Palatini-Born-Infeld Gravity and Black Hole Formation

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Introduction

Motivated with the accelerating expansion of the present universe, we are considering many kinds of gravity theories beyond the Einstein gravity.

Scalar-tensor theory, $F(R)$ gravity, the Gauss-Bonnet gravity, $F(G)$ gravity, massive gravity, bigravity, $\cdots$

Palatini-Born-Infeld Gravity

D. N. Vollick, "Palatini approach to Born-Infeld-Einstein theory and a geometric description of electrodynamics," Phys. Rev. D 69 (2004) 064030 [gr-qc/0309101].
M. Banados, "Eddington-Born-Infeld action for dark matter and dark energy," Phys. Rev. D 77 (2008) 123534 [arXiv:0801.4103 [hep-th]].
etc.

- No propagating mode except graviton.
- Bouncing universe.
- Minimal curvature?

The Born-Infeld gravity

The Born-Infeld gravity (*not Palatini*)

$$S = \frac{1}{\kappa^2 b} \int d^4x \left\{ \sqrt{|\det\ (g_{\mu\nu} + bR_{\mu\nu} + cX_{\mu\nu})|} - \sqrt{|\det\ (g_{\mu\nu})|} \right\} .$$

$X_{\mu\nu}$:
rank two tensor, which is the sum of the products of curvatures.

S. Deser and G. W. Gibbons, "Born-Infeld-Einstein actions?," Class. Quant. Grav. 15 (1998) L35 [hep-th/9803049].

Model including higher derivative terms generates ghost in general.
We can choose $X_{\mu\nu}$ to avoid the ghost.
(Although $X_{\mu\nu}$ is not uniquely determined.)

If $X_{\mu\nu} = 0$, there always appears a ghost.

The Palatini-Born-Infeld gravity

Starting action:

$$S = \frac{1}{\kappa^2 b} \int d^4x \left\{ \sqrt{|\det\,(g_{\mu\nu} + bR_{\mu\nu})|} - \sqrt{|\det\,(g_{\mu\nu})|} \right\} + S_{\text{matter}}\,.$$

S_{matter}**: matter action**
$R_{\mu\nu}$**:** $R_{\mu\nu} = -\Gamma^{\rho}_{\mu\rho,\nu} + \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\eta}_{\mu\rho}\Gamma^{\rho}_{\nu\eta} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\rho}_{\rho\eta}$

Connection $\Gamma^{\rho}_{\mu\nu}$ is a variable independent from the metric $g_{\mu\nu}$.

Variations of $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ $\Rightarrow$

$$0 = \sqrt{-P}\left(P^{-1}\right)^{\mu\nu} - \sqrt{-g}g^{\mu\nu}\,, \quad 0 = \nabla_\lambda\left(\sqrt{-P}\left(P^{-1}\right)^{\mu\nu}\right) = 0\,,$$
$$P_{\mu\nu} \equiv g_{\mu\nu} + bR_{\mu\nu}\,.$$

(The covarinat derivative ∇ is given in terms of $\Gamma^\lambda_{\mu\nu}$.)
$\Rightarrow$ Minkowski space-time is a solution.

By combining the equations, $0 = \nabla_\lambda\left(\sqrt{-g}g_{\mu\nu}\right)$

$$\Rightarrow \;\; \Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)\,,$$

identical with the expression in the Einstein gravity.

Perturbation from the Minowski background: $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$

$$\Rightarrow \ \Gamma^\lambda_{\mu\nu} = \frac{\kappa}{2}\left(\partial_\mu h^\lambda{}_\nu + \partial_\nu h^\lambda{}_\mu - \partial^\lambda h_{\mu\nu}\right) ,$$

$$R_{\mu\nu} = \frac{\kappa}{2}\left(\partial_\nu\partial_\mu h^\lambda{}_\lambda - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \partial^\lambda\partial_\lambda h_{\mu\nu}\right) ,$$

$$\Rightarrow \ 0 = b\kappa\left(\partial_\nu\partial_\mu h^\lambda{}_\lambda - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \partial^\lambda\partial_\lambda h_{\mu\nu}\right) ,$$

identical with the equation for the graviton in the Einstein gravity.

Only propagating mode is graviton
Any other propagating mode like ghost does not appear.

When we do not use the Palatini formulation, there appear higher derivative terms and ghost, in general.

FRW universe with dust

We consider the FRW cosmology:

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i\right)^2 ,$$

Assume that the non-vanishing components of the connection

$$\Gamma^t_{tt} = A(t) , \quad \Gamma^t_{ij} = a(t)^2 B(t) \delta_{ij} , \quad \Gamma^i_{jt} = \Gamma^i_{tj} = C(t) \delta^i{}_j .$$

In the Einstein gravity, $A = 0$, $B = C = H \equiv \dot{a}/a$.

$$\Rightarrow R_{tt} = -3 \left(\dot{C} + C^2 - AC \right) ,$$

$$R_{ij} = a^2 \left(\dot{B} + 2HB + BC + BA \right) \delta_{ij} , \quad R_{ti} = R_{it} = 0 ,$$

In the previous works, the FRW metric was assumed for $P_{\mu\nu} \equiv g_{\mu\nu} + bR_{\mu\nu}$,

$$ds_P^2 = \sum_{\mu,\nu=0}^{3} = -dt^2 + \tilde{a}(t)^2 \sum_{i=1,2,3} \left(dx^i\right)^2 \ ,$$

and $\Gamma^\lambda_{\mu\nu}$ is given by $P_{\mu\nu}$:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\left(P^{-1}\right)^{\lambda\rho}\left(\partial_\mu P_{\rho\nu} + \partial_\nu P_{\mu\rho} - \partial_\rho P_{\mu\nu}\right) \ ,$$

which, however, reduces the degrees of freedom in $\Gamma^\lambda_{\mu\nu}$ so that $A = 0$ and $B = C$.

Assume that matter is dust (pressure $p = 0$, energy density $\rho = \rho_0 a^{-3}$).

$$\begin{aligned} \Rightarrow \quad b\kappa^2\rho &= \left\{1 + b\left(\dot{B} + 2HB + BC + BA\right)\right\}^{\frac{3}{2}} \\ &\quad \times \left\{1 + 3b\left(\dot{C} + C^2 - AC\right)\right\}^{-\frac{1}{2}} - 1\,, \\ 0 &= \left\{1 + b\left(\dot{B} + 2HB + BC + BA\right)\right\}^{\frac{1}{2}} \\ &\quad \times \left\{1 + 3b\left(\dot{C} + C^2 - AC\right)\right\}^{\frac{1}{2}} - 1\,, \end{aligned}$$

$$
\begin{aligned}
A =& \frac{1}{2}\frac{d}{dt}\left\{\ln\left\{1+3b\left(\dot{C}+C^2-AC\right)\right\}\right\}\ , \\
B =& \frac{1}{2}\left\{1+3b\left(\dot{C}+C^2-AC\right)\right\}^{-1}\left\{2H+b\left\{4H\dot{B}\right.\right. \\
& +4H^2B+2HBC+2HBA+\ddot{B}+2\dot{H}B+\dot{B}\left(C+A\right) \\
& \left.\left.+B\left(\dot{C}+\dot{A}\right)\right\}\right\}\ , \\
C =& \frac{1}{2}\frac{d}{dt}\left\{\ln\left\{a^2+ba^2\left(\dot{B}+2HB+BC+BA\right)\right\}\right\}\ .
\end{aligned}
$$

$\Rightarrow$

$$A = H - C\,, \quad B = H + \frac{b\kappa^2}{4} H\rho\,, \quad C = H - \frac{3}{4} b\kappa^2 H\rho \left(1 + b\kappa^2\rho\right)^{-1}.$$

$\Rightarrow$ **Single equation with respect to the e-foldings N, $\left(a = \mathrm{e}^N\right)$.**

$$\ddot{N} = -\frac{3\dot{N}^2}{1 + \frac{b\kappa^2\rho_0}{4}\mathrm{e}^{-3N}} - \frac{1 - \sqrt{1 + b\kappa^2\rho_0\mathrm{e}^{-3N}}}{b\left(1 + \frac{b\kappa^2\rho_0}{4}\mathrm{e}^{-3N}\right)},$$

$\Rightarrow$

$$0 = \frac{d^2}{dt^2}\left(\frac{\mathrm{e}^{3N}}{3} + \frac{b\kappa^2\rho_0}{4}N\right) + \frac{\mathrm{e}^{3N}\left(1 - \sqrt{1 + b\kappa^2\rho_0\mathrm{e}^{-3N}}\right)}{b},$$

When $b < 0$, maximum of density, ρ_{max}: $\rho_{\mathrm{max}} = -\frac{1}{b\kappa^2}$.

Conserved quantity E ($\sim$ total energy in classical mechanics),

$$\begin{aligned} E =& \frac{1}{2}\left\{\frac{d}{dt}\left(\frac{\mathrm{e}^{3N}}{3}+\frac{b\kappa^2\rho_0}{4}N\right)\right\}^2 \\ & + \int^N dN \frac{\mathrm{e}^{3N}\left(1-\sqrt{1+b\kappa^2\rho_0\mathrm{e}^{-3N}}\right)\left(\mathrm{e}^{3N}+\frac{b\kappa^2}{4}\rho_0\right)}{b} \\ =& \frac{1}{2}\left\{\frac{d}{dt}\left(\frac{\mathrm{e}^{3N}}{3}+\frac{b\kappa^2\rho_0}{4}N\right)\right\}^2 + V(N)\,, \\ V(N) \equiv& \frac{\mathrm{e}^{6N}}{6b}\left(1+\frac{1}{2}b\kappa^2\rho_0\mathrm{e}^{-3N}\right)\left(1-\sqrt{1+b\kappa^2\rho_0\mathrm{e}^{-3N}}\right) \\ & - \frac{b\kappa^2\rho_0}{12b}\mathrm{e}^{3N}\sqrt{1+b\kappa^2\rho_0\mathrm{e}^{-3N}}\,. \end{aligned}$$

$V(N)$ is negative and monotonically decreasing function.

When N is positive and large,

$$V(N) \sim -\frac{\kappa^2 \rho_0 \mathrm{e}^{3N}}{6} \to -\infty \,.$$

In case $b > 0$, when N is negative and large,

$$V(N) \sim -\frac{\left(b\kappa^2\rho_0\right)^{\frac{3}{2}} \mathrm{e}^{\frac{3}{2}N}}{6b} \to 0- \ \,.$$

In case $b < 0$, there is a maximum in $V(N)$ when $1 + b\kappa^2\rho_0\mathrm{e}^{-3N} = 0$:

$$V(N) = V_{\mathrm{max}} \equiv \frac{\left(b\kappa^2\rho_0\right)^2}{12b} < 0 \,.$$

We now assume that the universe may have started from $N \to +\infty$ and after that they have stated to shrink.
$\Rightarrow$

- **In case $b > 0$, if $E < 0$, the shirinking of the universe will stop and turn to expand. On the other hand if $E > 0$, the universe will continue to shrink and the scale factor a vanishes in the infinite future.**

- **In case $b < 0$, if $E < V_{\rm max}$, the shirinking of the universe will stop and turn to expand. On the other hand if $E > V_{\rm max}$, the universe will reach the singular point at $1 + b\kappa^2\rho_0 {\rm e}^{-3N} = 0$.**

Estimation of E (Assume $N \gg 1$),

$$\ddot{N} + 3\dot{N}^2 - \frac{\kappa^2\rho_0 \mathrm{e}^{-3N}}{2} = \frac{3}{4} b\kappa^2\rho_0 \mathrm{e}^{-3N}\dot{N}^2 - \frac{\left(b\kappa^2\rho_0\right)^2}{4b}\mathrm{e}^{-6N} + \mathcal{O}\left(b^2\right) \ .$$

In the limit $b \to 0$**,** $N = \frac{2}{3}\ln\left|\frac{t}{t_0}\right|$ **and** $t_0^2 \equiv \frac{4}{3\kappa^2\rho_0}$

For the finite b**, by writing** $N = \frac{2}{3}\ln\frac{t}{t_0} + \delta N$**,**

$$\delta N = C_+ \left|t\right|^{\frac{-3+\sqrt{7}}{2}} + C_- \left|t\right|^{\frac{-3-\sqrt{7}}{2}} + \mathcal{O}\left(b^2\right) \ .$$

$C_\pm$**: arbitrary constants.**

If $C_+ \neq 0$, E diverges, which is not acceptable physically.
Even if keep C_-, this term does not contribute to E.

$$E = -\frac{\left(b\kappa^2\rho_0\right)^2}{16b}.$$

- **When $b > 0$, the shrinking of the universe will always stop and turn to expand, that is, we obtain the bouncing universe.**

- **When $b < 0$, the shrinking universe always reaches the singular point at $1 + b\kappa^2\rho_0\mathrm{e}^{-3N} = 0$.**

When $b > 0$, estimation of N when the shrinking universe turns to expand ($V(N) = E$).

When $b\kappa^2\rho_0 \gg 1$,

$$\mathrm{e}^{3N} \sim \frac{3}{8} b\kappa^2\rho_0 \, .$$

When $b\kappa^2\rho_0 \ll 1$,

$$\mathrm{e}^{3N} \sim \frac{9}{64} b\kappa^2\rho_0 \, .$$

$H = \dot{N} \Rightarrow$ **1st FRW-like equation**

$$\frac{3}{\kappa^2} H^2 = \frac{6}{\kappa^2} \mathrm{e}^{-6N} \left(1 + \frac{b\kappa^2 \rho_0}{4} \mathrm{e}^{-3N} \right)^2 (E - V(N)) \ .$$

For large N,

$$\frac{3}{\kappa^2} H^2 = \rho \left(1 - \frac{\rho}{\rho_l} \right) + \mathcal{O}\left(\mathrm{e}^{-9N}\right) \ , \quad \rho = \rho_0 \mathrm{e}^{-3N} \ , \quad \rho_l \equiv \frac{2}{b\kappa^2} \ .$$

Structure is similar to the loop quantum cosmology

$$\rho_l \neq \rho_c = \left\{ \begin{array}{ll} \frac{8}{3b\kappa^2} & \textbf{when } b\kappa^2 \rho_0 \gg 1 \\ \frac{64}{9b\kappa^2} & \textbf{when } b\kappa^2 \rho_0 \ll 1 \end{array} \right. \ .$$

Black hole formation by the collapse of dust

Collapse of a spherically symmetric and uniform ball made of dust.

This assumption is valid because the pressure of the dust vanishes nor the density of ball cannot be uniform because the pressure should vanish at the boundary between the ball and bulk, which is assumed to be vacuum.

Inside the ball: identified with the previous shrinking FRW universe.

Bouncing

If the radius of the ball at the bouncing is larger than the Schwarzschild radius, the black hole cannot be formed.

Ball of dust: radius R at $N = N_0 \gg 1$, Total mass M: $M = \frac{4\pi}{3} R^3 \rho_0 \mathrm{e}^{-3N_0}$.

$b > 0$ case:

Case I: $b\kappa^2\rho_0 = \frac{3b\kappa^2 M \mathrm{e}^{3N_0}}{4\pi R^3} \gg 1$.

$$N = N_b \text{ at the bouncing:} \quad \mathrm{e}^{3N_b} \sim \frac{9b\kappa^2 M \mathrm{e}^{3N_0}}{32\pi R^3}\,,$$

$$\text{Radius } R_b \text{ at the bouncing:} \quad R_b^3 = R^3 \mathrm{e}^{3(N_b - N_0)} = \frac{9b\kappa^2 M}{32\pi}\,,$$

$$\text{Schwarzschild radius } R_s\text{:} \quad R_s = \frac{\kappa^2 M}{4\pi} \quad \Rightarrow \quad \frac{R_b^3}{R_s^3} = \frac{18\pi b}{\kappa^4 M^2}\,.$$

Large black hole, where $M^2 \gg \frac{b}{\kappa^4}$, can be formed because $R_b \ll R_s$ and therefore the bouncing can occur after the formation of the horizon.

Case II: $b\kappa^2\rho_0 = \frac{3b\kappa^2 M \mathrm{e}^{3N_0}}{4\pi R^3} \ll 1$. **Bouncing when**

$$\mathrm{e}^{3N} \sim \mathrm{e}^{3\tilde{N}_b} \sim \frac{27b\kappa^2 M \mathrm{e}^{3N_0}}{256\pi R^3},$$

Radius $\tilde{R}_b$ at the bouncing:

$$R_b^3 = R^3 \mathrm{e}^{3(N_b - N_0)} = \frac{27b\kappa^2 M}{256\pi} \quad \Rightarrow \quad \frac{R_b^3}{R_s^3} = \frac{27\pi b}{4\kappa^4 M^2}.$$

Small black hole, where $M^2 \ll \frac{b}{\kappa^4}$, cannot be formed because $R_b \gg R_s$.

$b < 0$ case:

Maximum of density, $\rho_{\rm max}$: $\rho_{\rm max} = -\frac{1}{b\kappa^2}$

Consider collapse of star made of dust with radius r.
The energy density ρ: $\rho = \tilde\rho_0 r^{-3}$, $\tilde\rho_0$: a constant.
Mass M and Schwarzschild radius R_s

$$M = \frac{4\pi}{3}\rho r^3 = \frac{4\pi}{3}\tilde\rho_0\,, \quad R_s = \frac{\kappa^2 M}{4\pi} = \frac{\kappa^2\tilde\rho_0}{3}\,.$$

The minimum of r: $r_{\rm min} = (-3bR_s)^{\frac{1}{3}}$.

Black hole cannot be formed if $r_{\rm min} > R_s$, that is, $R_s^2 < -3b$.
Small black holes may be prohibited if $b < 0$ but large ones are not prohibited.

Summary

We considered the Palatini formulation of the Born-Infeld gravity

- In the Palatini formulation, the propagating mode is only graviton, whose situation is different from that in the metric formulation.
- We considered the FRW cosmology with flat spacial part.
  - We assumed that the matter is dust.
  - We obtain a conserved quantity E, which may correspond to the total energy in the classical mechanics.
  - We found that when $b > 0$, the shrinking of the universe will always stop and turn to expand, that is, we obtain the bouncing universe. On the other hand, when $b < 0$, the shrinking universe always reaches the singular point at $1 + b\kappa^2\rho_0 \mathrm{e}^{-3N} = 0$, where $\sqrt{|\det\,(g_{\mu\nu} + bR_{\mu\nu})|}$ vanishes.

- **We also considered if black hole can be formed by the collapse of dust.**
  - **We assumed there is a spherically symmetric and uniform ball made of dust and consider the collapse of ball. This assumption is valid because the pressure of the dust vanishes nor the density of ball cannot be uniform because the pressure should vanish at the boundary between the ball and bulk, which is assumed to be vacuum.**
  - **Inside the ball, the space-time can be regarded with the shrinking FRW universe. Then the results about the FRW universe tell that there could be a bouncing. If the radius of the ball at the bouncing is larger than the Schwarzschild radius, the black hole cannot be formed.**