Recent progress of homogeneous Einstein metrics on generalized flag manifolds

Yusuke Sakane

Kawanishi

March 6, 2013

outline

Homogeneous Einstein metrics on generalized flag manifolds

based on joint works with
A. Arvanitoyeorgos and I. Chrysikos

- introduction
- generalized flag manifolds
- Ricci tensor of a compact homogeneous space
- structures of generalized flag manifolds
- $\mathfrak{t}$-roots of generalized flag manifolds
- decomposition associated to generalized flag manifolds ($\mathfrak{t}$-roots and decompositions)
- invariant Einstein metrics on a generalized flag manifold

Introduction

(M, g): Riemannian manifold

- (M, g) is called Einstein if the Ricci tensor $\mathrm{r}(g)$ of the metirc g satisfies $\mathrm{r}(g) = cg$ for some constant c.

  We consider G-invariant Einstein metrics on a homogeneous space G/K.
- General Problem: Find G-invariant Einstein metrics on a homogeneous space G/K and classify them if it is not unique.
- Einstein homogeneous spaces can be diveded into three cases depending on Einstein constant c.
  Here we consider the case $c > 0$.

Introduction

1. Examples of the case $c > 0$
(G/K is compact and $\pi_1(G/K)$ is finite).

- Sphere $(S^n = SO(n+1)/SO(n), g_0)$,
  Complex Projective space $(\mathbb{C}P^n = SU(n+1)/(S(U(1)\times U(n)))$,
  Symmetric spaces of compact type,
  isotropy irreducible spaces (in these cases G-invariant Einstein metrics is unique)
- Compact semi-simple Lie groups (bi-invariant metric (negative of Killing form))
- Generalized flag manifolds (Kähler C-spaces) (if we fix a complex structure, it admits a unique Kähler-Einstein metric, but complex structure may not be unique)

Introduction

- (Wang-Ziller [17] 1986) There exist compact homogeneous space G/K with no G-invariant Einstein metrics.

  Example. Let $G = SU(4)$, $L = Sp(2)$, $K = SU(2)$ ($SU(2)$ is a maxmal subgroup of $Sp(2)$). Then G/K has no (G-)invariant Einstein metrics. Note that $\dim G/K = 12$.
- (Böhm-Kerr (2006)) For a simply connected compact homogeneous space G/K of $\dim G/K \leq 11$, there exists at least one G-invariant Einstein metric on G/K.

Introduction

- Problem: Find all G-invariant Einstein metrics on a compact homogeneous space G/K.
- (Nikonorov, Rodionov (2003)) For a simply connected compact homogeneous space G/K of $\dim G/K \leq 7$, all G-invariant Einstein metrics has been determined on G/K, except for $SU(2) \times SU(2)$.
- For $SU(2) \times SU(2)$, there exist at least two left-invariant Einstein metrics. The first is the standard metric, and the other was found by Jensen.
- In 2003 Nikonorov and Rodionov computed the scalar curvature of left-invariant metrics on $SU(2) \times SU(2)$, but these depend on 14 parameters and it is difficult to find critical points (Einstein metrics).

Introduction

- Open problem : How many left-invariant Einstein metrics are there on compact simple Lie groups G ($\dim G \geq 4$) ?
  (finite or infinite?)
- (Wang-Ziller (1990))
  The principal S^1-bundles over $\mathbb{C}P^1 \times \mathbb{C}P^1$ are all diffeomorphic to $S^2 \times S^3$, but as homogeneous spaces $(SU(2) \times SU(2))/S^1$ they are quite different. There are infinitely many ways to embed the group S^1 in $SU(2) \times SU(2)$. On $S^2 \times S^3$ the moduli space of Einstein metrics has infinitely many components.

Generalized flag manifolds

- A generalized flag manifold M is an adjoint orbit of a compact connected semi-simple Lie group G, and is a homogeneous space of the form $M = G/C(S)$, where $C(S)$ is the centralizer of a torus S in G.
- Generalized flag manifolds exhaust compact simply connected homogeneous Kähler manifolds.
- A generalized flag manifold admits a finite number of G-invariant complex structures. For each G-invariant complex structure there is a compatible Kähler-Einstein metric.
- Generalized flag manifolds can be classified by use of painted Dynkin diagrams.
- Generalized flag manifolds are also referred to as Kähler C-spaces.

Examples of Generalized flag manifolds

- Set $G = SU(n+1)$, $K = S(U(n) \times U(1))$. Then G/K is a complex projective space $\mathbb{C}P^n$.
- Set $G = SU(n+m)$, $K = S(U(n) \times U(m))$. Then G/K is a Grassmann manifold $G_{m+n,n}(\mathbb{C})$.
- Set $G = SU(n+m+\ell)$, $K = S(U(n) \times U(m) \times U(\ell))$. Then G/K is a generalized flag manifold.
- Set $G = Sp(n+1)$, $K = Sp(n) \times U(1)$. Then G/K is a a complex projective space $\mathbb{C}P^{2n-1}$.

Ricci tensor of a compact homogeneous space G/K

- Let G be a compact semi-simple Lie group and K a connected closed subgroup of G.
  Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B (= − Killing form of $\mathfrak{g}$). Then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, $[\,\mathfrak{k},\,\mathfrak{m}\,] \subset \mathfrak{m}$ and a decomposition of $\mathfrak{m}$ into irreducible Ad(K)-modules:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q.$$

- We assume that Ad(K)-modules $\mathfrak{m}_j$ $(j = 1, \cdots, q)$ are mutually non-equivalent.
  Then a G-invariant metric on G/K can be written as

$$<\ ,\ >= x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q}, \qquad (1)$$

  for positive real numbers $x_1, \cdots, x_q$.

Ricci tensor of a compact homogeneous space G/K

- Note that G-invariant symmetric covariant 2-tensors on G/K are the same form as the metrics.
  In particular, the Ricci tensor r of a G-invariant Riemannian metric on G/K is of the same form as (1).
- Let $\{e_\alpha\}$ be a B-orthonormal basis adapted to the decomposition of $\mathfrak{m}$, i.e., $e_\alpha \in \mathfrak{m}_i$ for some i, and $\alpha < \beta$ if $i < j$ (with $e_\alpha \in \mathfrak{m}_i$ and $e_\beta \in \mathfrak{m}_j$).
- We put $A^\gamma_{\alpha\beta} = B\left(\left[e_\alpha, e_\beta\right], e_\gamma\right)$, so that $\left[e_\alpha, e_\beta\right] = \sum_\gamma A^\gamma_{\alpha\beta} e_\gamma$, and set $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A^\gamma_{\alpha\beta})^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i,\ e_\beta \in \mathfrak{m}_j,\ e_\gamma \in \mathfrak{m}_k$.
- Notations $\begin{bmatrix} k \\ ij \end{bmatrix}$ are introduced by Wang and Ziller [17].

Ricci tensor of a compact homogeneous space G/K

- Then, the non-negative number $\begin{bmatrix} k \\ ij \end{bmatrix}$ is independent of the B-orthonormal bases chosen for $\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k$, and

$$\begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}. \qquad (2)$$

- Let $d_k = \dim \mathfrak{m}_k$. Then we have (cf. Park - S. [15])

Lemma

The components $r_1, \cdots, r_q$ of Ricci tensor r of the metric $<\ ,\ >= x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q}$ on G/K are given by

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} \begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} \begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = 1, \cdots, q) \qquad (3)$$

where the sum is taken over $i, j = 1, \cdots, q$.

Structures of generalized flag manifolds

- Let G be a compact semi-simple Lie group, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{h}$ a maximal abelian subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$ and $\mathfrak{h}$ respectively.
- We identify an element of the root system Δ of $\mathfrak{g}^{\mathbb{C}}$ relative to the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ with an element of $\mathfrak{h}_0 = \sqrt{-1}\mathfrak{h}$ by the duality defined by the Killing form of $\mathfrak{g}^{\mathbb{C}}$. Let $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ be a fundamental system of Δ and $\{\Lambda_1, \cdots, \Lambda_l\}$ the fundamental weights of $\mathfrak{g}^{\mathbb{C}}$ corresponding to Π, that is

$$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \qquad (1 \le i, j \le \ell).$$

- Let Π_0 be a subset of Π and $\Pi - \Pi_0 = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$ $(1 \le \alpha_{i_1} < \cdots < \alpha_{i_r} \le \ell)$. We put $[\Pi_0] = \Delta \cap \{\Pi_0\}_{\mathbb{Z}}$, where $\{\Pi_0\}_{\mathbb{Z}}$ denotes the subspace of $\mathfrak{h}_0$ generated by Π_0.

Structures of generalized flag manifolds

- Consider the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha\in\Delta} \mathfrak{g}^{\mathbb{C}}_{\alpha}.$$

For a subset Π_0 of Π, we define a parabolic subalgebra $\mathfrak{u}$ of $\mathfrak{g}^{\mathbb{C}}$ by

$$\mathfrak{u} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha\in[\Pi_0]\cup\Delta^+} \mathfrak{g}^{\mathbb{C}}_{\alpha},$$

where Δ^+ is the set of all positive roots relative to Π.
- Note that the nilradical $\mathfrak{n}$ of $\mathfrak{u}$ is given by

$$\mathfrak{n} = \sum_{\alpha\in\Delta^+-[\Pi_0]} \mathfrak{g}^{\mathbb{C}}_{\alpha}.$$

We put $\Delta^+_{\mathfrak{m}} = \Delta^+ - [\Pi_0]$.

Structures of generalized flag manifolds

- Let $G^{\mathbb{C}}$ be a simply connected complex semi-simple Lie group whose Lie algebra is $\mathfrak{g}^{\mathbb{C}}$ and U the parabolic subgroup of $G^{\mathbb{C}}$ generated by $\mathfrak{u}$. Then the complex homogeneous manifold $G^{\mathbb{C}}/U$ is compact simply connected and G acts transitively on $G^{\mathbb{C}}/U$. Note also that $K = G \cap U$ is a connected closed subgroup of G, $G^{\mathbb{C}}/U = G/K$ as C^∞-manifolds, and $G^{\mathbb{C}}/U$ admits a G-invariant Kähler metric.
Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{k}^{\mathbb{C}}$ the complexification of $\mathfrak{k}$. Then we have a direct decomposition

$$\mathfrak{u} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{n}, \qquad \mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha\in[\Pi_0]} \mathfrak{g}^{\mathbb{C}}_{\alpha}.$$

- We put $\mathfrak{t} = \left\{H \in \mathfrak{h}_0 \mid (H, \Pi_0) = (0)\right\}$. Then $\{\Lambda_{i_1}, \cdots, \Lambda_{i_r}\}$ is a basis of $\mathfrak{t}$. Put $\mathfrak{s} = \sqrt{-1}\mathfrak{t}$. Then the Lie algebra $\mathfrak{k}$ is given by $\mathfrak{k} = \mathfrak{z}(\mathfrak{s})$ (the Lie algebra of centralizer of a torus S in G).

$\mathfrak{t}$-roots of generalized flag manifolds

- We consider the restriction map

$$\kappa : \mathfrak{h}_0^* \to \mathfrak{t}^* \quad \alpha \mapsto \alpha|_{\mathfrak{t}}$$

and set $\Delta_{\mathfrak{t}} = \kappa(\Delta)$. The elements of $\Delta_{\mathfrak{t}}$ are called $\mathfrak{t}$-roots.
(The notion of $\mathfrak{t}$-roots is introduced by Alekseevky and Perelomov [2] around 1985 to study invariant Kähler-Einstein metrics of generalized flag manifolds.)
- There exists a 1-1 correspondence between $\mathfrak{t}$-roots ξ and irreducible submodules $\mathfrak{m}_\xi$ of the $\mathrm{Ad}_G(K)$-module $\mathfrak{m}^{\mathbb{C}}$ that is given by

$$\Delta_{\mathfrak{t}} \ni \xi \mapsto \mathfrak{m}_\xi = \sum_{\kappa(\alpha)=\xi} \mathfrak{g}^{\mathbb{C}}_{\alpha}.$$

- Thus we have a decomposition of the $\mathrm{Ad}_G(K)$-module $\mathfrak{m}^{\mathbb{C}}$:

$$\mathfrak{m}^{\mathbb{C}} = \sum_{\xi\in\Delta_{\mathfrak{t}}} \mathfrak{m}_\xi.$$

Decomposition associated to generalized flag manifolds

- Denote by $\Delta_{\mathfrak{t}}^{+}$ the set of all positive $\mathfrak{t}$-roots, that is, the restricton of the system Δ^{+}. Then $\mathfrak{n} = \sum_{\xi \in \Delta_{\mathfrak{t}}^{+}} \mathfrak{m}_{\xi}$.
- Denote by τ the complex conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$ (note that τ interchanges $\mathfrak{g}_{\alpha}^{\mathbb{C}}$ and $\mathfrak{g}_{-\alpha}^{\mathbb{C}}$) and by $\mathfrak{v}^{\tau}$ the set of fixed points of τ in a (complex) vector subspace $\mathfrak{v}$ of $\mathfrak{g}^{\mathbb{C}}$. Thus we have a decomposition of $\mathrm{Ad}_G(K)$-module $\mathfrak{m}$ into irreducible submodules:

$$\mathfrak{m} = \sum_{\xi \in \Delta_{\mathfrak{t}}^{+}} \left(\mathfrak{m}_{\xi} + \mathfrak{m}_{-\xi}\right)^{\tau}.$$

Decomposition associated to generalized flag manifolds

- For integers $j_1, \cdots, j_r$ with $(j_1, \cdots, j_r) \neq (0, \cdots, 0)$, we put $\Delta(j_1, \cdots, j_r) = \left\{ \sum_{j=1}^{\ell} m_j \alpha_j \in \Delta^{+} \;\middle|\; m_{i_1} = j_1, \cdots, m_{i_r} = j_r \right\}$.
  There exists a natural 1-1 correspondence between $\Delta_{\mathfrak{t}}^{+}$ and the set $\{\Delta(j_1, \cdots, j_r) \neq \emptyset\}$
- For a generalized flag manifold G/K, we have a decomposition of $\mathfrak{m}$ into mutually non-equivalent irreducible $\mathrm{Ad}_G(H)$-modules :

$$\mathfrak{m} = \sum_{\xi \in \Delta_{\mathfrak{t}}^{+}} \left(\mathfrak{m}_{\xi} + \mathfrak{m}_{-\xi}\right)^{\tau} = \sum_{j_1, \cdots, j_r} \mathfrak{m}(j_1, \cdots, j_r).$$

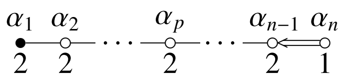
  Thus a G-invariant metric g on G/K can be written as

$$g = \sum_{\xi \in \Delta_{\mathfrak{t}}^{+}} x_{\xi} B|_{(\mathfrak{m}_{\xi} + \mathfrak{m}_{-\xi})^{\tau}} = \sum_{j_1, \cdots, j_r} x_{j_1 \cdots j_r} B|_{\mathfrak{m}(j_1, \cdots, j_r)} \qquad (4)$$

  for positive real numbers x_{ξ}, $x_{j_1 \cdots j_r}$.

$\mathfrak{t}$-roots and decompositions

- From now on we assume that the Lie group G is simple. We denote by q the number of elements of $\Delta_{\mathfrak{t}}^{+}$ for a generalized flag manifold G/K, that is, the number of irreducible components of $\mathrm{Ad}_G(K)$-module $\mathfrak{m}$.
- If $q = 1$, then $\Delta_{\mathfrak{t}}^{+} = \{\xi\}$ and G/K is an irreducible Hermitian symmetric space with the symmetric pair $(\mathfrak{g}, \mathfrak{k})$.
- If $q = 2$, then we see that $r = b_2(G/K) = 1$ and $\mathfrak{m} = \mathfrak{m}(1) \oplus \mathfrak{m}(2)$, that is, $\Delta_{\mathfrak{t}}^{+} = \{\xi, 2\xi\}$. We say this case that $\mathfrak{t}$-roots system is of type $A_1(2)$.
- Example. $\mathbb{C}P^{2n-1} = Sp(n)/(Sp(n-1) \times U(1))$



Ricci tensor for case $q = 2$

- Note that only $\begin{bmatrix} 2 \\ 11 \end{bmatrix}$ is non-zero.
  Put $d_1 = \dim \mathfrak{m}(1)$ and $d_2 = \dim \mathfrak{m}(2)$.
  For a G-invariant metric $< \, , \, >= x_1 \cdot B|_{\mathfrak{m}(1)} + x_2 \cdot B|_{\mathfrak{m}(2)}$, components r_1, r_2 of Ricci tensor r of the metric $< \, , \, >$ are given by

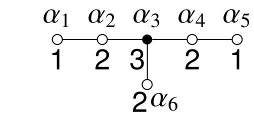
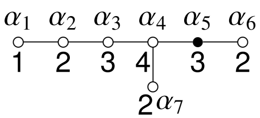
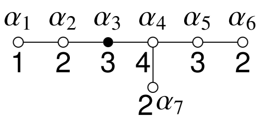
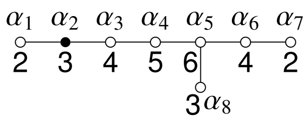
$$\begin{cases} r_1 = \dfrac{1}{2x_1} - \dfrac{x_2}{2\, d_1\, {x_1}^2} \begin{bmatrix} 2 \\ 11 \end{bmatrix} \\[2ex] r_2 = \dfrac{1}{2x_2} - \dfrac{1}{2\, d_2\, x_2} \begin{bmatrix} 1 \\ 21 \end{bmatrix} + \dfrac{x_2}{4\, d_2\, {x_1}^2} \begin{bmatrix} 2 \\ 11 \end{bmatrix}. \end{cases}$$

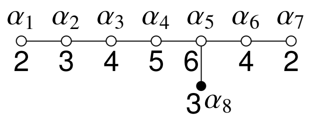
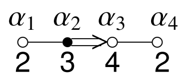
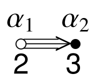
- Note that Kähler-Einstein metric is given by $< \, , \, >= 1 \cdot B|_{\mathfrak{m}(1)} + 2 \cdot B|_{\mathfrak{m}(2)}$ and thus we can determine the value $\begin{bmatrix} 2 \\ 11 \end{bmatrix}$ and find 2 Einstein metrics.

The case $q = 3$

- If $q = 3$, then we see that either $r = b_2(G/K) = 1$ or $r = b_2(G/K) = 2$.
- Einstein metrics of case $q = 3$ was studied by Masahiro Kimura [13] and A. Arvanitoyeorgos [3] independently (around 1990).
- We say the case of $r = b_2(G/K) = 1$ and $q = 3$ that t-roots system is of type $A_1(3)$, that is, $\Delta_t^+ = \{\xi, 2\xi, 3\xi\}$. There are 7 cases and the Lie group G is always exceptional, that is, E_6, E_7, E_8, F_4 and G_2 (for E_7, E_8, there are 2 cases.)
- We say the case of $r = b_2(G/K) = 2$ and $q = 3$ that t-roots system is of type A_2, that is, $\Delta_t^+ = \{\xi_1, \xi_2, \xi_1 + \xi_2\}$. There are 3 cases.

The case $q = 3$ and $b_2(G/K) = 1$

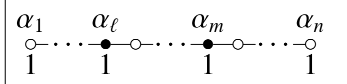
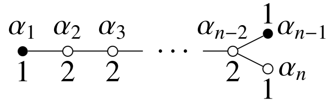
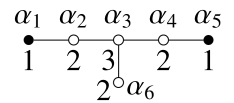
E_6	
E_7	
E_7	
E_8	

E_8	
F_4	
G_2	

Kähler Einstein 1
non-Kähler Einstein 2

- The system of equations $r_1 = r_2 = r_3$ reduces to a polynomial equation of degree 5.

The case $q = 3$ and $b_2(G/K) = 2$

Flag manifold	Painted Dynkin diagram	number of Einstein metrics up to isometry
$SU(n)/S(U(\ell)\times U(m)\times U(k))$ $(n = \ell + m + k)$		Kähler 3*) non-Kähler 1
$SO(2n)/(U(n-1)\times U(1))$		Kähler 2 non-Kähler 1
$E_6/(SO(8)\times U(1)\times U(1))$		Kähler 1 non-Kähler 1 (normal metric)

*) If ℓ, m and k are mutually different, there exist 3 different complex strucutres.

The case $q = 4$ and $b_2(G/K) = 1$

- The case $q = 4$ has started to study by A. Arvanitoyeorgos and I. Chrysikos around 2009 [4].
  We see that either $r = b_2(G/K) = 1$ or $r = b_2(G/K) = 2$ also occur in this case and we divide into 2 cases.
- We call the case of $r = b_2(G/K) = 1$ that t-roots system is of type $A_1(4)$, that is, $\Delta_t^+ = \{\xi, 2\xi, 3\xi, 4\xi\}$. There are 4 cases and G is always exceptional Lie group.
- We call the case of $r = b_2(G/K) = 2$ that t-roots system is of type B_2, that is, $\Delta_t^+ = \{\xi_1, \xi_2, \xi_1 + \xi_2, \xi_1 + 2\xi_2\}$. There are 6 cases.

The case $q = 4$ and $b_2(G/K) = 1$

Flag manifold	Painted Dynkin diagram	number of Einstein metrics up to isometry
$F_4/$ $(SU(3) \times SU(2) \times U(1))$	$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4$ 2 3 4 2	Kähler 1 non-Kähler 2
$E_7/(SU(4){\times}SU(3)$ ${\times}SU(2){\times}U(1))$	$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6$ 1 2 3 4 3 2 $2\alpha_7$	Kähler 1 non-Kähler 2
$E_8/$ $(SO(10){\times}SU(3){\times}U(1))$	$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7$ 2 3 4 5 6 4 2 $3\alpha_8$	Kähler 1 non-Kähler 2
$E_8/$ $(SU(7){\times}SU(2){\times}U(1))$	$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7$ 2 3 4 5 6 4 2 $3\alpha_8$	Kähler 1 non-Kähler 4

The case $q = 4$ and $b_2(G/K) = 2$ (B_2)

Flag manifold	Painted Dynkin diagram	number of Einstein metrics up to isometry
$SO(2n+1)/$ $(SO(2n-3){\times}U(1){\times}U(1))$	$\alpha_1\ \alpha_2\ \alpha_3\ \cdots\ \alpha_{n-1}\ \alpha_n$ 1 2 2 2 2	Kähler 1 non-Kähler 3
$SO(2n)/$ $(SO(2n-4){\times}U(1){\times}U(1))$	$\alpha_1\ \alpha_2\ \alpha_3\ \cdots\ \alpha_{n-2}\ \alpha_{n-1}\ \alpha_n$ 1 2 2 2 1 1	Kähler 1 non-Kähler 3
$E_6/(SU(5){\times}U(1){\times}U(1))$	$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5$ 1 2 3 2 1 $2\alpha_6$	Kähler 2 non-Kähler 4
$E_7/(SO(10)){\times}U(1){\times}U(1))$	$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6$ 1 2 3 4 3 2 $2\alpha_7$	Kähler 2 non-Kähler 4

The case $q = 4$ B_2

$SO(2n)/(U(p){\times}U(n-p))$	$\alpha_1\ \alpha_2\ \cdots\ \alpha_p\ \cdots\ \alpha_{n-2}\ \alpha_{n-1}\ \alpha_n$ 1 2 2 2 1 1 $(2 \leq p \leq n-2)$	Kähler 2 $(n \neq 2p)$ non-Kähler 2
$Sp(n)/(U(p){\times}U(n-p))$	$\alpha_1\ \alpha_2\ \cdots\ \alpha_p\ \cdots\ \alpha_{n-1}\ \alpha_n$ 2 2 2 2 1 $(1 \leq p \leq n-1)$	Kähler 2 $(n \neq 2p)$ non-Kähler 1

- Einstein metrics for the case of $r = b_2(G/K) = 1$ has been studied by A. Arvanitoyeorgos and I. Chrysikos [4]. Einstein metrics for the case of $r = b_2(G/K) = 2$, that is, $\mathfrak{t}$-roots system is of type B_2, has been studied by A. Arvanitoyeorgos and I. Chrysikos [4] and A. Arvanitoyeorgos, I. Chrysikos and Y. S. [5], [6], [7].

The case $q = 5$

- For the case $q = 5$ we also see that either $r = b_2(G/K) = 1$ or $r = b_2(G/K) = 2$.
- We call the case of $r = b_2(G/K) = 1$ that $\mathfrak{t}$-roots system is of type $A_1(5)$, that is, $\Delta_{\mathfrak{t}}^{+} = \{\xi, 2\xi, 3\xi, 4\xi, 5\xi\}$. There is only one case, $G = E_8$ and $K = SU(4) \times SU(5) \times U(1)$ is the case.
- We call the cases of $r = b_2(G/K) = 2$ that $\mathfrak{t}$-roots system is of "extended" type B_2, that is,
  type A : $\Delta_{\mathfrak{t}}^{+} = \{\xi_1, \xi_2, 2\xi_2, \xi_1 + \xi_2, 2\xi_1 + \xi_2\}$, or
  type B : $\Delta_{\mathfrak{t}}^{+} = \{\xi_1, \xi_2, \xi_1 + \xi_2, 2\xi_1 + \xi_2, 2\xi_1 + 2\xi_2\}$.
  There are 4 cases for each. We can show there is an isometry between homogeneous spaces of type A and of type B.
- Einstein metrics for the case of $r = b_2(G/K) = 1$ is studied by I. Chrysikos and Y. S. [11], and for the case of $r = b_2(G/K) = 2$ is studied by A. Arvanitoyeorgos, I. Chrysikos and Y. S. [10] recently.

The case $q = 6$

- For the case $q = 6$ we also see that either $r = b_2(G/K) = 1$, $r = b_2(G/K) = 2$ or $r = b_2(G/K) = 3$.
- We call the case of $r = b_2(G/K) = 1$ that $\mathfrak{t}$-roots system is of type $A_1(6)$, that is, $\Delta_{\mathfrak{t}}^{+} = \{\xi, 2\xi, 3\xi, 4\xi, 5\xi, 6\xi\}$. There is only one case, $G = E_8$ and $K = SU(5) \times SU(3) \times SU(2) \times U(1)$.
- For $r = b_2(G/K) = 2$, we have 4 cases:
  $\Delta_{\mathfrak{t}}^{+} = \{\xi_1, \xi_2, 2\xi_1, \xi_1 + \xi_2, 2\xi_1 + \xi_2, 2\xi_1 + 2\xi_2\}$, of type BC_2,
  $\Delta_{\mathfrak{t}}^{+} = \{\xi_1, \xi_2, \xi_1 + \xi_2, 2\xi_1 + \xi_2, 3\xi_1 + \xi_2, 3\xi_1 + 2\xi_2\}$, of type G_2,
  $\Delta_{\mathfrak{t}}^{+} = \{\xi_1, \xi_2, \xi_1 + \xi_2, 2\xi_1 + \xi_2, 2\xi_1 + 2\xi_2, 3\xi_1 + 2\xi_2\}$,
  $\Delta_{\mathfrak{t}}^{+} = \{\xi_1, \xi_2, \xi_1 + \xi_2, 2\xi_2, \xi_1 + 2\xi_2, \xi_1 + 3\xi_2\}$,
  $\Delta_{\mathfrak{t}}^{+} = \{\xi_1, \xi_2, \xi_1 + \xi_2, 2\xi_1 + \xi_2, \xi_1 + 2\xi_2, 2\xi_1 + 2\xi_2\}$.
- For $r = b_2(G/K) = 3$, we have only one case of $\mathfrak{t}$-roots system with $q = 6$, that is, of type A_3.

The case of G_2/T

- We first consider the case of full flag manifold G_2/T. Note that the highest root $\widetilde{\alpha}$ of $\mathfrak{g}_2^{\mathbb{C}}$ is given by $\widetilde{\alpha} = 3\alpha_1 + 2\alpha_2$ and G_2/T has a $\mathfrak{t}$-roots system of type G_2.
- Note that G_2/T has only one complex strucure and thus, up to isometry, there exist only one Kähler-Einstein metric. There exits exactly two non-Kähler Einstein metrics up to isometry. These are obtained from solutions of polynomial of degree 14. (A. Arvanitoyeorgos, I. Chrysikos and Y. S. [9])
- There is four other generalized flag manifolds (all exceptional Lie groups, F_4, E_6, E_7, E_8) with $\mathfrak{t}$-roots of type G_2. There are only one Kähler-Einstein metric and 6 non-Kähler Einstein metrics up to isometry. Recently M. Graev [12] has studied also these cases and he obtained one non-Kähler Einstein metric by a different method.

The case of $\mathfrak{t}$-roots of type G_2

Flag manifold	Painted Dynkin diagram	number of Einstein metrics up to isometry
$F_4/(U(3)\times U(1))$	$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4$; 2 3 4 2	Kähler 1 non-Kähler 6
$E_6/(U(3)\times U(3))$	$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5$; 1 2 3 2 1; $2\alpha_6$	Kähler 1 non-Kähler 6
$E_7/(U(6)\times U(1))$	$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6$; 1 2 3 4 3 2; $2\alpha_7$	Kähler 1 non-Kähler 6
$E_8/(E_6\times U(1)\times U(1))$	$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7$; 2 3 4 5 6 4 2; $3\alpha_8$	Kähler 1 non-Kähler 6

The case of flag manifold $SU(4)/T$ and $SU(10)/S(U(1) \times U(2) \times U(3) \times U(4))$

- Note that for these cases $q = 6$ and the system of $\mathfrak{t}$-roots is of type A_3.
- For the case $SU(4)/T$, there is only one complex strucure and thus, up to isometry, there exist only one Kähler-Einstein metric. There exits 3 non-Kähler Einstein metrics up to isometry, one of them is normal. (cf. Sakane [16] Lobachevskii J. Math. 4 (1999))
- For the case $SU(10)/S(U(1) \times U(2) \times U(3) \times U(4))$, There are 12 complex strucure and thus, up to isometry, there exist 12 Kähler-Einstein metrics. There exits 12 non-Kähler Einstein metrics up to isometry. These are obtained from solutions of polynomial of degree 68.

Kähler-Einstein metric of a generalized flag manifold

- Put $Z_{\mathfrak{t}} = \left\{ \Lambda \in \mathfrak{t} \;\middle|\; \dfrac{2(\Lambda,\ \alpha)}{(\alpha,\ \alpha)} \in \mathbb{Z} \text{ for each } \alpha \in \Delta \right\}$.
  Then $Z_{\mathfrak{t}}$ is a lattice of $\mathfrak{t}$ generated by $\{\Lambda_{i_1}, \cdots, \Lambda_{i_r}\}$.
- Put $Z_{\mathfrak{t}}^{+} = \left\{ \lambda \in Z_{\mathfrak{t}} \;\middle|\; (\lambda, \alpha) > 0 \text{ for } \alpha \in \Pi - \Pi_0 \right\}$. Then we have $Z_{\mathfrak{t}}^{+} = \sum_{\alpha \in \Pi - \Pi_0} \mathbb{Z}^{+} \Lambda_{\alpha}$. We define an element $\delta_{\mathfrak{m}} \in \mathfrak{h}_0 = \sqrt{-1}\mathfrak{h}$ by

$$\delta_{\mathfrak{m}} = \frac{1}{2} \sum_{\alpha \in \Delta_{\mathfrak{m}}^{+}} \alpha.$$

  Let $c_1(M)$ be the first Chern class of M. Then $2\delta_{\mathfrak{m}} \in Z_{\mathfrak{t}}^{+}$ corresponds to $c_1(M)$.
- Note that 2^{nd} Betti number $b_2(M)$ of M is given by

$$b_2(M) = \dim \mathfrak{t} = \text{ the cardinality of } \Pi - \Pi_0 = r.$$

Kähler-Einstein metric of a generalized flag manifold

- Put $k_{\alpha} = \dfrac{2(2\delta_{\mathfrak{m}}, \alpha)}{(\alpha, \alpha)}$ for $\alpha \in \Pi - \Pi_0$. Then

$$2\delta_{\mathfrak{m}} = \sum_{\alpha \in \Pi - \Pi_0} k_{\alpha} \Lambda_{\alpha} = k_{\alpha_{i_1}} \Lambda_{\alpha_{i_1}} + \cdots + k_{\alpha_{i_r}} \Lambda_{\alpha_{i_r}}$$

  and each $k_{\alpha_{i_s}}$ is a positive integer.
- The G-invariant metric $g_{2\delta_{\mathfrak{m}}}$ on G/K corresponding to $2\delta_{\mathfrak{m}}$, which is a Kähler-Einstein metric, is given by

$$g_{2\delta_{\mathfrak{m}}} = \sum_{\xi \in \Delta_{\mathfrak{t}}^{+}} (2\delta_{\mathfrak{m}}, \xi) B|_{(\mathfrak{m}_{\xi} + \mathfrak{m}_{-\xi})^{\tau}} = \sum_{j_1, \cdots, j_r} \left(\sum_{\ell=1}^{r} k_{i_\ell} j_\ell \frac{(\alpha_{i_\ell}, \alpha_{i_\ell})}{2} \right) B|_{\mathfrak{m}(j_1, \cdots, j_r)}.$$

Riemannian submersion

- Let G be a compact semi-simple Lie group and K, L two closed subgroups of G with $K \subset L$. Then we have a natural fibration $\pi : G/K \to G/L$ with fiber L/K.
- With respect to B (- Killing form of $\mathfrak{g}$),
  $\mathfrak{p} = \mathfrak{l}^{\perp}$ in $\mathfrak{g}$: the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{g}$,
  $\mathfrak{n} = \mathfrak{k}^{\perp}$ in $\mathfrak{l}$: the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{l}$.
  Then $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{n} \oplus \mathfrak{p}$.
- Denote
  a G-invariant metric $\check{g}$ on G/L defined by an $\mathrm{Ad}_G(L)$-invariant scalar product on $\mathfrak{p}$,
  an L-invariant metric $\hat{g}$ on L/K defined by an $\mathrm{Ad}_L(K)$-invariant scalar product on $\mathfrak{n}$ and
  a G-invariant metric g on G/K defined by the orthogonal direct sum for these scalar products on $\mathfrak{n} \oplus \mathfrak{p}$.

Riemannian submersion

Theorem

The map π is a Riemannian submersion from $(G/K,\ g)$ to $(G/L,\ \check{g})$ with totally geodesic fibers isometric to $(L/K,\ \hat{g})$.

Note that $\mathfrak{n}$ is the vertical subspace of the submersion and $\mathfrak{p}$ is the horizontal subspace.
For a Riemannian submersion, O'Neill [14] has introduced two tensors A and T. In our case we have $T = 0$, because the fibers are totally geodesic. We also have

$$A_X Y = \frac{1}{2}[X,\ Y]_{\mathfrak{n}} \quad \text{for } X, Y \in \mathfrak{p}.$$

Riemannian submersion

Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{p}$ and $\{U_j\}$ an orthonormal basis of $\mathfrak{n}$. We put for $X, Y \in \mathfrak{p}$, $g(A_X,\ A_Y) = \sum_i g(A_X X_i, A_Y X_i)$. Then we have

$$g(A_X,\ A_Y) = \frac{1}{4}\sum_i \check{g}([X,\ X_i]_{\mathfrak{n}},\ [Y,\ X_i]_{\mathfrak{n}}).$$

Let r, $\check{r}$ be the Ricci tensor of the metric g, $\check{g}$ respectively. Then we have

$$r(X,Y) = \check{r}(X,Y) - 2g(A_X, A_Y) \quad \text{for} \quad X, Y \in \mathfrak{p}.$$

Riemannian submersion

- $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_\ell$: a decomposition of $\mathfrak{p}$ into irreducible Ad(L)-modules
  $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_s$: a decomposition of $\mathfrak{n}$ into irreducible Ad(K)-modules
- Note that each irreducible component $\mathfrak{p}_j$ (as Ad(L)-module) can be decomposed into irreducible Ad(K)-modules.
- We consider a G-invariant metric on G/K defined by a Riemannian submersion $\pi : (G/K,\ g) \to (G/L,\ \check{g})$ of the form

$$g = y_1 B|_{\mathfrak{p}_1} + \cdots + y_\ell B|_{\mathfrak{p}_\ell} + z_1 B|_{\mathfrak{n}_1} + \cdots + z_s B|_{\mathfrak{n}_s} \tag{5}$$

  for positive real numbers $y_1, \cdots, y_\ell, z_1, \cdots, z_s$.

Riemannian submersion

We decompose each irreducible component $\mathfrak{p}_j$ into irreducible Ad(K)-modules:

$$\mathfrak{p}_j = \mathfrak{m}_{j,1} \oplus \cdots \oplus \mathfrak{m}_{j,k_j}.$$

As before we assume that Ad(K)-modules $\mathfrak{m}_{j,t}$ $(j = 1, \cdots, \ell,\ t = 1, \cdots, k_j)$ are mutually non-equivalent. Note that the metric of the form (5) can be written as

$$g = y_1 \sum_{t=1}^{k_1} B|_{\mathfrak{m}_{1,t}} + \cdots + y_\ell \sum_{t=1}^{k_\ell} B|_{\mathfrak{m}_{\ell,t}} + z_1 B|_{\mathfrak{n}_1} + \cdots + z_s B|_{\mathfrak{n}_s} \tag{6}$$

and this is a special case of the metric of the form (1).

Riemannian submersion

Lemma

Let $d_{j,t} = \dim \mathfrak{m}_{j,t}$. The components $r_{(j,t)}$ $(j = 1, \cdots, \ell,\ t = 1, \cdots, k_j)$ of Ricci tensor r for the metric (6) on G/K are given by

$$r_{(j,t)} = \check{r}_j - \frac{1}{2d_{j,t}} \sum_i \sum_{j',t'} \frac{z_i}{y_j y_{j'}} \begin{bmatrix} i \\ (j,t)\ (j',t') \end{bmatrix}, \tag{7}$$

where $\check{r}_j$ are the components of Ricci tensor $\check{r}$ for the metric $\check{g}$ on G/L.

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