if it is not unique.

- Einstein homogeneous spaces can be divided into three cases depending on Einstein constant c.
  Here we consider the case $c > 0$.

Examples of the case $c > 0$
(G/K is compact and $\pi_1(G/K)$ is finite).

- Sphere $(S^n = SO(n+1)/SO(n), g_0)$,
  Complex Projective space $(\mathbb{C}P^n = SU(n+1)/(S(U(1)\times U(n)))$,
  Symmetric spaces of compact type,
  isotropy irreducible spaces (in these cases G-invariant Einstein metrics is unique)
- Generalized flag manifolds (Kähler C-spaces) (if we fix a complex structure, it admits a unique Kähler-Einstein metric, but complex structure may not be unique)

- Compact semi-simple Lie groups
  (bi-invariant metric = negative of Killing form)
- Concerning left-invariant Einstein metrics on compact simple Lie groups,
  D' Atri and Ziller classified all naturally reductive metrics on G in " *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Memoirs Amer. Math. Soc. 19 (215) (1979)"

  and have found a large number of Einstein metrics on compact simple Lie groups G which are naturally reductive.
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  Are there non-naturally reductive Einstein metrics on the compact simple Lie groups ?

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Introduction

- For this problem, K. Mori (1996) has shown that there exist non-naturally reductive Einstein metrics on $\mathrm{SU}(n)$ $(n \geq 6)$.

  A. Arvanitoyeorgos, K. Mori and S. (2012) have shown existence of non-naturally reductive Einstein metrics on the compact simple Lie groups $\mathrm{SO}(n)$ $(n \geq 11)$, $\mathrm{Sp}(n)$ $(n \geq 3)$, E_6, E_7 and E_8, by using Generalized flag manifolds.
- Here we consider the case of $\mathrm{SO}(n)$ $(n \geq 7)$.
- Recently we can also show that, for compact simple Lie groups F_4, G_2 and $\mathrm{SU}(n)$ ($n \geq 3$), there exist non-naturally reductive Einstein metrics, that is, except $\mathrm{SO}(5)$, there exist non-naturally reductive Einstein metrics on compact simple Lie groups G with $\dim G > 3$. (Note that $\mathrm{SU}(2) = S^3$.)

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- A little history on homogeneous Einstein metrics on Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$:

  In 1963, S. Kobayashi proved
  existence of a homogeneous Einstein metric on the unit tangent bundle $T_1S^n = \mathrm{SO}(n)/\mathrm{SO}(n-2) = V_2\mathbb{R}^n$ (as S^1-bundle over a Kähler manifold $\mathrm{SO}(n)/(\mathrm{SO}(n-2)\times\mathrm{SO}(2))$).

  In 1970, A. Sagle proved that
  the Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$ for $k \geq 3$ admits an homogeneous Einstein metric.

  In 1973, G. Jensen proved that
  the Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$ for $k \geq 3$ admits at least two homogeneous Einstein metric.

- $V_2\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-2)$
  By a result of A. Back and W.Y. Hsiang (1987) and M. Kerr (1998), for $n \geq 5$ it is known that $\mathrm{SO}(n)/\,\mathrm{SO}(n-2)$ admits exactly one homogeneous Einstein metric.
  (the diagonal metrics are only homogeneous Einstein metrics)

- $V_2\mathbb{R}^4 = \mathrm{SO}(4)/\,\mathrm{SO}(2)$
  By a result of D. V. Alekseevsky, I. Dotti and C. Ferraris, $\mathrm{SO}(4)/\,\mathrm{SO}(2)$ admits exactly two invariant Einstein metrics (one is diagonal metric and the other is non-diagonal metric).
  Jensen's metric is a diagonal metric. Note that $\mathrm{SO}(4)/\,\mathrm{SO}(2)$ is diffeomorphic to $S^3 \times S^2$. The non-diagonal Einstein metric comes from the product metric on $S^3 \times S^2$.

- For the tangent space $\mathfrak{p}$ of Stiefel manifold $\mathrm{SO}(n)/\mathrm{SO}(n-4)$

$$\mathfrak{p} = \left(\begin{array}{cccc|c} 0 & a_{12} & a_{13} & a_{14} & A_{15} \\ -a_{12} & 0 & a_{23} & a_{24} & A_{25} \\ -a_{13} & -a_{23} & 0 & a_{34} & A_{35} \\ -a_{14} & -a_{24} & a_{34} & 0 & A_{45} \\ \hline -{}^tA_{15} & -{}^tA_{25} & -{}^tA_{35} & -{}^tA_{45} & * \end{array}\right),$$

  we have a decomposition $\mathfrak{p} = \sum_{i<j} \mathfrak{p}_{ij} \oplus \sum_{k<5} \mathfrak{p}_{k5}$ as $\mathrm{Ad}(\mathrm{SO}(n-4))$-submodules, where $\dim \mathfrak{p}_{ij} = 1$ and $\dim \mathfrak{p}_{k5} = n-4$. Note that submodules $\mathfrak{p}_{ij}$ are equivalent each other and submodules $\mathfrak{p}_{k5}$ are equivalent each other.
- For general invariant metrics on the Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$, it would be difficult to find all homogeneous Einstein metrics.

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- For general invariant metrics on the Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$, it would be difficult to find all homogeneous Einstein metrics.

- Open problem : How many homogeneous Einstein metrics are there on Stiefel manifold $V_k\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-k)$?

  Are there homogeneous Einstein metrics on Stiefel manifold $\mathrm{SO}(n)/\,\mathrm{SO}(n-k)$ other than Jensen's Einstein metrics ?

Main results

- Now we state our main results:

Theorem

There exist new homogeneous Einstein metrics on the Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-k)$ for $k \geq 4$ and $n \geq 6$.

These are not Jensen's Einstein metrics.
(For $k = 4$, A. Arvanitoyeorgos, S. and M. Statha: New homogeneous Einstein metrics on Stiefel manifolds, Differential Geom. Appl. (2014), in press. Available online 5th February 2014, http://dx.doi.org/10.1016/j.difgeo.2014.01.007)

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Related known other results

- A. Arvanitoyeorgos, V.V. Dzhepko and Yu. G. Nikonorov (2009) proved that
  for $s > 1$ and $\ell > k \geq 3$, the Stiefel manifold $\mathrm{SO}(sk + \ell)/\mathrm{SO}(\ell)$ admits at least four $\mathrm{SO}(sk + \ell)$-invariant Einstein metrics that are also $\mathrm{Ad}(\mathrm{SO}(k)^s \times \mathrm{SO}(\ell))$-invariant and two of which are Jensen's metrics.

  In 2007, they have treated $\mathrm{SO}(k + k + \ell)/\mathrm{SO}(\ell)$ ($\ell > k \geq 3$) as the special case of above and proved that the Stiefel manifold $\mathrm{SO}(k + k + \ell)/\mathrm{SO}(\ell)$ admits at least four $\mathrm{SO}(k + k + \ell)$-invariant Einstein metrics that are also
  $\mathrm{Ad}(\mathrm{SO}(k) \times \mathrm{SO}(k) \times \mathrm{SO}(\ell))$-invariant.

- Let G be a compact semi-simple Lie group and K a connected closed subgroup of G.
  Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B (= − Killing form of $\mathfrak{g}$). Then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, $[\,\mathfrak{k}, \mathfrak{m}\,] \subset \mathfrak{m}$ and a decomposition of $\mathfrak{m}$ into irreducible Ad(K)-modules:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q.$$

- We assume that Ad(K)-modules $\mathfrak{m}_j$ $(j = 1, \cdots, q)$ are mutually non-equivalent.
  Then a G-invariant metric on G/K can be written as

$$<\,,\,>= x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q}, \tag{1}$$

  for positive real numbers $x_1, \cdots, x_q$.

Ricci tensor of a compact homogeneous space G/K

- Note that G-invariant symmetric covariant 2-tensors on G/K are the same form as the metrics.
  In particular, the Ricci tensor r of a G-invariant Riemannian metric on G/K is of the same form as (1).

- Let $\{e_\alpha\}$ be a B-orthonormal basis adapted to the decomposition of $\mathfrak{m}$, i.e., $e_\alpha \in \mathfrak{m}_i$ for some i, and $\alpha < \beta$ if $i < j$ (with $e_\alpha \in \mathfrak{m}_i$ and $e_\beta \in \mathfrak{m}_j$).
- We put $A^\gamma_{\alpha\beta} = B\left(\left[e_\alpha, e_\beta\right], e_\gamma\right)$, so that $\left[e_\alpha, e_\beta\right] = \sum_\gamma A^\gamma_{\alpha\beta} e_\gamma$, and set $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A^\gamma_{\alpha\beta})^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i,\ e_\beta \in \mathfrak{m}_j,\ e_\gamma \in \mathfrak{m}_k$.
- Notations $\begin{bmatrix} k \\ ij \end{bmatrix}$ are introduced by Wang and Ziller in 1986.

- Then, the non-negative number $\begin{bmatrix} k \\ ij \end{bmatrix}$ is independent of the B-orthonormal bases chosen for $\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k$, and

$$\begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}. \qquad (2)$$

- Let $d_k = \dim \mathfrak{m}_k$. Then we have (cf. Park - S. (1997))

Lemma

The components $r_1, \cdots, r_q$ *of Ricci tensor* r *of the metric* $< \,,\, >= x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q}$ *on* G/K *are given by*

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} \begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} \begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = 1, \cdots, q) \qquad (3)$$

where the sum is taken over $i, j = 1, \cdots, q$.

The Stiefel manifolds $V_{k_1+k_2}\mathbb{R}^{k_1+k_2+k_3} = \mathrm{SO}(k_1+k_2+k_3)/\mathrm{SO}(k_3)$

- We consider the homogeneous space $G/K = \mathrm{SO}(k_1+k_2+k_3)/\mathrm{SO}(k_1)\times\mathrm{SO}(k_2)\times\mathrm{SO}(k_3)$, where the embedding of K in G is diagonal. The tangent space $\mathfrak{m}$ of G/K decomposes into three $\mathrm{Ad}(K)$-submodules

$$\mathfrak{m} = \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23},$$

where

$$\mathfrak{m}_{12} = \begin{pmatrix} 0 & A_{12} & 0 \\ -{}^tA_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathfrak{m}_{13} = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & 0 & 0 \\ -{}^tA_{13} & 0 & 0 \end{pmatrix}, \mathfrak{m}_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & -{}^tA_{23} & 0 \end{pmatrix}$$

and $A_{12} \in M(k_1,k_2), A_{13} \in M(k_1,k_3), A_{23} \in M(k_2,k_3)$ ($M(p,q)$ the set of all $p\times q$ matrices). Note that the irreducible $\mathrm{Ad}(K)$-submodules $\mathfrak{m}_{12}$, $\mathfrak{m}_{13}$ and $\mathfrak{m}_{23}$ are mutually non-equivalent.

The Stiefel manifolds $V_{k_1+k_2}\mathbb{R}^{k_1+k_2+k_3} = \mathrm{SO}(k_1+k_2+k_3)/\mathrm{SO}(k_3)$

- For the tangent space $\mathfrak{p}$ of the Stiefel manifold $G/H = \mathrm{SO}(k_1+k_2+k_3)/\mathrm{SO}(k_3)$, we consider the decomposition

$$\mathfrak{p} = \mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2) \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}$$

where corresponding $\mathrm{Ad}(\mathrm{SO}(k_1)\times\mathrm{SO}(k_2)\times\mathrm{SO}(k_3))$-submodules are non-equivalent.
We consider G-invariant metrics on the Stiefel manifold G/H determined by the $\mathrm{Ad}(\mathrm{SO}(k_1)\times\mathrm{SO}(k_2)\times\mathrm{SO}(k_3))$-invariant scalar products on $\mathfrak{p}$ given by

$$\begin{aligned}\langle\ ,\ \rangle &= x_1\,(-B)|_{\mathfrak{so}(k_1)} + x_2\,(-B)|_{\mathfrak{so}(k_2)} \\ &\quad + x_{12}\,(-B)|_{\mathfrak{m}_{12}} + x_{13}\,(-B)|_{\mathfrak{m}_{13}} + x_{23}\,(-B)|_{\mathfrak{m}_{23}}\end{aligned} \quad (4)$$

for $k_1 \geq 2$, $k_2 \geq 2$ and $k_3 \geq 1$.

Lemma

The components of the Ricci tensor r *for the left-invariant metric* $\langle\ ,\ \rangle$ *on* G/H *defined by* (4), are given as follows

$$
\begin{aligned}
r_1 &= \frac{k_1-2}{4(n-2)x_1} + \frac{1}{4(n-2)}\left(k_2\frac{x_1}{{x_{12}}^2} + k_3\frac{x_1}{{x_{13}}^2}\right),\\
r_2 &= \frac{k_2-2}{4(n-2)x_2} + \frac{1}{4(n-2)}\left(k_1\frac{x_2}{{x_{12}}^2} + k_3\frac{x_2}{{x_{23}}^2}\right),\\
r_{12} &= \frac{1}{2x_{12}} + \frac{k_3}{4(n-2)}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right)\\
&\quad - \frac{1}{4(n-2)}\left((k_1-1)\frac{x_1}{{x_{12}}^2} + (k_2-1)\frac{x_2}{{x_{12}}^2}\right),
\end{aligned}
$$

Lemma

$$\begin{aligned} r_{13} &= \frac{1}{2x_{13}} + \frac{k_2}{4(n-2)}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) \\ &\quad - \frac{1}{4(n-2)}\left((k_1-1)\frac{x_1}{{x_{13}}^2}\right) \\ r_{23} &= \frac{1}{2x_{23}} + \frac{k_1}{4(n-2)}\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right) \\ &\quad - \frac{1}{4(n-2)}\left((k_2-1)\frac{x_2}{{x_{23}}^2}\right), \end{aligned}$$

where $n = k_1 + k_2 + k_3$.

The Stiefel manifolds $V_{1+k_2}\mathbb{R}^{1+k_2+k_3} = \mathrm{SO}(1+k_2+k_3)/\mathrm{SO}(k_3)$

For $k_1 = 1$ and $k_2 \geq 2$, we have the Stiefel manifold $G/H = \mathrm{SO}(1+k_2+k_3)/\mathrm{SO}(k_3)$ with corresponding decomposition

$$\mathfrak{p} = \mathfrak{so}(k_2) \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}.$$

We consider G-invariant metrics on G/H determined by the $\mathrm{Ad}(\mathrm{SO}(k_2)\times\mathrm{SO}(k_3))$-invariant scalar products on $\mathfrak{p}$ given by

$$\langle\ ,\ \rangle = x_2\,(-B)|_{\mathfrak{so}(k_2)} + x_{12}\,(-B)|_{\mathfrak{m}_{12}} + x_{13}\,(-B)|_{\mathfrak{m}_{13}} + x_{23}\,(-B)|_{\mathfrak{m}_{23}}. \quad (5)$$

For simplicity we use the notation

$$\langle\ ,\ \rangle = \begin{pmatrix} 0 & x_{12} & x_{13} \\ x_{12} & x_2 & x_{23} \\ x_{13} & x_{23} & * \end{pmatrix}.$$

Lemma

The components of the Ricci tensor r for the left-invariant metric $\langle\ ,\ \rangle$ on G/H defined by (5), are given as follows

$$r_2 = \frac{k_2-2}{4(n-2)x_2} + \frac{1}{4(n-2)}\left(\frac{x_2}{{x_{12}}^2} + k_3\frac{x_2}{{x_{23}}^2}\right),$$

$$r_{12} = \frac{1}{2x_{12}} + \frac{k_3}{4(n-2)}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) - \frac{1}{4(n-2)}\left((k_2-1)\frac{x_2}{{x_{12}}^2}\right),$$

$$r_{23} = \frac{1}{2x_{23}} + \frac{1}{4(n-2)}\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right) - \frac{1}{4(n-2)}\left((k_2-1)\frac{x_2}{{x_{23}}^2}\right),$$

$$r_{13} = \frac{1}{2x_{13}} + \frac{k_2}{4(n-2)}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right),$$

where $n = 1 + k_2 + k_3$.

We consider the system of equations

$$r_2 = r_{12}, \quad r_{12} = r_{13}, \quad r_{13} = r_{23}. \tag{6}$$

Then finding Einstein metrics of the form (5) reduces to finding positive solutions of system (6), and we normalize our equations by putting $x_{23} = 1$.
For simplicity, we consider the case when $k_2 = 3$ and $k_3 = n - 4$.
Now the system (6) reduces to the system of equations:

$$\left.\begin{array}{l}
f_1 = -(n-4){x_{12}}^3 x_2 + (n-4){x_{12}}^2 x_{13} x_2^2 + (n-4) x_{12} {x_{13}}^2 x_2 \\
-2(n-2) x_{12} x_{13} x_2 + (n-4) x_{12} x_2 + {x_{12}}^2 x_{13} + 3 x_{13} x_2^2 = 0, \\
f_2 = (n-3){x_{12}}^3 - 2(n-2){x_{12}}^2 x_{13} - (n-5) x_{12} {x_{13}}^2 \\
+2(n-2) x_{12} x_{13} + (3-n) x_{12} + 2{x_{12}}^2 x_{13} x_2 - 2 x_{13} x_2 = 0, \\
f_3 = (n-2) x_{12} x_{13} - (n-2) x_{12} + {x_{12}}^2 - x_{12} x_{13} x_2 \\
-2{x_{13}}^2 + 2 = 0.
\end{array}\right\} \tag{7}$$

We consider a polynomial ring $R = \mathbb{Q}[z, x_2, x_{12}, x_{13}]$ and an ideal I generated by $\{f_1,\ f_2,\ f_3,\ z\,x_2\,x_{12}\,x_{13} - 1\}$ to find non-zero solutions of equations (7). We take a lexicographic order $>$ with $z > x_2 > x_{12} > x_{13}$ for a monomial ordering on R. Then, by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $(x_{13} - 1)\,h_1(x_{13})$, where $h_1(x_{13})$ is a polynomial of x_{13} with degree 10 given by

$$
\begin{aligned}
& h_1(x_{13}) \\
&= (n-1)^3(5n-11)^2\left(n^3 - 10n^2 + 33n - 35\right)\left(n^3 - 6n^2 + 9n - 3\right)x_{13}{}^{10} \\
&-2(n-1)^2(5n-11)\Big(17n^8 - 356n^7 + 3221n^6 - 16396n^5 + 51159n^4 \\
&-99720n^3 + 117862n^2 - 76568n + 20649\Big)x_{13}{}^9 \\
&+(n-1)\Big(4n^{11} + 389n^{10} - 11430n^9 + 136940n^8 - 946084n^7 + 4220820n^6 \\
&-12735744n^5 + 26330445n^4 - 36830352n^3 + 33361745n^2 - 17678114n \\
&+4164053\Big)x_{13}{}^8
\end{aligned}
$$

$$
\begin{aligned}
&-4\big(8n^{12} - 38n^{11} - 2320n^{10} + 43360n^{9} - 379590n^{8} + 2055155n^{7} \\
&-7507061n^{6} + 19112638n^{5} - 34063584n^{4} + 41706995n^{3} - 33417851n^{2} \\
&+15765962n - 3316050\big)\,{x_{13}}^{7} \\
&+\big(112n^{12} - 2718n^{11} + 27906n^{10} - 149523n^{9} + 354855n^{8} + 588726n^{7} \\
&-7694150n^{6} + 29295831n^{5} - 65164167n^{4} + 92342878n^{3} - 82220114n^{2} \\
&+41992646n - 9373722\big)\,{x_{13}}^{6} \\
&-2\big(112n^{12} - 3338n^{11} + 45506n^{10} - 376557n^{9} + 2113393n^{8} - 8496684n^{7} \\
&+25132832n^{6} - 55172371n^{5} + 89317711n^{4} - 104159676n^{3} + 83190848n^{2} \\
&-40884390n + 9337014\big)\,{x_{13}}^{5} \\
&+\big(280n^{12} - 8710n^{11} + 123662n^{10} - 1060617n^{9} + 6124653n^{8} \\
&-25086974n^{7} + 74662934n^{6} - 162341127n^{5} + 255246159n^{4} \\
&-282268554n^{3} + 208035522n^{2} - 91729890n + 18337990\big)\,{x_{13}}^{4}
\end{aligned}
$$

$$
\begin{aligned}
&-4\Big(56n^{12} - 1710n^{11} + 23600n^{10} - 194131n^9 + 1056185n^8 - 3982619n^7\\
&+10582237n^6 - 19666327n^5 + 24629929n^4 - 18903391n^3 + 6556083n^2\\
&+985682n - 1096346\Big)\,{x_{13}}^3\\
&+(n-1)\Big(112n^{11} - 3115n^{10} + 38156n^9 - 268869n^8 + 1189262n^7\\
&-3348224n^6 + 5627178n^5 - 3967104n^4 - 3831854n^3 + 11143963n^2\\
&-9643014n + 3094229\Big)\,{x_{13}}^2\\
&-2(n-5)(n-3)(n-1)^2(n+1)\Big(16n^7 - 275n^6 + 1868n^5 - 6039n^4\\
&+7372n^3 + 7943n^2 - 31120n + 23163\Big)\,x_{13}\\
&+(n-5)^2(n-3)^2(n-1)^3(n+1)^2\Big(4n^3 - 23n^2 - 10n + 161\Big).
\end{aligned}
$$

Note that, if the equation $h_1(x_{13}) = 0$ has real solutions, then these are positive.

If $x_{13} = 1$ we see that $f_3 = x_{12}(x_{12} - x_2) = 0$ and thus the system of equations (7) reduces the system of equations

$$x_{12} = x_2, \quad (n-1){x_2}^2 - 2(n-2)x_2 + 2 = 0.$$

Thus we obtain two solutions for the system of equations (7) :

$$x_{12} = x_2 = (n - 2 - \sqrt{n^2 - 6n + 6})/(n-1),\ \ x_{13} = x_{23} = 1 \quad \text{and}$$

$$x_{12} = x_2 = (n - 2 + \sqrt{n^2 - 6n + 6})/(n-1),\ \ x_{13} = x_{23} = 1.$$

These are known as the Jensen's Einstein metrics on Stiefel manifolds.

Consider the case when $x_{13} \neq 1$. Then we have $h_1(x_{13}) = 0$ and we claim that the equation $h_1(x_{13}) = 0$ has at least two positive roots.

- At first we consider the value $h_1(x_{13})$ at $x_{13} = 1$. We have

$$h_1(1) = -800\left(6n^5 - 88n^4 + 476n^3 - 1175n^2 + 1274n - 490\right)$$
$$= -800\Big(6(n-6)^5 + 92(n-6)^4 + 524(n-6)^3 + 1345(n-6)^2$$
$$+1430(n-6) + 278\Big).$$

  Thus we see that $h_1(1) < 0$ for $n \geq 6$.

- Secondly we consider the value $h_1(x_{13})$ at $x_{13} = 0$. Then we have

$$h_1(0) = (n-5)^2(n-3)^2(n-1)^3(n+1)^2\left(4n^3 - 23n^2 - 10n + 161\right)$$
$$= (n-5)^2(n-3)^2(n-1)^3(n+1)^2\times$$
$$\Big(4(n-5)^3 + 37(n-5)^2 + 60(n-5) + 36\Big).$$

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$$\Big(4(n-5)^3 + 37(n-5)^2 + 60(n-5) + 36\Big).$$

  Thus we see that $h_1(0) > 0$ for $n \geq 6$.

- Thirdly we consider the value $h_1(x_{13})$ at $x_{13} = 2$. We have

$$h_1(2) = (n-5)\Big(4n^{11} + 313n^{10} - 2902n^9 - 11175n^8 + 334728n^7 \\ -2555222n^6 + 10151316n^5 - 22397134n^4 + 25374596n^3 \\ -8599331n^2 - 8372942n + 6279733\Big) \\ = (n-5)\Big(4(n-5)^{11} + 533(n-5)^{10} + 18248(n-5)^9 + 292860(n-5)^8 \\ +2795928(n-5)^7 + 17723008(n-5)^6 + 77831856(n-5)^5 \\ +236200016(n-5)^4 + 477068416(n-5)^3 + 593616384(n-5)^2 \\ +389736448(n-5) + 89941248\Big).$$

Thus we see that $h_1(0) > 0$, $h_1(1) < 0$ and $h_1(2) > 0$ for $n \geq 6$. Hence, we obtain two solutions $x_{13} = \alpha_{13},\ \beta_{13}$ of the equation $h_1(x_{13}) = 0$ between $0 < \alpha_{13} < 1$ and $1 < \beta_{13} < 2$.
More precisely, for $n \geq 9$, we have

$$1 - 2/n - 6/n^2 < \alpha_{13} < 1 - 2/n - 7/(2n^2),$$

$$1 + 50/(63n^2) < \beta_{13} < 1 + 3/n^2.$$

We consider a polynomial ring $R = \mathbb{Q}[z, x_2, x_{12}, x_{13}]$ and an ideal J generated by $\{f_1,\ f_2,\ f_3,\ z\,x_2\,x_{12}\,x_{13}\,(x_{13}-1)-1\}$ and take a lexicographic order $>$ with $z > x_2 > x_{12} > x_{13}$ for a monomial ordering on R. Then, by the aid of computer, we see that a Gröbner basis for the ideal J contains the polynomials $h_1(x_{13})$ and

$$8(n-3)(n-2)^3(n-1)^2 a(n)\,x_{12} - w_{12}(x_{13}),$$

Also, for the same ideal J and the lexicographic order $>$ with $z > x_{12} > x_2 > x_{13}$ for monomials on R, we see that a Gröbner basis for J contains the polynomial

$$8(n-5)(n-2)^3(n-1)^3(n+1)\left(4n^3 - 23n^2 - 10n + 161\right) a(n)x_2 - w_2(x_{13}).$$

where $a(n)$ is a polynomial of n of degree 43 with integer coefficients and $w_{12}(x_{13})$, $w_2(x_{13})$ are polynomials of x_{13} (and n) with integer coefficients.

It is easy to check that $a(n) > 0$ for $n \geq 6$. Thus for the positive values $x_{13} = \alpha_{13}, \beta_{13}$ found above we obtain real values $x_2 = \alpha_2, \beta_2$ and $x_{12} = \alpha_{12}, \beta_{12}$ as solutions of the system of equations (7).

- We claim that $\alpha_2, \beta_2, \alpha_{12}, \beta_{12}$ are positive.
  Consider the ideal J generated by $\{f_1,\ f_2,\ f_3,\ z\,x_2\,x_{12}\,x_{13}\,(x_{13}-1)-1\}$ and take a lexicographic order $>$ with $z > x_2 > x_{13} > x_{12}$ for a monomial ordering on R.
  Then a Gröbner basis for the ideal J contains the polynomial $h_2(x_{12})$ of x_{12} of the form

$$h_2(x_{12}) = \sum_{k=0}^{10} b_k(n) {x_{12}}^k$$

  where, for $n \geq 6$, $b_k(n)$ are positive for even k and negative for odd. Thus if the equation $h_2(x_{12}) = 0$ has real solutions, then these are positive.

- For the same ideal J take a lexicographic order $>$ with $z > x_{12} > x_{13} > x_2$ for a monomial ordering on R. Then a Gröbner basis for the ideal J contains the polynomial $h_3(x_2)$ of x_2 of the form

$$h_3(x_2) = \sum_{k=0}^{10} c_k(n){x_2}^k$$

where, for $n \geq 6$, $c_k(n)$ are positive for even k and negative for odd. Thus if the equation $h_3(x_2) = 0$ has real solutions, then these are positive.

We see that, for $n \geq 16$,

$$1 - \frac{2}{n} - \frac{6}{n^2} < \alpha_{13} < 1 - \frac{2}{n} - \frac{7}{2n^2}, \qquad 1 + \frac{50}{63n^2} < \beta_{13} < 1 + \frac{3}{n^2},$$

$$2 - \frac{2}{n} - \frac{6}{n^2} < \alpha_{12} < 2 - \frac{4}{n} - \frac{31}{4n^2}, \qquad \frac{5}{3n} + \frac{815}{162n^2} < \beta_{12} < \frac{5}{3n} + \frac{10}{n^2},$$

$$\frac{1}{2n} + \frac{13}{8n^2} < \alpha_2 < \frac{1}{2n} + \frac{11}{5n^2}, \qquad \frac{5}{9n} + \frac{23}{20n^2} < \beta_2 < \frac{5}{9n} + \frac{10}{n^2}.$$

Jensen's metrics on Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$

$$\langle\ ,\ \rangle = \begin{pmatrix} 0 & a & 1 \\ a & a & 1 \\ 1 & 1 & * \end{pmatrix},\ \mathrm{Ad}(\mathrm{SO}(4)) \times \mathrm{SO}(n-4)\text{-invariant.}$$

Our Einstein metrics

$$\langle\ ,\ \rangle = \begin{pmatrix} 0 & \beta & \gamma \\ \beta & \alpha & 1 \\ \gamma & 1 & * \end{pmatrix},\ \mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(n-4))\text{-invariant}$$

($\alpha, \beta, \gamma \neq 1$ are all different).

For the Stiefel manifolds $V_\ell\mathbb{R}^{k+k+\ell} = \mathrm{SO}(2k+\ell)/\mathrm{SO}(\ell)$ ($\ell > k \geq 3$)
Einstein metrics of Arvanitoyeorgos, Dzhepko and Nikonorov

$$\langle\ ,\ \rangle = \begin{pmatrix} \alpha & \beta & 1 \\ \beta & \alpha & 1 \\ 1 & 1 & * \end{pmatrix} \quad (\ \alpha, \beta \text{ are different }).$$

For general Stiefel manifolds $V_{1+k_2}\mathbb{R}^{1+k_2+k_3}$ ($k_2 \geq 3$ and $k_3 \geq 2$), the system of equations (6) reduces to the system of polynomial equations of three variables $\{x_2, x_{12}, x_{13}\}$ similar to $\{f_1 = 0, f_2 = 0, f_3 = 0\}$.

Again we compute a Gröbner basis and we obtain a polynomial $q_1(x_{13})$ of x_{13} which is a product of a polynomial $p_1(x_{13})$ of degree 10 with parameter k_2 and k_3 (similar to $h_1(x_{13})$) and $(x_{13} - 1)$.

Then we can show that at least two positive solutions for $p_1(x_{13}) = 0$ and also we can see that there exist at least two positive solutions of Einstein equations (6).

For $x_{13} = 1$ we obtain Jensen's Einstein metrics.

- For the tangent space $\mathfrak{so}(k_1 + k_2 + k_3)$ of the Lie group $G = \mathrm{SO}(k_1 + k_2 + k_3)$, we consider the decomposition

$$\mathfrak{so}(k_1 + k_2 + k_3) = \mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2) \oplus \mathfrak{so}(k_3) \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}, \quad (8)$$

where corresponding $\mathrm{Ad}(K)$-submodules are non-equivalent. By taking into account the diffeomorphism

$$G/\{e\} \cong (G \times \mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3))/\mathrm{diag}(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3)),$$

left-invariant metrics on G which are $\mathrm{Ad}(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3))$-invariant are given by

$$\begin{aligned} \langle\ ,\ \rangle \ = \ & x_1\,(-B)|_{\mathfrak{so}(k_1)} + x_2\,(-B)|_{\mathfrak{so}(k_2)} + x_3\,(-B)|_{\mathfrak{so}(k_3)} \\ & + x_{12}\,(-B)|_{\mathfrak{m}_{12}} + x_{13}\,(-B)|_{\mathfrak{m}_{13}} + x_{23}\,(-B)|_{\mathfrak{m}_{23}} \end{aligned} \quad (9)$$

for $k_1 \geq 2$, $k_2 \geq 2$ and $k_3 \geq 2$.

Lemma

The components of the Ricci tensor r for the left-invariant metric $\langle\ ,\ \rangle$ on G defined by (9), are given as follows

$$
\begin{aligned}
r_1 &= \frac{k_1 - 2}{4(n-2)x_1} + \frac{1}{4(n-2)}\left(k_2\frac{x_1}{{x_{12}}^2} + k_3\frac{x_1}{{x_{13}}^2}\right),\\
r_2 &= \frac{k_2 - 2}{4(n-2)x_2} + \frac{1}{4(n-2)}\left(k_1\frac{x_2}{{x_{12}}^2} + k_3\frac{x_2}{{x_{23}}^2}\right),\\
r_3 &= \frac{k_3 - 2}{4(n-2)x_3} + \frac{1}{4(n-2)}\left(k_1\frac{x_3}{{x_{13}}^2} + k_2\frac{x_3}{{x_{23}}^2}\right),
\end{aligned}
$$

Lemma

$$
\begin{aligned}
r_{12} &= \frac{1}{2x_{12}} + \frac{k_3}{4(n-2)}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) \\
&\quad - \frac{1}{4(n-2)}\left((k_1-1)\frac{x_1}{{x_{12}}^2} + (k_2-1)\frac{x_2}{{x_{12}}^2}\right), \\
r_{13} &= \frac{1}{2x_{13}} + \frac{k_2}{4(n-2)}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) \\
&\quad - \frac{1}{4(n-2)}\left((k_1-1)\frac{x_1}{{x_{13}}^2} + (k_3-1)\frac{x_3}{{x_{13}}^2}\right), \\
r_{23} &= \frac{1}{2x_{23}} + \frac{k_1}{4(n-2)}\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right) \\
&\quad - \frac{1}{4(n-2)}\left((k_2-1)\frac{x_2}{{x_{23}}^2} + (k_3-1)\frac{x_3}{{x_{23}}^2}\right).
\end{aligned}
$$

We consider the system of equations

$$r_1 = r_2, \quad r_2 = r_3, \quad r_3 = r_{12}, \quad r_{12} = r_{13}, \quad r_{13} = r_{23}. \tag{10}$$

Then finding Einstein metrics of the form (9) reduces to finding positive solutions of system (10).
We put $k_1 = k_2 = 3$ and consider our equations by putting

$$x_{13} = x_{23} = 1, \quad x_2 = x_1.$$

Then the system of equations (10) reduces to the system of equations:

$$\left.\begin{array}{rcl} g_1 & = & n{x_1}^2{x_{12}}^2x_3 - nx_1{x_{12}}^2 - 6{x_1}^2{x_{12}}^2x_3 + 3{x_1}^2x_3 \\ & & -6x_1{x_{12}}^2{x_3}^2 + 8x_1{x_{12}}^2 + {x_{12}}^2x_3 = 0, \\ g_2 & = & -nx_12^3x_3 + n{x_{12}}^2 + 4x_1x_3 + 6{x_{12}}^3x_3 \\ & & +6{x_{12}}^2{x_3}^2 - 8{x_{12}}^2 - 8x_{12}x_3 = 0, \\ g_3 & = & nx_12^3 + n{x_{12}}^2x_3 - 2n{x_{12}}^2 + 2x_1{x_{12}}^2 - 4x_1 \\ & & -3{x_{12}}^3 - 7{x_{12}}^2x_3 + 4{x_{12}}^2 + 8x_{12} = 0. \end{array}\right\} \tag{11}$$

By computing a Gröbner basis for $\{g_1, g_2, g_3\}$, we see that there exist at least two positive solutions of the system for $n \geq 10$ of the form

$$\langle\ ,\ \rangle = \begin{pmatrix} \alpha & \beta & 1 \\ \beta & \alpha & 1 \\ 1 & 1 & \gamma \end{pmatrix} \quad (\alpha, \beta \text{ are different, } \beta \neq 1).$$

We can see that these metrics are not naturally reductive.

- For $G = \mathrm{SO}(7), \mathrm{SO}(8), \mathrm{SO}(9)$, we consider separately and we see that $\mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(3))$-invariant Einstein metrics on $\mathrm{SO}(7)$, $\mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathrm{SO}(2))$-invariant Einstein metrics on $\mathrm{SO}(8)$ and $\mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathrm{SO}(3))$-invariant Einstein metrics on $\mathrm{SO}(9)$ which are not naturally reductive.

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