Higher Derivative Quantum Mechanics

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KMI film: https://www.youtube.com/watch?v=TP2WFsN_k6w

Introduction

Higher derivative (gravity) models,
$F(R)$ gravity, the Gauss-Bonnet gravity, Galileon models,
Horndeski Models,

J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, "Exploring gravitational theories beyond Horndeski," JCAP **1502** (2015) 018 [arXiv:1408.1952 [astro-ph.CO]].

J. Ben Achour, D. Langlois and K. Noui, "Degenerate higher order scalar-tensor theories beyond Horndeski and disformal transformations," Phys. Rev. D **93** (2016) no.12, 124005 [arXiv:1602.08398 [gr-qc]].

Stability, consistency $\Leftrightarrow$ no ghost, bounded potential in Hamiltonian
How to construct the Hamiltonian?

$\Rightarrow$ Higher derivative quantum mechanics

M. Ostrogradsky, "Mémoires sur les équations différentielles, relatives au problème des isopérim ètres," Mem. Acad. St. Petersbourg **6** (1850) no.4, 385.

In French, 133 pages
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$\star$ How to define the canonical momenta and Hamiltonian?
Hamilton's equations of motion $\Leftrightarrow$ Lagrange's equations of motion

Higher derivative model does not always generate instability or breakdown of unitarity,
An Example: $F(R)$ gravity,

$$S = \int d^4x\sqrt{-g}F(R) \Leftrightarrow S = \int d^4\sqrt{-g}\left(\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) .$$

Model without higher derivative term is not always stable nor consistent,
An example: scalar model,

$$S = \int d^4x\sqrt{-g}\left(-\frac{\omega(\phi)}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) .$$

If $\omega(\phi) < 0$, ghost and non-unitary.
If $V(\phi)$ is unbounded below, unstable.

Another example: 3rd Galileon model

$$S = \int \left\{ \bar{\alpha} \left(\partial_\mu \phi \partial^\mu \phi \Box \phi - \partial^\mu \phi \partial^\nu \phi \partial_\mu \partial_\nu \phi \right) \right\} .$$

Because

$$\begin{aligned}
& \partial_\mu \phi \partial^\mu \phi \Box \phi - \partial^\mu \phi \partial^\nu \phi \partial_\mu \partial_\nu \phi \\
& = \left(-\dot{\phi}^2 + g^{ij} \partial_i \phi \partial_j \phi \right) \left(-\ddot{\phi} + \triangle \phi \right) - \left(\dot{\phi}^2 \ddot{\phi} - 2 \dot{\phi} \partial^i \phi \partial_i \dot{\phi} + \partial^i \phi \partial^j \phi \partial_i \partial_j \phi \right) \\
& = - \triangle \phi \dot{\phi}^2 + 4 \dot{\phi} \partial^i \phi \partial_i \dot{\phi} + g^{ij} \partial_i \phi \partial_j \phi \triangle \phi - \partial^i \phi \partial^j \phi \partial_i \partial_j \phi - \partial_t (\dot{\phi} (\partial \phi)^2) \\
& = - 3 \triangle \phi \dot{\phi}^2 + \partial_i \phi \partial^i \phi \triangle \phi - \partial^i \phi \partial^j \phi \partial_i \partial_j \phi + 2 \partial_i \left(\dot{\phi}^2 \partial^i \phi \right) - \partial_t (\dot{\phi} (\partial \phi)^2) \\
& \sim - 3 \triangle \phi \dot{\phi}^2 + \partial_i \phi \partial^i \phi \triangle \phi - \partial^i \phi \partial^j \phi \nabla_i \nabla_j \phi + 2 \partial_i \left(\dot{\phi}^2 \partial^i \phi \right) .
\end{aligned}$$

Canonical momentum π conjugate to ϕ

$$\pi = -6\alpha \triangle \phi\dot{\phi} \,.$$

Hamiltonian density

$$\mathcal{H} = -\frac{\pi^2}{12\alpha \triangle \phi} - \alpha \left(\partial_i\phi\partial^i\phi \triangle \phi - \partial^i\phi\partial^j\phi\partial_i\partial_j\phi \right) \,.$$

In order to avoid the ghost, we require

$$\alpha \triangle \phi < 0 \,,$$

for stability,

$$V \equiv -\alpha \left(\partial_i\phi\partial^i\phi \triangle \phi - \partial^i\phi\partial^j\phi\nabla_i\nabla_j\phi \right) \,.$$

should be bounded below.
The above conditions are not always satisfied because they are the odd function of ϕ.

1. Introduction
2. Energy in Higher Derivative Theory
3. Construction of Hamiltonian
4. Degenerate Case
5. Eliminating Ghost
6. Summary

Energy in Higher Derivative Theory

1st example $L = L(q, \dot{q}, \ddot{q})$.
Assume that the Lagrangian does not depend on time t explicitly.
Euler-Lagrange equation:

$$
\begin{aligned}
0 =& \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) \,, \\
\Rightarrow \quad \frac{dL}{dt} =& \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial \ddot{q}}\dddot{q} \\
=& \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} - \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial \ddot{q}}\dddot{q} \\
=& \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}}\dot{q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\dot{q} + \frac{\partial L}{\partial \ddot{q}}\ddot{q}\right\} \,.
\end{aligned}
$$

Conserved quantity $\Leftrightarrow$ Energy

$$
E = \frac{\partial L}{\partial \dot{q}}\dot{q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\dot{q} + \frac{\partial L}{\partial \ddot{q}}\ddot{q} - L \,.
$$

More general Lagrangian: $L = L\left(q, \dot q, \cdots, q^{(N)}\right)$.
Euler-Lagrange equation:

$$\begin{aligned}
0 =& \frac{\partial L}{\partial q} + \sum_{n=1}^{N} (-)^n \frac{d^n}{dt^n}\left(\frac{\partial L}{\partial q^{(n)}}\right) , \\
\Rightarrow \quad \frac{dL}{dt} =& \frac{\partial L}{\partial q}\dot q + \sum_{n=1}^{N} \frac{\partial L}{\partial q^{(n)}} q^{(n+1)} \\
=& \sum_{n=1}^{N} \left\{ \frac{\partial L}{\partial q^{(n)}} q^{(n+1)} - (-)^n \frac{d^n}{dt^n}\left(\frac{\partial L}{\partial q^{(n)}}\right)\dot q \right\} \\
=& \sum_{n=1}^{N} \left\{ \frac{\partial L}{\partial q^{(n)}} q^{(n+1)} - (-)^n \frac{d}{dt}\left(\frac{d^{n-1}}{dt^{n-1}}\left(\frac{\partial L}{\partial q^{(n)}}\right)\dot q\right) - (-)^{n-1}\frac{d^{n-1}}{dt^{n-1}}\left(\frac{\partial L}{\partial q^{(n)}}\right)\ddot q \right\} \\
=& \frac{d}{dt}\left\{ \sum_{n=1}^{N}\sum_{k=1}^{n} (-)^{k-1}\left(\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial q^{(n)}}\right) q^{(n-k+1)}\right)\right\} .
\end{aligned}$$

Energy:

$$
\begin{aligned}
E &= \sum_{n=1}^{N} \sum_{k=1}^{n} (-)^{k-1} \frac{d^{k-1}}{dt^{k-1}} \left(\frac{\partial L}{\partial q^{(n)}} \right) q^{(n-k+1)} - L \\
&= \sum_{l=1}^{N} \sum_{k=0}^{N-l} (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial q^{(l+k)}} \right) q^{(l)} - L \,.
\end{aligned}
$$

Construction of Hamiltonian

Based on Principle of least Action

$$S = \int dtL \,.$$

Standard Lagrangian $L = L\left(q, \dot q\right)$.
Auxiliary variables p and q_1,

$$\Rightarrow \ S = \int dt \left(p\left(\dot q - q_1\right) + L\left(q, q_1\right)\right) \,.$$

$\delta p \ \Rightarrow \ q_1 = \dot q$.
By substituting the expression $q_1 = \dot q$ into the action, we obtain the original action.

$\delta q_1 \Rightarrow$

$$p = \frac{\partial L}{\partial q_1} \, .$$

If the above equation can be solved with respect to q_1 as a function of q and p, $q_1 = q_1(q, p)$, by substituting the obtained expression, we rewrite the action as

$$S = \int dt \left(p\dot{q} - H(q,p) \right) \, , \quad H(q,p) \equiv pq_1(q,p) - L(q, q_1(q,p)) \, .$$

$\Rightarrow$ Standard canonical formulation by using the Hamiltonian.

Case of $L = L(q, \dot{q}, \ddot{q})$.
Auxiliary variables p, p_1, q_1, and q_2,

$$S = \int dt \left(p\left(\dot{q} - q_1\right) + p_1\left(\dot{q}_1 - q_2\right) + L\left(q, q_1, q_2\right)\right) ,$$

$$\delta q_1,\ \delta q_2 \Rightarrow p + \dot{p}_1 = \frac{\partial L}{\partial q_1} , \quad p_1 = \frac{\partial L}{\partial q_2} ,$$

$$\Rightarrow p = \frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_2}\right) .$$

Because δp, $\delta p_1 \Rightarrow q_1 = \dot{q}$, $q_2 = \dot{q}_1 = \ddot{q}$, and $\delta q \Rightarrow 0 = \dot{p} - \frac{\partial L}{\partial q}$

$$\begin{aligned} 0 =& \frac{d}{dt}\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_2}\right)\right) - \frac{\partial L}{\partial q} \\ =& \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{\partial L}{\partial q} . \end{aligned}$$

$\Rightarrow$ Euler-Lagrange equation.

Solving $p_1 = \frac{\partial L}{\partial q_2}$ with respect to q_2 as a function of q, q_1, and p_1 as $q_2 = q_2(q, q_1, p_1)$.

$$\Rightarrow \ S = \int dt \left\{ p\dot{q} + p_1 \dot{q}_1 - H(q, q_1, p, p_1) \right\} ,$$
$$H(q, q_1, p, p_1) = p q_1 + p_1 q_2(q, q_1, p_1) - L(q, q_1, q_2(q, q_1, p_1)) .$$

By identifying q_1 and q_2 with $\dot{q}$ and $\ddot{q}$, we find H coincides with the energy,

$$E = \frac{\partial L}{\partial \dot{q}} \dot{q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{q}} \right) \dot{q} + \frac{\partial L}{\partial \ddot{q}} \ddot{q} - L ,$$
$$p = \frac{\partial L}{\partial q_1} - \frac{d}{dt} \left(\frac{\partial L}{\partial q_2} \right) , \quad p_1 = \frac{\partial L}{\partial q_2} .$$

Hamilton's equations of motion

$$\dot{q} = \frac{\partial H}{\partial p} = q_1 \,, \quad \dot{q}_1 = \frac{\partial H}{\partial p_1} = q_2 + p_1 \frac{\partial q_2}{\partial p_1} - \frac{\partial L}{\partial q_2}\frac{\partial q_2}{\partial p_1} \,,$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -p_1 \frac{\partial q_2}{\partial q} + \frac{\partial L}{\partial q} + \frac{\partial L}{\partial q_2}\frac{\partial q_2}{\partial q} \,,$$

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -p - p_1 \frac{\partial q_2}{\partial q_1} + \frac{\partial L}{\partial q_1} + \frac{\partial L}{\partial q_2}\frac{\partial q_2}{\partial q_1} \,,$$

By using $p_1 = \frac{\partial L}{\partial q_2}$,

$$\dot{q} = q_1 \,, \quad \dot{q}_1 = q_2 \,, \quad \dot{p} = \frac{\partial L}{\partial q} \,, \quad \dot{p}_1 = -p + \frac{\partial L}{\partial q_1} \,,$$

$\Rightarrow$ Equivalence with Euler-Lagrange equation.

More general case: $L = L\left(q, \dot{q}, \cdots q^{(N)}\right)$.
Auxiliary variables p_{i-1} and q_i $(i = 1, 2, \cdots, N)$,

$$S = \int dt \left(\sum_{i=1}^{N} p_{i-1} \left(\dot{q}_{i-1} - q_i \right) + L\left(q_0, q_1, \cdots, q_N\right) \right) , \quad \left(q_0 \equiv q\right) ,$$

$$\delta q_i \ \Rightarrow p_{i-1} + \dot{p}_i = \frac{\partial L}{\partial q_i} \ \Rightarrow \ p_{i-1} = \sum_{k=0}^{N-i} (-)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial q_{i+k}} \right) , \ (i = 1, \cdots N-1) ,$$

$$\delta q_N \ \Rightarrow p_{N-1} = \frac{\partial L}{\partial q_N} \ \Rightarrow \ q_N = q_N \left(q_0, q_1, \cdots, q_{N-1}, p_{N-1}\right) .$$

$$\Rightarrow S = \int dt \left(\sum_{i=1}^{N} p_{i-1} \dot{q}_{i-1} - H\left(q_0, q_1, \cdots, q_{N-1}, p, p_1, \cdots p_{N-1}\right) \right) ,$$

$$\begin{aligned} & H\left(q_0, q_1, \cdots q_{N-1}, p, p_1, \cdots, p_{N-1}\right) \\ & \quad \equiv \sum_{i=1}^{N-1} p_{i-1} q_i + p_{N-1} q_N \left(q_0, q_1, \cdots, q_{N-1}, p_{N-1}\right) \\ & \qquad - L\left(q_0, q_1, \cdots, q_{N-1}, q_N\left(q_0, q_1, \cdots, q_{N-1}, p_{N-1}\right)\right) . \end{aligned}$$

Degenerate Case

When $p_1 = \frac{\partial L}{\partial q_2}$ cannot be solved with respect to q_2,

$$L(q, q_1, q_2) = L_1(q, q_1) + q_2 L_2(q, q_1) \ .$$

$\delta q_2 \Rightarrow$ constraint

$$\Phi_1 = p_1 - L_2(q, q_1) = 0 \, .$$

$\delta q_1 \Rightarrow$

$$p + \dot{p}_1 = \frac{\partial L_1(q, q_1)}{\partial q_1} + q_2 \frac{\partial L_2(q, q_1)}{\partial q_1} \, ,$$

and δp, $\delta p_1 \Rightarrow \dot{q} = q_1$, $\dot{q}_1 = q_2$, $\Rightarrow$ secondary constraint

$$\begin{aligned}
\dot{\Phi} &= -p + \frac{\partial L_1(q, q_1)}{\partial q_1} + q_2 \frac{\partial L_2(q, q_1)}{\partial q_1} - q_1 \frac{\partial L_2(q, q_1)}{\partial q} - q_2 \frac{\partial L_2(q, q_1)}{\partial q_1} \\
&= -p + \frac{\partial L_1(q, q_1)}{\partial q_1} - q_1 \frac{\partial L_2(q, q_1)}{\partial q} \, ,
\end{aligned}$$

$\dot{\Phi} = 0 \Rightarrow q_1 = q_1(q, p)$.

$\Rightarrow$

$$S=\int dt\left(p\left(\dot{q}-q_1\left(q,p\right)\right)+L_2\left(q,q_1\left(q,p\right)\right)\left(\frac{\partial q_1}{\partial q}\dot{q}+\frac{\partial q_1}{\partial p}\dot{p}\right)+L_1\left(q,q_1\left(q,p\right)\right)\right) .$$

Assume

$$L_2\left(q,q_1\left(q,p\right)\right)\frac{\partial q_1}{\partial p}=\sum_{l=0}^{\infty}f_l\left(q\right)p^l .$$

$$\begin{aligned}S&=\int dt\left(p\left(\dot{q}-q_1\left(q,p\right)\right)+L_2\left(q,q_1\left(q,p\right)\right)\frac{\partial q_1}{\partial q}\dot{q}+\sum_{l=0}^{\infty}\frac{f_l\left(q\right)}{l+1}\frac{dp^{l+1}}{dt}+L_1\left(q,q_1\left(q,p\right)\right)\right)\\&=\int dt\left(\left(p+L_2\left(q,q_1\left(q,p\right)\right)\frac{\partial q_1}{\partial q}-\sum_{l=0}^{\infty}\frac{f_l'\left(q\right)}{l+1}p^{l+1}\right)\dot{q}-pq_1\left(q,p\right)+L_1\left(q,q_1\left(q,p\right)\right)\right) .\end{aligned}$$

$$p+L_2\left(q,q_1\left(q,p\right)\right)\frac{\partial q_1}{\partial q}-\sum_{l=0}^{\infty}\frac{f_l'\left(q\right)}{l+1}p^{l+1}\to p ,$$

$p=p\left(q,p\right)$

$$S=\int dt\left(p\dot{q}-H\left(q,p\right)\right) ,\quad H\left(q,p\right)\equiv p\left(q,p\right)q_1\left(q,p\left(q,p\right)\right)-L_1\left(q,q_1\left(q,p\left(q,p\right)\right)\right) .$$

$H\left(q,p\right)$: Hamiltonian

Example:

$$L = -\frac{1}{2}\omega^2 q^2 - \frac{1}{2} q\ddot{q} \Rightarrow L_1(q) + q_2 L_2(q)\,,\ L_1(q) = -\frac{1}{2}\omega^2 q^2\,,\ L_2(q) = -\frac{1}{2} q\,.$$

$$S = \int dt\left(p\left(\dot{q} - q_1\right) + p_1\left(\dot{q}_1 - q_2\right) + L_1(q) + q_2 L_2(q)\right)$$

$\delta q_2 \ \Rightarrow\ 0 = \Phi = -p_1 + L_2(q) = -p_1 - \frac{1}{2} q$
$0 = \dot{\Phi} = -p + \frac{\partial L_1(q, q_1)}{\partial q_1} - q_1 \frac{\partial L_2(q, q_1)}{\partial q} = -p + \frac{1}{2} q_1 \ \Rightarrow$

$$\begin{aligned} S &= \int dt\left(p\left(\dot{q} - 2p\right) - q\dot{p} + q_2\left(-p_1 + L_2(q)\right) - \frac{1}{2}\omega^2 q^2\right) \\ &= \int dt\left(2p\left(\dot{q} - p\right) - \frac{1}{2}\omega^2 q^2\right)\,. \end{aligned}$$

If $2p \to p \Rightarrow$

$$H = \frac{1}{2} p^2 + \frac{1}{2}\omega^2 q^2\,,$$

Eliminating Ghost

In order to avoid the ghosts, we may introduce ghost c_i $(i = 1, 2, \cdots, N)$ and anti-ghost b_i $(i = 1, 2, \cdots, N)$ and define the BRS transformation by

$$\delta q_i = 0\,, \quad \delta p_{i-1} = \epsilon c_i\,, \quad \delta c_i = 0\,, \delta b_i = \epsilon \left(q_i - \dot{q}_{i-1} \right)\,,$$

Anti-commuting parameter ϵ.

$$\delta \left(\sum_{i=1}^{N} b_i p_{i-1} \right) = \epsilon \left(L_{\mathrm{GF}} + L_{\mathrm{ghost}} \right)\,,$$

$$L_{\mathrm{GF}} \equiv \sum_{i=1}^{N} \left(q_i - \dot{q}_{i-1} \right) p_{i-1}\,,$$

$$L_{\mathrm{ghost}} \equiv - \sum_{i=i}^{N} b_i c_i\,,$$

$\Rightarrow$

$$
\begin{aligned}
S &= \int dt \left(\sum_{i=1}^{N} p_{i-1} \left(\dot{q}_{i-1} - q_i \right) + L \left(q_0, q_1, \cdots, q_N \right) \right) \\
&= \int dt \left(L_{\mathrm{GF}} + L \left(q_0, q_1, \cdots, q_N \right) \right) .
\end{aligned}
$$

Then if we consider the action,

$$
S = \int dt \left(L_{\mathrm{GF}} + L_{\mathrm{ghost}} + L \left(q_0, q_1, \cdots, q_N \right) \right) ,
$$

Action: BRS invariant.
If we define the physical states as the BRS invariant states, the unitarity could not be broken.
BRS invariance $\Rightarrow \langle \delta b_i \rangle = \epsilon \langle (q_i - \dot{q}_{i-1}) \rangle = 0$.
Eq. of motion from $\delta p_{i-1} \Rightarrow 0 = q_i - \dot{q}_{i-1}$.
Equivalent to the classical theory but no ghost???.

Summary

M. Ostrogradsky, "Mémoires sur les équations différentielles, relatives au problème des isopérim ètres," Mem. Acad. St. Petersbourg **6** (1850) no.4, 385.