微分方程式系に対する Klein Cartan プログラム

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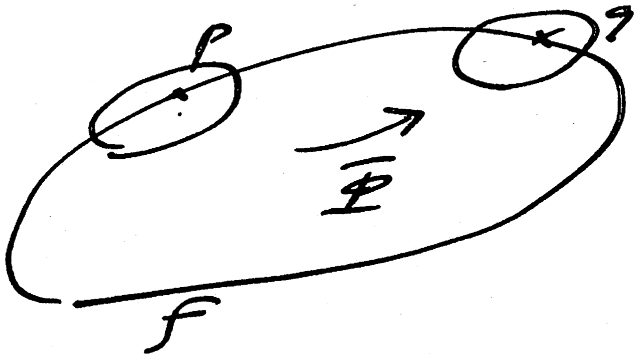
Introduction.

My friend Izumi asked me a question:

What does look like ~~a~~ an affinely homogeneous curves in the plane?



This is one of the typical concrete problems that have been treated by many mathematicians since more than 100 years and that I have recently been considering in much more general framework.

I would like to give an introductory talk on this subject.

1. Extrinsic geometry of curves in the projective plane.

Two curves in the projective plane $\mathbb{P}^2$; $f_i : M_i \to \mathbb{P}^2$ $(i=1,2)$ are called equivalent if

$\exists\, \Phi$: projective transformation
$\exists\, \varphi$: diffeomorphism

s.t.
$$\begin{tikzcd} M_1 \arrow[r, "f_1"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{P}^2 \arrow[d, "\Phi"] \\ M_2 \arrow[r, "f_2"] & \mathbb{P}^2 \end{tikzcd}$$

A curve $f : M \to \mathbb{P}^2$ is called homogeneous if for any $x_1, x_2 \in M$ the germs of mapping (f, M, x_1) and (f, M, x_2) are equivalent.

Problems

- Find complete invariants of $f : M \to \mathbb{P}^2$ which enable us to recognize whether two curves are equivalent or not.
- Classify homogeneous curves.

2. Geometry of 3rd order linear ODE's.

Two 3rd order linear ODE's

(R) $$\frac{d^3y}{dx^3} + \binom{3}{1} p_1(x) \frac{d^2y}{dx^2} + \binom{3}{2} p_2(x) \frac{dy}{dx} + p_3(x)\, y = 0$$

(R′) $$\frac{d^3\eta}{d\xi^3} + \binom{3}{1} \pi_1(\xi) \frac{d^2\eta}{d\xi^2} + \binom{3}{2} \pi_2(\xi) \frac{d\eta}{d\xi} + \pi_3(\xi)\, \eta = 0$$

are called equivalent if a change of variables

(T) $$\begin{cases} y = \lambda(\xi)\,\eta \\ x = \varphi(\xi) \end{cases}$$

transforms (R) and (R′) to each other

The equation (R) is called homogeneous if the germs of equations (R, x_1) and (R, x_2) are equivalent for any x_1, x_2.

<u>Problems.</u>

- Find the complete invariants of (R)
- Classify the homogeneous 3rd ord. homogeons ODE's.

3. Categorical isomorphism

Category of 3rd order linear ODE's

$\updownarrow$

Category of curves in $\mathbb{P}^2$

Let $\{f_1, f_2, f_3\}$ be a fundamental system of solutions of (R) defined on a 1-dim mfd M.

$$f : M \ni x \longmapsto [f_1(x), f_2(x), f_3(x)] \in \mathbb{P}^2.$$

The equivalence class of f does not depends on the choice of the fundamental system $\{f_1, f_2, f_3\}$.

Conversely the mapping f defines a ODE whose solution space is spanned by f_1, f_2, f_3.

A little more sophisticated formulation may be:

Given a linear ODE $\underset{(R)}{}$ (of 3rd ord.) defined on a line bundle E over 1-dim mfd M.

$$0 \to Sol(R)^0_x \to Sol(R) \xrightarrow{\text{eval.}} E_x \to 0$$

$$M \ni x \longmapsto Sol(R)^0_x \in Gr(V, 2)$$

$$\longmapsto (Sol(R)^0_x)^\perp \in \mathbb{P}(V^*),$$

where $V = Sol(R)$.

Conversely, given a mapping $f : M \to P(V^*)$. Then we have a line bundle $E = f^* \mathbb{T}^*_{P(V^*)}$ and an embedding

$$V \to T(E),$$

which defines a 3rd ord. linear ODE (R):

$$\begin{tikzcd} V \arrow[r] & \underline{E} \arrow[r] \arrow[dr] & J^3E \\ & & R \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$

where τ is the tautological line bundle

$$\begin{array}{ccc} \tau & & \tau_L = L \\ \downarrow & & \\ P(V^*) \ni L & & \end{array}$$

$$V \longrightarrow \Gamma(\tau^*)$$

$$\begin{array}{l} \ \ \ \Psi \\ \ \ \ v \longmapsto \sigma(v) \end{array}$$

$$\sigma(v)(L) = v|_L$$

$$\in L^*$$

$$\tau^*_L.$$

. regarding $V = (V^*)^*$..

4. Invariants of a linear 3rd order ODE (R).

By a change of variables

$$(T) \qquad \begin{cases} y = \lambda(\xi)\,\eta \\ x = \varphi(\xi) \end{cases},$$

we have:

$$(4.1) \qquad \begin{cases} y = \lambda\eta \\ y' = (\lambda\dot{\eta} + \dot{\lambda}\eta)\,\xi' \\ y'' = \left(\lambda\ddot{\eta} + \binom{2}{1}\dot{\lambda}\dot{\eta} + \ddot{\lambda}\eta\right)(\xi')^2 \\ \qquad + (\lambda\dot{\eta} + \dot{\lambda}\eta)\,\xi'' \\ y''' = \left(\lambda\dddot{\eta} + \binom{3}{1}\dot{\lambda}\ddot{\eta} + \binom{3}{2}\ddot{\lambda}\dot{\eta} \right. \\ \qquad \left. + \dddot{\lambda}\eta\right)(\xi')^3 \\ \qquad + 3(\lambda\ddot{\eta} + 2\dot{\lambda}\dot{\eta} + \ddot{\lambda}\eta)\,\xi'\xi'' \\ \qquad + (\lambda\dot{\eta} + \dot{\lambda}\eta)\,\xi''' \end{cases}$$

Substituting this into (R), we have:

$$\dddot{\eta}\left(\lambda(\xi')^3\right)$$

$$+\binom{3}{1}\ddot{\eta}\left(\dot{\lambda}(\xi')^3+\lambda\xi'\xi''+p_1\lambda(\xi')^2\right)$$

$$+\binom{3}{2}\dot{\eta}\Big\{\ddot{\lambda}(\xi')^3+2\dot{\lambda}\xi'\xi''+\frac{1}{3}\lambda\xi'''$$

$$+2p_1\left(2\dot{\lambda}(\xi')^2+\lambda\xi''\right)+p_2\lambda\xi'\Big\}$$

$$+\eta\Big\{\dddot{\lambda}(\xi')^3+3\ddot{\lambda}\xi'\xi''+\dot{\lambda}\xi'''$$

$$+3p_1\left(\ddot{\lambda}(\xi')^2+\dot{\lambda}\xi''\right)$$

$$+3p_2\left(\dot{\lambda}\xi'\right)+p_3\lambda\Big\}.$$

Then we have:

$$(4.2)\quad \left\{\begin{aligned}
\pi_1 &= \frac{\dot{\lambda}}{\lambda}+\frac{\xi''}{(\xi')^2}+\frac{1}{\xi'}p_1\\
\pi_2 &= \frac{\ddot{\lambda}}{\lambda}+2\left(\frac{\xi''}{(\xi')^2}+\frac{1}{\xi'}p_1\right)\frac{\dot{\lambda}}{\lambda}\\
&\quad+\frac{1}{3}\frac{\xi'''}{(\xi')^3}+\frac{\xi''}{(\xi')^3}p_1+\frac{1}{(\xi')^2}p_2\\
\pi_3 &= \frac{\dddot{\lambda}}{\lambda}+3\left(\frac{\xi''}{(\xi')^2}+\frac{1}{\xi'}p_1\right)\frac{\ddot{\lambda}}{\lambda}\\
&\quad+\frac{1}{(\xi')^3}\left(\xi'''+3\xi''p_1+3\xi'p_2\right)\frac{\dot{\lambda}}{\lambda}
\end{aligned}\right.$$

$$+\frac{1}{(\xi')^3}p_3,$$

where $\xi'=\dfrac{d\xi}{dx}$, $\dot{\lambda}=\dfrac{d\lambda}{dt}$ ...

<u>Proposition</u> (Laguerre - Forsyth)
There exists a change of variables (T.) such that the transformed eq. (R′) satisfies:

$$\pi_1 = \pi_2 = 0 .$$

<u>In fact</u>, from (4.2) we see that we can transform (R) to (R′) with $\pi_1 = 0$.

Next to transform (R) with $p_1 = 0$ into (R′) with $\pi_1 = \pi_2 = 0$, we have only to solve:

$$\left\{\begin{aligned} \pi_1 &= \frac{\dot\lambda}{\lambda} + \frac{\xi''}{(\xi')^2} = 0 \\ \pi_2 &= \frac{\ddot\lambda}{\lambda} + 2\frac{\xi''}{(\xi')^2}\frac{\dot\lambda}{\lambda} + \frac{1}{3}\frac{\xi'''}{(\xi')^3} + \frac{1}{(\xi')^2}p_2 = 0 , \end{aligned}\right.$$

which reduces to

$$(4.3)\quad \left\{\begin{aligned} &\frac{\dot\lambda}{\lambda} + \frac{\xi''}{(\xi')^2} = 0 \\ &\frac{\xi'''}{\xi'} - \frac{3}{2}\left(\frac{\xi''}{\xi'}\right)^2 = \frac{3}{2}p_2 \end{aligned}\right.$$

what is obviously integrable

<u>Remark</u>: The left hand side of the second equation of (4.3) is nothing but the <u>Schwarzian derivative</u> of ξ w.r.t. x.

Now let us determine the transformations which maps $\mathcal{R})$ with $\varphi_1 = \varphi_2 = 0$ into $\mathcal{R}')$ with $\pi_1 = \pi_2 = 0$.

They are solutions of:

$$(4.4)\begin{cases} \text{i)} & \dfrac{\dot{\lambda}}{\lambda} + \dfrac{\xi''}{(\xi')^2} = 0 \\ \text{ii)} & \dfrac{\xi'''}{\xi'} - \dfrac{3}{2}\left(\dfrac{\xi''}{\xi'}\right)^2 = 0 \end{cases}$$

From ii) of (4.4), we have:

$$\xi = \frac{a'x + b'}{c'x + d'} \qquad (a'd' - b'c' = 1)$$

or

$$x = \frac{a\xi + b}{c\xi + d} \qquad (ad - bc = 1)$$

Then, integrating λ, we have:

$$(4.5)\quad \begin{cases} x = \dfrac{a\xi + b}{c\xi + d} = \varphi(\xi) \\ y = k\,(c\xi + d)^{-2}\,\eta, \end{cases}$$

and by this transformation, we have:

$$\pi_3 = \frac{1}{(c\xi + d)^6}\, p_3.$$

If we set

$$\begin{cases} \theta_x = p_3(x)\,(dx)^3 \\ \theta_\xi = \pi_3(\xi)\,(d\xi)^3 \end{cases}$$

then

$$\varphi^* \theta_x = \theta_\xi,$$

that is, $\theta_x = p_3(x)(dx)^3$ is an invariant of (R).

(This and its higher order covariant derivatives fulfill the complete invariants

5. Homogeneous 3rd order linear ODE's.

Consider 3rd ord. linear ODE of Laguerre Forsyth canonical form:

(5.1) $$y^{(3)} + p_3(x)\, y = 0$$

The automorphism group of (5.1) acts on the $\overbrace{x\text{-space}}^{M}$ G 1-dimensional projective space, as projective transformation, so there is an homomorphism

$$G \to G' \subset SL(2).$$

Now assume that G' is transitive on M.

Let $A \in$ Lie alg G', and

$g_t = \exp tA$, then

$$g_t^{*}\left(p(x)(dx)^3\right) = p(x)(dx)^3.$$

By differentiating w.r.t. t and putting $t=0$, we get

$$(\beta + 2\alpha x - \gamma x^2)\frac{dp}{dx} - 6(-\alpha + \gamma x)p = 0$$

where $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

By integrating, we have

$$p = \frac{C}{(\beta + 2\alpha x - \gamma x^2)^3}.$$

Theorem. The 3rd order linear homogeneous equations may be expressed in the form:

$$y^{(3)} + \frac{C}{(\beta + 2\alpha x - \gamma x^2)^3} y = 0,$$

whose invariant is

$$C\left(\frac{dx}{\beta + 2\alpha x - \gamma x^2}\right)^3.$$

Note that, for

$$x = \varphi(\xi) = \frac{a\xi + b}{c\xi + d}$$

we have $(ad - bc = 1)$

$$\varphi^*\left(\frac{dx}{\beta + 2\alpha x - \gamma x^2}\right)$$

$$= \frac{d\xi}{\beta(c\xi+d)^2 + 2\alpha(a\xi+b)(c\xi+d) - \gamma(a\xi+b)^2}$$

The action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overset{SL}{\cancel{GL}}(2, \mathbb{R})$:

$$\beta + 2\alpha x - \gamma x^2 \rightsquigarrow \beta(c\xi+d)^2 + 2\alpha(a\xi+b)(c\xi+d) - \gamma(a\xi+b)^2$$

is nothing but the representation of $\underset{SL}{\cancel{GL}}(2, \mathbb{R})$ on $Sym^2(\mathbb{R}^2)$.

Under the action of ~~GL~~ $SL(2)$

$\beta + 2\alpha x - \gamma x^2$ is equivalent

to one of the following:

0) 0

i) ± 1

ii) Cx

iii) $\pm C(x^2+1)$.

Therefore we have:

<u>Theorem</u> A homogeneous 3rd ord linear ODE is equivalent to one of the following:

0) $y''' = 0$

i) $y''' \pm y = 0$

ii) $y''' + \dfrac{C}{x^3} y = 0$

iii) $y''' + \dfrac{C}{(x^2+1)^3} y = 0$

6. Klein-Cartan program for differential equations

Klein model	Cartan Geometry
$G \xrightarrow{H} G/H$, ω: Maurer-Cartan form.	$P \xrightarrow{H} M$, $\omega: TP \to \mathfrak{g}$ Cartan connection

e.g.

$$E(n) \xrightarrow{O(n)} \underline{E}^n \qquad\qquad O(M) \xrightarrow{O(n)} (M, g)$$

orthogonal frame bundle with $\omega = \omega_{gr} + \omega_{a}$

Levi-Civita connection.

3-ord. linear ODE.

$f: M \to \mathbb{P}^3$

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

$$\mathrm{Sym}^2(\mathbb{R}^2) = V = V_{-3} \oplus V_{-2} \oplus V_{-1}$$

In general, given a

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p \quad : \text{graded Lie alg}$$

$$V = \bigoplus_{q \in \mathbb{Z}} V_q \quad : \text{graded } \mathfrak{g} \text{ module},$$

then

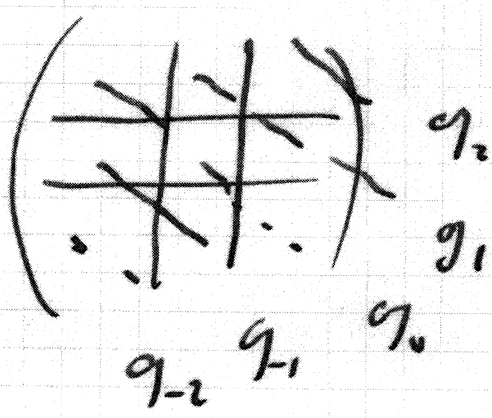
$$\left\{ \begin{array}{l} \text{holonomic systems} \\ \text{of diff eqs of} \\ \text{type } (\mathfrak{g}, V) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} f : M \to \mathrm{Flag}(V) \\ \text{of type} \\ (\mathfrak{g}, V) \end{array} \right\}$$

In the case when $\mathfrak{g}$ is simple, V irreducible, we ~~can~~ have a general scheme to find the invariants of such structures.

- algebraic harmonic theory
- Cartan's moving frame method (generalized)

Example

$$\mathfrak{g} = \mathfrak{sl}(3)$$
$$= \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$$



$V = \mathfrak{sl}(3)$

$\circlearrowleft \mathfrak{g}$ adjoint rep.

G/F^0G : 3-dim Heisenberg Lie gp

(x, y, z)

$$X = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}$$
$$Y = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}$$
$$Z = \frac{\partial}{\partial z} \qquad [X, Y] = Z$$

$$(R) \begin{cases} X^2 u = (a_1 X + b_1 Y + c_1) u \\ Y^2 u = (a_2 X + b_2 Y + c_2) u \end{cases}$$

$$\dim Sol(R) \leq 8$$

$f : M \longrightarrow P^2$