佐々木多様体のチャーン・サイモンズ型不変量の局所化について

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Introduction

(M, g) : a (connected compact) Riemannian manifold.
η : a contact form on M.

(M, g, η) is a Sasakian manifold iff
$(M \times \mathbb{R}_{>0}, r^2 g + dr \otimes dr, d(r^2\eta))$ is a Kähler manifold.

Definition

The Reeb vector field ξ of η is defined by $\iota_\xi d\eta = 0$ and $\eta(\xi) = 1$.

The Reeb vector field of a Sasakian manifold is Killing, i.e., $L_\xi g = 0$. The flow generated by ξ is called the Reeb flow. In this talk, we will uss its orbit foliation.

Example

- S^{2n+1},
- certain S^1-bundles over Kähler manifolds,
- contact toric manifolds,
- the link of certain isolated singularities of complex varieties.

Introduction

$S^{2n+1} \subset \mathbb{R}^{2n+2}$: the unit sphere
$w = (w_0, \dots, w_n) \in (\mathbb{R}_{>0})^{n+1}$

Consider

$$\eta_w = \frac{\sum_{i=0}^n (y_i dx_i - x_i dy_i)}{\sum_{i=0}^n w_i (x_i^2 + y_i^2)} \in \Omega^1(S^{2n+1}).$$

Here S^{2n+1} admits a Sasakian structure (η_w, g_w), where the metric g_w is determined by η_w and the standard CR structure on S^{2n+1}.

The Reeb vector field ξ_w of η_w is

$$\xi_w = \sum_{i=0}^n w_i \left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right).$$

Geometry of Sasaki(-Einstein) manifolds have been studied with motivation in

- AdS_5/CFT_4 correspondence and
- Construction of Einstein metrics.

Several conjectures by physists remain open.

c.f. D. Martelli, J. Sparks and S.-T. Yau, *Sasaki-Einstein manifolds and volume minimisation*, Comm. Math. Phys. **280** (2008), no. 3, 611–673.

Martelli-Sparks-Yau's theorems

M^{2n+1} : a closed manifold,
$\mathcal{S}$: the space of Sasakian metrics on M.

$$\begin{array}{rccc} \mathrm{Vol}: & \mathcal{S} & \longrightarrow & \mathbb{R} \\ & g & \longmapsto & \mathrm{Vol}(M,g) \end{array}$$

It is easy to see that $\mathrm{Vol}(M,g) = \frac{1}{2^n n!} \int_M \eta \wedge (d\eta)^n$.

Proposition (Martelli-Sparks-Yau)

Vol *is equal to the Einstein-Hilbert action up to a constant on* $\mathcal{S}$*. In particular, Sasaki-Einstein metrics are critical points of* Vol*.*

Martelli-Sparks-Yau's theorems

$T \subset \mathrm{Isom}(M, g)$: a torus, $\exists b \in \mathrm{Lie}(T)$ s.t. $b^{\#} = \xi$.

$X = (M \times \mathbb{R}_{\geq 0})/(M \times \{0\})$: the cone of M.

$\exists$ a T-equivariant resolution of singularity $\widetilde{X} \to X$.

Theorem (Martelli-Sparks-Yau)

$\mathrm{Vol}(M, g)$ *is localized to the fixed point set* C *of the* T*-action on* $\widetilde{X}$ *so that*

$$\mathrm{Vol}(M, g) = \frac{1}{2^n n!} \sum_{F \in \pi_0 C} \frac{1}{d_F \det\left(\frac{L\xi - \Omega_{\nu F}}{2\pi\sqrt{-1}}\right)}. \tag{1}$$

They assume the existence of a good metric on $\widetilde{X}$ to prove this theorem.

式 (1) の $\det\left(\frac{L\xi - \Omega_{\nu F}}{2\pi\sqrt{-1}}\right)$ を $\det\left(\frac{L\xi - \Omega_{\nu F}}{2\pi\sqrt{-1}}\right)^{-1}$ としていましたが，藤井先生のご指摘通り誤りでした．上のように訂正します．

Main result

(M^{2n+1}, g, η) : a Sasakian manifold.

$\mathcal{F}$: the orbit foliation of the Reeb flow.

We will decompose

$$\mathrm{Vol}(M, g) = \frac{1}{2^n n!} \int_M \eta \wedge (d\eta)^n.$$

into

$$\eta : \text{a tangential 1-form on } \mathcal{F}$$
$$+$$
$$(d\eta)^n : \text{a vol form on the leaf space } M/\mathcal{F}.$$

Main result

The basic de Rham complex $\Omega^\bullet(M, \mathcal{F})$ of $(M, \mathcal{F})$ is defined by

$$\Omega^k(M, \mathcal{F}) := \{\alpha \in \Omega^k(M) \mid \iota_\xi \alpha = 0, L_\xi \alpha = 0\}.$$

Example

$M = \mathbb{R}^n$, $\mathcal{F} = \{\mathbb{R} \times \{y\}\}_{y \in \mathbb{R}^{n-1}}$.

$\omega \in \Omega^k(M)$ is basic $\iff$

ω is of the form

$$\omega = \sum_{I \subset \{2,\ldots,n\}} f_I(x_2, \ldots, x_n) dx_I.$$

Main result

The basic cohomology of $(M, \mathcal{F})$ is defined by

$$H^k(M, \mathcal{F}) := H^k(\Omega^\bullet(M, \mathcal{F}), d).$$

Its dimension is a topological invariant for the Reeb flow of Sasakian manifolds.

Theorem

- (El Kacimi-Hector-Sergiescu) $\dim H^k(M, \mathcal{F}) < \infty$.
- (Kamber-Tondeur) $H^{2n}(M, \mathcal{F}) \cong \mathbb{R}$.
- (Boyer-Galicki) $\dim H^k(M, \mathcal{F})$ *is determined only by* M.
- (G.-N.-T.) *If closed Reeb orbits are isolated, then*

$$\#\{\text{closed Reeb orbits}\} = \sum_{k=0}^{2n} \dim H^k(M, \mathcal{F}).$$

Main result

Definition

$\alpha \in \Omega^1(M)$ on M is *relatively closed* if $\iota_X d\alpha = 0$ for any X tangent to $\mathcal{F}$.

Proposition

η is relatively closed and the map

$$\int_\eta : \Omega^{2n}(M, \mathcal{F}) \longrightarrow \mathbb{R}; \quad \sigma \longmapsto \int_M \eta \wedge \sigma.$$

descends to a map $\int_\eta : H^{2n}(M, \mathcal{F}) \to \mathbb{R}$.

Proof. $\int_{\mathcal{F}} d\omega = \int_M d(\eta \wedge \omega) - \int_M d\eta \wedge \omega = 0$. □

Main result

Let $\Xi(\mathcal{F}) = C^\infty(T\mathcal{F})$.
A vector field X on M is said to be foliated if $[X, Y] \in \Xi(\mathcal{F})$ for all $Y \in \Xi(\mathcal{F})$. Let

$$L(M, \mathcal{F}) = \{ \text{ foliated vector field on } (M, \mathcal{F})\},$$
$$\ell(M, \mathcal{F}) = L(M, \mathcal{F})/\Xi(\mathcal{F}).$$

$\ell(M, \mathcal{F})$ is a Lie algebra with Lie bracket induced from $L(M, \mathcal{F})$.

Definition

For a Lie algebra $\mathfrak{g}$, a transverse action of $\mathfrak{g}$ on $(M, \mathcal{F})$ is a Lie algebra homomorphism $\mathfrak{g} \to \ell(M, \mathcal{F})$.

Main result

(M, η, g) : a Sasakian manifold,
T : the closure of the Reeb flow, $\mathfrak{t} = \mathrm{Lie}(T)$,
$\mathcal{F}$: the orbit foliation of the Reeb flow.

Let $\mathfrak{a} = \mathfrak{t}/\mathbb{R}\xi$.
$\mathfrak{a}$ transversely acts on $(M, \mathcal{F})$.
$\mathfrak{a}$ acts on $\Omega^\bullet(M, \mathcal{F})$.

Define the equivariant basic cohomology of $(M, \mathcal{F})$ as follows:

$$\Omega^\bullet_\mathfrak{a}(M, \mathcal{F}) := (\Omega^\bullet(M, \mathcal{F}) \otimes S\mathfrak{a}^*)^\mathfrak{a},$$
$$H^\bullet_\mathfrak{a}(M, \mathcal{F}) := H^\bullet(\Omega_\mathfrak{a}(M, \mathcal{F}), d_\mathfrak{a}).$$

Idea due to Goertsches-Töben (2016): Use the Cartan model.

Main result

Since $[d\eta] \in H^2(M, \mathcal{F})$, we get

$$\mathrm{Vol}(M, g) = \frac{1}{2^n n!} \int_M \eta \wedge (d\eta)^n = \frac{1}{2^n n!} \int_\eta [d\eta]^n.$$

Here $\int_\eta$ extends to $\int_\eta : H^{2n}_{\mathfrak{a}}(M, \mathcal{F}) \longrightarrow S\mathfrak{a}^*$, and $[d\eta] \in H^2(M, \mathcal{F})$ extends to $[\omega] \in H^2_{\mathfrak{a}}(M, \mathcal{F})$ so that

$$\mathrm{Vol}(M, g) = \frac{1}{2^n n!} \int_\eta [d\eta]^n = \frac{1}{2^n n!} \int_\eta [\omega]^n.$$

Let $L_1, \ldots, L_N$ be closed Reeb orbits and $i : \sqcup_k L_k \to M$ be the inclusion. We show that i_* is invertible on $Q(\mathfrak{a}^*) \otimes H_{\mathfrak{a}}(M, \mathcal{F})$ and

$$[\omega]^n = i_* S[\omega]^n = \sum_k \frac{i_*^{L_k} i_{L_k}^* [\omega]^n}{e_{\mathfrak{a}}(\nu L_k, \mathcal{F})}.$$

Main result

Theorem (G.-N.-T.)

Let (M, g, η) be a $(2n+1)$-dimensional compact Sasakian manifold with only finitely many closed Reeb orbits $L_1, \ldots, L_N$. Denote the weights of the isotropy $\mathfrak{a}$-representation on νL_i by $\alpha^i_j \in \mathfrak{a}^*$, $i = 1, \ldots, N$ and $j = 1, \ldots, n$. Then

$$\mathrm{Vol}(M, g) = \frac{\pi^n}{n!} \sum_{i=1}^{N} \ell_i \cdot \frac{\eta|_{L_i}(v^{\#})^n}{\prod_j \alpha^i_j(v + \mathbb{R}\xi)},$$

where $\ell_i = \int_{L_i} \eta$ is the length of the closed Reeb orbit L_i.

Main result

- This is a formula of the volume only with intrinsic data of (M, η, g).
- This is a special case of a general Atiyah-Bott-Berline-Vergne type localization formula for equivariant basic cohomology of Killing foliations.
- Casselman-Fisher showed a localization formula for K-contact manifolds, which implies this formula.
- Töben showed a similar localization formula with slightly different formulation of integration operators.

Examples

Recall the example (S^{2n+1}, g_w, η_w) in the page 4.

Corollary

We have

$$\mathrm{Vol}(S^{2n+1}, g_w) = \frac{2\pi^{n+1}}{n!} \sum_{i=0}^{n} \frac{1}{w_i^{n+1}} \frac{\beta_i^n}{\prod_{j=0,\dots,n,\, j\neq i}(\frac{\beta_i}{w_i} w_j - \beta_j)}$$
$$= \frac{2\pi^{n+1}}{n!} \cdot \frac{1}{w_0 \cdot \cdots \cdot w_n}.$$

The second equality follows from the identity $\sum_{i=0}^{n} \frac{x_i^n}{\prod_{j\neq i}(x_i - x_j)} = 1$. (教えて頂き、ありがとうございます.)

Main result

It is easy to see that $\int_{L_i} \eta_w = \frac{2\pi}{w_i}$.
The weights $\{\alpha^i_j\}_{j=0,\dots,\hat{i},\dots,n}$ of the isotropy $\mathfrak{a}$-representation at L_i are given by $\alpha^i_j(e_k + \mathbb{R}b_w) = -\delta_{jk}$ for $j, k \neq i$.
Writing $v = \sum_{i=0}^n \beta_j e_j$, we calculate
$v + \mathbb{R}b_w = \sum_{j\neq i} \beta_j(e_j + \mathbb{R}b_w) - \frac{\beta_i}{w_i} \sum_{k\neq i} w_k(e_k + \mathbb{R}b_w)$.
Then the denominator is given by

$$\prod_{j\neq i} \alpha^i_j(v + \mathbb{R}b_w) = \prod_{j\neq i} \left(\frac{\beta_i}{w_i} w_j - \beta_j \right).$$

The numerator is therefore given by

$$\eta_w|_{L_i}(v^{\#})^n = \left(\frac{\beta_i}{w_i} \right)^n.$$

Examples: Toric cases

Consider the toric case where $\dim T = n + 1$. The momentum polytope of a toric Sasakian manifold of Reeb type is the image of the contact moment map:

$$\begin{array}{rccc} \Phi : & M & \longrightarrow & \mathrm{Lie}(T)^* \\ & x & \longmapsto & (X \mapsto \eta(X^{\#})(x)) \end{array}$$

When the close Reeb orbits are isolated, the inverse image of the vertices of Δ is the union of closed Reeb orbits.

Take $b \in \mathfrak{t}$ so that $b^{\#} = \xi$.

Corollary

Assume that (M, η, g) is a toric Sasakian manifold of Reeb type with isolated Reeb closed orbits. For a Reeb closed orbit L, $\Phi(L)$ is a vertex of the momentum polytope. Let $v_1^L, \ldots, v_n^L$ be the face vectors of Δ of faces which contains $\Phi(L)$. We have

$$\mathrm{Vol}(M, \eta) = \frac{1}{2^n n!} \sum_L \frac{1}{\det(b, v_1^L, \ldots, v_n^L)} \cdot \frac{\det(v, v_1^L, \ldots, v_n^L)^n}{\prod_{i=1}^n \det(b, v_1^L, \ldots, v_{i-1}^L, v, v_{i+1}^L, \cdots, v_n^L)}$$

for any $v \in \mathfrak{t}$.

Examples

Proposition (Martelli-Sparks-Yau)

$$\mathrm{Vol}(M, \eta_b) = 2\pi^{n+1}\,\mathrm{Vol}_H(\Delta_1).$$

Theorem (Lawrence)

Take $u \in \mathfrak{t}$ and $d \in \mathbb{R}$ so that the function $f(x) = u(x) + d$ on $\mathfrak{t}^*$ is nonconstant on each edge of Δ_1. Then

$$\mathrm{Vol}_H(\Delta_1) = \frac{1}{n!}\sum_L \frac{f(\Phi(L))^n}{\delta^L \gamma_1^L \cdot \ldots \cdot \gamma_n^L},$$

where L runs over the set of closed Reeb orbits, $\gamma_i^L \in \mathbb{R}$ is determined by

$$u = \gamma_0^L b + \gamma_1^L v_1^L + \cdots + \gamma_n^L v_n^L,$$

and $\delta^L = \det(b, v_1^L, \ldots, v_n^L)$.

Examples

G/K : a homogeneous Sasaki manifold.
$\pi : G/K \to G/H$ an S^1-bundle.
T : a maximal torus of H.

Corollary

$$\mathrm{Vol}(G/K, \eta') = \frac{2\pi^{n+1}}{n!} \cdot \sum_{\gamma \in W(G)/W(H)} \frac{1}{p(\mathrm{Ad}_{\gamma^{-1}}(R'))^{n+1}} \cdot \frac{p(\mathrm{Ad}_{\gamma^{-1}} v)^n}{\prod_{\alpha \in \Delta_G \setminus \Delta_H} \alpha \left(\mathrm{Ad}_{\gamma^{-1}} \left(v - \frac{p(\mathrm{Ad}_{\gamma^{-1}} v)}{p(\mathrm{Ad}_{\gamma^{-1}} R')} \cdot R' \right) \right)},$$

where $W(G)$ (resp. $W(H)$) is the Weyl group of G (resp. H), Δ_G (resp. Δ_H) is the roots of G (resp. H) with respect to $\mathfrak{t}$ and $p : \mathfrak{t} \to \mathbb{R}\xi \cong \mathbb{R}$ is an orthogonal projection.

$M^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$ admits a homogeneous Sasakian structure.
$T^3 = T^2 \times T^1 \subset \mathrm{SO}(5) \times \mathrm{SO}(2)$ acts on M.
Choose a generic $w = (a, b, c) \in \mathfrak{t}^3$.
g_w: a Sasakian metric whose Reeb vector field is w.

Corollary

$$\mathrm{Vol}(M, g_w) = \frac{2\pi^4}{3} \frac{1}{(c^2 - b^2)(c^2 - a^2)}.$$

Characteristic classes of the Reeb flow

- Secondary characteristic classes of foliations are Chern-Simons type cobordism invariants of foliations.
- They are cohomology classes of classifying spaces, $B\Gamma_q$, $R\Gamma_q$, $K\Gamma_q$, ...
- It is difficult to compute.
- They can be used to produce infinite families of foliations which are not mutullay cobordant. It implies certain infinite aspect of the homotopy type of classifying spaces.

We will discuss computation of characteristic classes of transversely Kähler foliations by localization.

Characteristic Classes of the Reeb flow

(M, η, g) : a Sasakian manifold
Let $H = (T\mathcal{F})^{\perp} = \ker \eta$.
The Reeb flow preserves H, g and $d\eta$.
$\mathcal{F}$ admits a transversely Kähler structure induced from $(g, d\eta)$ and the projection $H \to TM/T\mathcal{F}$.

Characteristic Classes of the Reeb flow

$S^{2n+1} \subset \mathbb{R}^{2n+2}$: the unit sphere
$w = (w_0, \ldots, w_n) \in (\mathbb{R}_{>0})^{n+1}$
Consider

$$\eta_w = \frac{\sum_{i=0}^{n}(y_i dx_i - x_i dy_i)}{\sum_{i=0}^{n} w_i(x_i^2 + y_i^2)} \in \Omega^1(S^{2n+1}).$$

Here S^{2n+1} admits a Sasakian structure (η_w, g_w) where the metric g_w is determined by η_w and the standard CR structure on S^{2n+1}.
The Reeb vector field ξ is

$$\xi = \sum_{i=0}^{n} w_i \left(-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i} \right).$$

A *transversely Kähler Haefliger cocycle* of complex codimension m on a manifold M is a quintuple $(\{U_i\}, \{g_i\}, \{\omega_i\}, \{\pi_i\}, \{\gamma_{ij}\})$ consisting of

1. an open covering $\{U_i\}$ of M,
2. submersions $\pi_i : U_i \to \mathbb{R}^{2m}$,
3. Riemannian metrics g_i on $\pi_i(U_i)$,
4. symplectic form ω_i on $\pi_i(U_i)$ such that g_i and ω_i determine an integrable complex structure J_i on $\pi_i(U_i)$ by the equation $g_i(X, J_i Y) = \omega_i(X, Y)$,
5. transition maps $\gamma_{ij} : \pi_j(U_i \cap U_j) \to \pi_i(U_i \cap U_j)$ such that $\pi_i = \gamma_{ij} \circ \pi_j$, $\gamma_{ij}^* g_i = g_j$ and $\gamma_{ij}^* \omega_i = \omega_j$.

Characteristic Classes of the Reeb flow

$(M, \mathcal{F})$: a manifold with a transversely Kähler foliation of complex codimension m with topologically trivial $\wedge^{m,0}\nu^*\mathcal{F}$.
Fix a topological trivialization φ of the $U(1)$-bundle $U(\wedge^{m,0}\nu^*\mathcal{F}) \to M$ associated to $\wedge^{m,0}\nu^*\mathcal{F}$. Define $u_1(\mathcal{F})$ by

$$u_1(\mathcal{F}) = \frac{\sqrt{-1}}{2\pi}\varphi^*\theta,$$

where $\theta \in \Omega^1(U(\wedge^{m,0}\nu^*\mathcal{F})) \otimes \mathfrak{u}(1)$ is the connection form of the Chern connection on $\wedge^{m,0}\nu^*\mathcal{F}$ induced from $\nu^{1,0}\mathcal{F}$.

Lem

$du_1(\mathcal{F}) = s_1(\mathcal{F})$, where $s_1(\mathcal{F})$ is the first Chern form of $\nu^{1,0}\mathcal{F}$.

Lem

$d(u_1(\mathcal{F})s_1(\mathcal{F})^m) = 0$.

Proof. $d(u_1(\mathcal{F})s_1(\mathcal{F})^m) = s_1(\mathcal{F})^{m+1} = 0$. □

Definition

The characteristic class $[u_1(\mathcal{F})s_1(\mathcal{F})^m]$ is called the Bott class of $\mathcal{F}$.

Corollary (Bott, Baum-Bott)

We have

$$\int_{S^{2m+1}} u_1 s_J(\mathcal{F}_w) = \frac{s_1 s_J}{s_{m+1}}(w_0, \ldots, w_m), \tag{2}$$

where s_J in the RHS are symmetric polynomials.

Corollary

Let $(M, \mathcal{F})$ be a compact manifold with an orientable taut transversely Kähler foliation of dimension one and complex codimension m. Assume that $\wedge^{m,0}\nu^*\mathcal{F}$ is trivial as a topological line bundle and $(M, \mathcal{F})$ admits only finite closed leaves L_1, ..., L_N. For a given multi-index $J = (j_1, \ldots, j_l)$ with $j_1 + \cdots + j_l = 2m$ we have

$$\int_M u_1 s_J(\mathcal{F}) = \sum_{k=1}^{N} \left(\int_{L_k} u_1(\mathcal{F}) \right) \frac{i_{L_k}^* s_{J,\mathfrak{a}}(\nu\mathcal{F}, \mathcal{F})}{i_{L_k}^* s_{m,\mathfrak{a}}(\nu\mathcal{F}, \mathcal{F})},$$

where the L_k are the isolated closed leaves of $\mathcal{F}$ and $s_{J,\mathfrak{a}}(\nu\mathcal{F}, \mathcal{F})$ is the equivariant characteristic form of $\mathcal{F}$ associated to s_J. In particular, in the case where $J = \{m\}$, we obtain

$$\int_M u_1 s_m(\mathcal{F}) = \sum_k \int_{L_k} u_1(\mathcal{F}).$$

Thank you for your attention!!