

1 S-Matrix

$$\phi^{\text{out}} = S^\dagger \phi^{\text{in}} S, \quad S^\dagger S = S S^\dagger = \mathbf{1} \quad (1.1)$$

$\phi^{\text{in}}, \phi^{\text{out}}$; Asymptotic Fields, (describing Free Particles.)
(World of Special Relativity)

$$|\text{out}, \alpha\rangle \equiv S |\text{in}, \alpha\rangle \quad (1.2)$$

$$S_{\alpha\beta} = \langle \text{in}, \alpha | \text{out}, \beta \rangle = \langle \text{in}, \alpha | S | \text{in}, \beta \rangle \quad (1.3)$$

$\{ | \text{in}, \alpha \rangle \} \Leftrightarrow$ Fock Space

(Note: $c = \hbar = 1$, Pauli metric; e.g. $T_{\dots 4 \dots} = iT_{\dots 0 \dots}$)

2 Massless Particles ($\mathbf{k}^2 = k_0^2$)

One-Body Amplitude $\phi_S^{(\pm)}(\mathbf{k}) \in$ Unitary Irrep. of Poincaré Group

$$S = 0, \pm 1/2, \pm 1, \pm 3/2, \dots$$

$$S \geq 0 \sim \begin{array}{l} \text{right} \\ \text{left} \end{array} \text{ polarization}$$

$$(\pm) \sim k_0 \geq 0 \sim \begin{array}{l} \text{positive} \\ \text{negative} \end{array} \text{ frequency}$$

i.e.,

$$k_0 \phi_S^{(\pm)}(\mathbf{k}) = \pm |\mathbf{k}| \phi_S^{(\pm)}(\mathbf{k}) \quad (2.1)$$

1. Displacement: (b_μ)

$$x'_\mu \rightarrow x_\mu - b_\mu$$

$$\phi_S'^{(\pm)}(\mathbf{k}) = e^{i(\mathbf{k}b \mp |\mathbf{k}|b_0)} \phi_S^{(\pm)}(\mathbf{k}) \quad (2.2)$$

2. (Infinitesimal) Space Rotation: ($\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$)

$$\mathbf{x}' = \mathbf{x} + \mathbf{x} \times \boldsymbol{\theta}, \quad x'_0 = x_0$$

$$\phi_S'^{(\pm)}(\mathbf{k}) = (1 + i\mathbf{J} \cdot \boldsymbol{\theta}) \phi_S^{(\pm)}(\mathbf{k}) \quad (2.3)$$

where

$$\left\{ \begin{array}{l} J_1 = \frac{1}{i} \left(\mathbf{k} \times \frac{\partial}{\partial \mathbf{k}} \right)_1 + \frac{k_1}{|\mathbf{k}| + k_3} S \\ J_2 = \frac{1}{i} \left(\mathbf{k} \times \frac{\partial}{\partial \mathbf{k}} \right)_2 + \frac{k_2}{|\mathbf{k}| + k_3} S \\ J_3 = \frac{1}{i} \left(\mathbf{k} \times \frac{\partial}{\partial \mathbf{k}} \right)_3 + S \end{array} \right.$$

Note:

$$\mathbf{J} \cdot \frac{\mathbf{k}}{|\mathbf{k}|} = S \quad (2.4)$$

3. (Infinitesimal) Lorentz Boost: ($\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$)

$$\mathbf{x}' = \mathbf{x} - \boldsymbol{\tau} x_0, \quad x'_0 = x_0 - \mathbf{x} \cdot \boldsymbol{\tau}$$

$$\phi_S'^{(\pm)}(\mathbf{k}) = (1 + i\mathbf{K} \cdot \boldsymbol{\tau}) \phi_S^{(\pm)}(\mathbf{k}) \quad (2.5)$$

$$\left\{ \begin{array}{l} K_1 = -k_0 \left(\frac{1}{i} \frac{\partial}{\partial k_1} - \frac{k_2}{|\mathbf{k}|(|\mathbf{k}| + k_3)} S \right) \\ K_2 = -k_0 \left(\frac{1}{i} \frac{\partial}{\partial k_2} + \frac{k_1}{|\mathbf{k}|(|\mathbf{k}| + k_3)} S \right) \\ K_3 = -\frac{1}{i} k_0 \frac{\partial}{\partial k_3} \end{array} \right.$$

3 Covariant Amplitudes for $n \equiv 2|S| \geq 1$

Case for $|S| = 1/2$

$$\varphi^{R(+)}(\mathbf{k}) = \begin{pmatrix} \phi_{1/2}^{(+)}(\mathbf{k}) \\ 0 \end{pmatrix}, \quad \varphi^{R(-)}(\mathbf{k}) = \begin{pmatrix} 0 \\ \phi_{-1/2}^{(-)}(\mathbf{k}) \end{pmatrix}$$

$$\varphi^{L(+)}(\mathbf{k}) = \begin{pmatrix} 0 \\ \phi_{-1/2}^{(+)}(\mathbf{k}) \end{pmatrix}, \quad \varphi^{L(-)}(\mathbf{k}) = \begin{pmatrix} \phi_{1/2}^{(-)}(\mathbf{k}) \\ 0 \end{pmatrix}$$

$$\varphi^R(\mathbf{k}) = \varphi^{R(+)}(\mathbf{k}) + \varphi^{R(-)}(\mathbf{k})$$

$$\varphi^L(\mathbf{k}) = \varphi^{L(+)}(\mathbf{k}) + \varphi^{L(-)}(\mathbf{k})$$

$$k_0 \varphi^R(\mathbf{k}) = |\mathbf{k}| \sigma_3 \varphi^R(\mathbf{k}), \quad k_0 \varphi^L(\mathbf{k}) = -|\mathbf{k}| \sigma_3 \varphi^L(\mathbf{k}) \quad (3.1)$$

Using the unitary matrix

$$U(\mathbf{k}) = \frac{1}{\sqrt{2|\mathbf{k}|(|\mathbf{k}| + k_3)}} \{|\mathbf{k}| + k_3 + i(\sigma_1 k_2 - \sigma_2 k_1)\} \quad (3.2)$$

we define

$$\psi^{R/L}(\mathbf{k}) = \sqrt{|\mathbf{k}|} U(\mathbf{k}) \varphi^{R/L}(\mathbf{k}). \quad (3.3)$$

Then

$$k_0 \psi^{R/L}(\mathbf{k}) = (+/-) \mathbf{k} \boldsymbol{\sigma} \psi^{R/L}(\mathbf{k}). \quad (3.4)$$

Furthermore

1. Displacement:

$$\psi'^{R/L}(\mathbf{k}) = e^{ik_\mu b_\mu} \psi^{R/L}(\mathbf{k}) \quad (3.5)$$

2. Space Rotation:

$$\psi'^{R/L}(\mathbf{k}) = \left(1 + \frac{i}{2} \boldsymbol{\theta} \cdot \boldsymbol{\sigma}\right) \psi^{R/L}(\mathbf{k} - \mathbf{k} \times \boldsymbol{\theta}) \quad (3.6)$$

3. Lorentz Boost:

$$\psi'^{R/L}(\mathbf{k}) = \left(1 + (-/+)\frac{1}{2} \boldsymbol{\tau} \cdot \boldsymbol{\sigma}\right) \psi^{R/L}(\mathbf{k} + \boldsymbol{\tau} k_0) \quad (3.7)$$

Case for $|\mathcal{S}| = n/2$

$$\rho_1, \rho_2, \dots, \rho_n = 1, 2$$

$$\varphi_{(\rho_1 \dots \rho_n)}^{R(+)}(\mathbf{k}) = \begin{cases} \phi_{n/2}^{(+)}(\mathbf{k}) & (\rho_1 = \rho_2 = \dots = \rho_n = 1) \\ 0 & (\text{otherwise}) \end{cases} \quad (3.8)$$

$$\varphi_{(\rho_1 \dots \rho_n)}^{R(-)}(\mathbf{k}) = \begin{cases} \phi_{-n/2}^{(-)}(\mathbf{k}) & (\rho_1 = \rho_2 = \dots = \rho_n = 2) \\ 0 & (\text{otherwise}) \end{cases} \quad (3.9)$$

$$\varphi_{(\rho_1 \dots \rho_n)}^{L(+)}(\mathbf{k}) = \begin{cases} \phi_{-n/2}^{(+)}(\mathbf{k}) & (\rho_1 = \rho_2 = \dots = \rho_n = 2) \\ 0 & (\text{otherwise}) \end{cases} \quad (3.10)$$

$$\varphi_{(\rho_1 \dots \rho_n)}^{L(-)}(\mathbf{k}) = \begin{cases} \phi_{n/2}^{(-)}(\mathbf{k}) & (\rho_1 = \rho_2 = \dots = \rho_n = 1) \\ 0 & (\text{otherwise}) \end{cases} \quad (3.11)$$

$$\varphi_{(\dots)_n}^{R/L}(\mathbf{k}) = \varphi_{(\dots)_n}^{R/L(+)}(\mathbf{k}) + \varphi_{(\dots)_n}^{R/L(-)}(\mathbf{k}) \quad (3.12)$$

\Updownarrow

$$k_0 \varphi_{(\dots)_n}^{R/L}(\mathbf{k}) = (+/-) |\mathbf{k}| \sigma_3^{(i)} \varphi_{(\dots)_n}^{R/L}(\mathbf{k}), \quad (i = 1, 2, \dots, n) \quad (3.13)$$

Then for

$$\psi_{(\dots)_n}^{R/L}(\mathbf{k}) = |\mathbf{k}|^{n/2} \prod_{i=1}^n U^{(i)}(\mathbf{k}) \varphi_{(\dots)_n}^{R/L}(\mathbf{k}) \quad (3.14)$$

we obtain

$$k_0 \psi_{(\dots)_n}^{R/L}(\mathbf{k}) = (+/-) \mathbf{k} \boldsymbol{\sigma}^{(i)} \psi_{(\dots)_n}^{R/L}(\mathbf{k}), \quad (i = 1, 2, \dots, n) \quad (3.15)$$

together with the followings:

1. Displacement

$$\psi'_{(\dots)_n}{}^{R/L}(\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{b}} \psi_{(\dots)_n}^{R/L}(\mathbf{k}) \quad (3.16)$$

2. Space Rotation

$$\psi'_{(\dots)_n}{}^{R/L}(\mathbf{k}) = \left(1 + \frac{i}{2} \boldsymbol{\theta} \cdot \sum_{i=1}^n \boldsymbol{\sigma}^{(i)}\right) \psi_{(\dots)_n}^{R/L}(\mathbf{k} - \mathbf{k} \times \boldsymbol{\theta}) \quad (3.17)$$

3. Lorentz Boost

$$\psi'_{(\dots)_n}{}^{R/L}(\mathbf{k}) = \left(1 + (-/+)\frac{1}{2} \boldsymbol{\tau} \cdot \sum_{i=1}^n \boldsymbol{\sigma}^{(i)}\right) \psi_{(\dots)_n}^{R/L}(\mathbf{k} + \boldsymbol{\tau} k_0) \quad (3.18)$$

x -Representation:

$$\psi_{(\dots)_n}^{R/L}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{2}|\mathbf{k}|} e^{i\mathbf{k}_\mu x_\mu} \psi_{(\dots)_n}^{R/L}(\mathbf{k}) S \quad (3.19)$$

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Equations of Motion:

$$\left(\frac{\partial}{\partial t} + (+/-)\boldsymbol{\sigma}^{(i)} \boldsymbol{\nabla} \right) \psi_{(\dots)_n}^{R/L}(x) = 0, \quad (i = 1, 2, \dots, n), \quad (3.20)$$

or equivalently

$$\gamma_\mu^{(i)} \partial_\mu \left[\prod_{j=1}^n \frac{1}{2} (1 + (-/+) \gamma_5^{(j)}) \right] \psi_{(\dots)_n}^{R/L}(x) = 0, \quad (i = 1, 2, \dots, n)$$

where

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \gamma_5 \equiv \gamma_1 \gamma_2 \gamma_3 \gamma_4.$$

Note:

$$\psi_{(\dots)_n}^{R/L}(x) \iff \psi_{(\dots)_n}^{R/L}(\mathbf{k}) \iff \varphi_{(\dots)_n}^{R/L}(\mathbf{k}) \iff \phi_{n/2}^{(\pm)}(\mathbf{k}), \phi_{-n/2}^{(\pm)}(\mathbf{k}) \quad (3.21)$$

4 Quantized Fields

$$\phi_{n/2}^{(\pm)}(\mathbf{k}), \phi_{-n/2}^{(\pm)}(\mathbf{k}) \implies \hat{\phi}_{n/2}^{(\pm)}(\mathbf{k}), \hat{\phi}_{-n/2}^{(\pm)}(\mathbf{k}) \quad (4.1)$$

Poincaré Invariant Nonvanishing Commutators:

$$\begin{aligned} [\hat{\phi}_{\pm n/2}^{(+)}(\mathbf{k}), \hat{\phi}_{\pm n/2}^{(+)\dagger}(\mathbf{k}')]_{\epsilon_n} &= [\hat{\phi}_{\pm n/2}^{(-)\dagger}(\mathbf{k}), \hat{\phi}_{\pm n/2}^{(-)}(\mathbf{k}')]_{\epsilon_n} = |\mathbf{k}| \delta^3(\mathbf{k} - \mathbf{k}'), \\ \epsilon &= (-)^{n+1} \end{aligned} \quad (4.2)$$

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$$\left\{ \begin{aligned} &[\hat{\psi}_{(\rho_1 \rho_2 \dots \rho_n)}^R(x), \hat{\psi}_{(\rho'_1 \rho'_2 \dots \rho'_n)}^R(x')]_{\epsilon_n} = 0, \\ &[\hat{\psi}_{(\rho_1 \rho_2 \dots \rho_n)}^R(x), \hat{\psi}_{(\rho'_1 \rho'_2 \dots \rho'_n)}^{R\dagger}(x')]_{\epsilon_n} \\ &= \frac{i^{n+1}}{n!} \sum_{\tau \in S_n} \prod_{j=1}^n [\partial_t - (\boldsymbol{\sigma}^{(j)} \boldsymbol{\nabla})]_{\rho_j \rho'_{\tau j}} D(x - x') \end{aligned} \right. \quad (4.3)$$

and

$$\left\{ \begin{aligned} &[\hat{\psi}_{(\rho_1 \rho_2 \dots \rho_n)}^L(x), \hat{\psi}_{(\rho'_1 \rho'_2 \dots \rho'_n)}^L(x')]_{\epsilon_n} = 0, \\ &[\hat{\psi}_{(\rho_1 \rho_2 \dots \rho_n)}^L(x), \hat{\psi}_{(\rho'_1 \rho'_2 \dots \rho'_n)}^{L\dagger}(x')]_{\epsilon_n} \\ &= \frac{i^{n+1}}{n!} \sum_{\tau \in S_n} \prod_{j=1}^n [\partial_t + (\boldsymbol{\sigma}^{(j)} \boldsymbol{\nabla})]_{\rho_j \rho'_{\tau j}} D(x - x') \end{aligned} \right. \quad (4.4)$$

where

$$D(x) = -\frac{i}{(2\pi)^3} \int d^4k e^{ikx} \varepsilon(k_0) \delta(\mathbf{k}^2 - k_0^2). \quad (4.5)$$

(vanishing for space-like x)

5 Local Observables:

$$\hat{Q} = \int d^3\mathbf{x} \hat{Q}(x) + \text{const} \quad (5.1)$$

$$[\hat{Q}(x), \hat{Q}(x')] = 0 \quad (\text{for spacelike } (x - x')) \quad (5.2)$$

Especially, for massless free particles

$$\hat{Q}(x); \begin{cases} \text{gauge-independent} \\ \text{bilinear in } \partial^\eta \hat{\psi}_{(\dots)_n}^{R/L}(x) \text{ and } \partial^{\eta'} \hat{\psi}_{(\dots)_n}^{R/L\dagger}(x) \end{cases} \quad (5.3)$$

Hence

$$\hat{Q}(x) \sim \sum_{(\rho)_n, (\rho')_n, \eta+\eta'=N} c_{(\rho)_n, (\rho')_n}^{\eta\eta'} \partial^\eta \hat{\psi}_{(\rho)_n}^{R/L\dagger}(x) \partial^{\eta'} \hat{\psi}_{(\rho')_n}^{R/L}(x), \quad (5.4)$$

where

$$\begin{aligned} c_{(\rho)_n, (\rho')_n}^{\eta\eta'} &: \text{dimensionless consts.}, \\ (\rho)_n &= (\rho_1, \rho_2, \dots, \rho_n), \\ (\rho')_n &= (\rho'_1, \rho'_2, \dots, \rho'_n). \end{aligned}$$

Note:

$$[D(x)] = \text{L}^{-2} \quad \Rightarrow \quad [\hat{\psi}_{(\rho)_n}^{R/L}(x)] = \text{L}^{-(n/2+1)} \quad (5.5)$$

Thus

$$[\hat{Q}(x)] = \text{L}^{-(n+2+N)} = \text{L}^{-(2|S|+2+N)} \quad (5.6)$$

1. Energy-Momentum Density $\hat{P}_\mu^S(x)$

$$\begin{aligned}\hat{P}_\mu &= \int d^3\mathbf{x} \hat{P}_\mu^S(x) \\ &= \int \frac{d^3\mathbf{k}}{|\mathbf{k}|} k_\mu (\hat{\phi}_S^{(+)\dagger}(\mathbf{k})\hat{\phi}_S^{(+)}(\mathbf{k}) - \hat{\phi}_S^{(-)}(\mathbf{k})\hat{\phi}_S^{(-)\dagger}(\mathbf{k})),\end{aligned}\quad (5.7)$$

$$[\hat{P}_\mu^S(x)] = L^{-4} \quad \Rightarrow \quad |S| = 1 - \frac{N}{2}.\quad (5.8)$$

Cases $|S| \geq 3/2$ are not allowed.

2. Noether Charge Density $\hat{\rho}^S(x)$

Similarly,

$$\begin{aligned}\hat{\rho}^S &= \int d^3\mathbf{x} \hat{\rho}^S(x) \\ &= \int \frac{d^3\mathbf{k}}{|\mathbf{k}|} (\hat{\phi}_S^{(+)\dagger}(\mathbf{k})\hat{\phi}_S^{(+)}(\mathbf{k}) - \hat{\phi}_S^{(-)}(\mathbf{k})\hat{\phi}_S^{(-)\dagger}(\mathbf{k})).\end{aligned}\quad (5.9)$$

$$[\hat{\rho}^S(x)] = L^{-3} \quad \Rightarrow \quad |S| = \frac{(1-N)}{2}.\quad (5.10)$$

Cases $|S| \geq 1$ are not allowed.

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GRAVITON (neutral, massless, spin 2): ~ NON-EXISTENT ?!

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Then what happens !

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A plausible interpretation would be
CONFINEMENT OF GRAVITONS