例外型コンパクト Lie 群 E_8 における位数 4 の内部自己同型写像 $\tilde{\tau}_4$ とそれによる固定点部分群 $(E_8)^{\tau_4}$ の実現

-横田流によって-

宮下敏一

March 10, 2020　Shizuoka Univ.

講演内容 (目次風に)

- はじめに $\cdots$ 動機 (興味による)，本講演について
- これまで $\cdots$ 有限位数の自己同型写像による固定点部分群の実現の足跡 (2 sheets)
- これから $\cdots$ 位数 5 の自己同型写像による固定点部分群の実現 (2 sheets)
- 例外 Lie 群 $\cdots$ 複素及びコンパクト例外 Lie 群 G_2, F_4, E_6, E_7, E_8 の定義と周辺に関して (11 sheets)
- 例外群実現の記録 $\cdots$ コンパクト及び非コンパクトの E_6, E_7, E_8 について
- 横田流とよく利用する定理等
- 結果 $\cdots$7 つの場合について，位数 4 の自己同型写像とそれによる固定点部分群の群構造
- 固定点部分群 $(E_8)^{\sigma'_4}$ $\cdots \sigma'_4$ の定義，定理：$({E_8}^C)^{\sigma'_4} \Rightarrow$ 主定理 $(E_8)^{\sigma'_4}$(11 sheets)
- Epilogue $\cdots Y$ ゼミについて

Lie 環による分類結果 $\Longrightarrow$ 群化

■ 本講演において

- J. A. Jiménez, Riemannian 4-symmetric spaces, Trans. Amer. Math. Soc.306,(1988) 715-734

$$\Downarrow$$

- 上記論文の Lie 環による分類のコンパクト Lie 環 $\mathfrak{e}_8$ の部分の結果に対応して，例外型連結コンパクト Lie 群 E_8 における，位数 4 の内部自己同型写像 $\tilde{\tau}_4$ とそれによる固定点部分群 $(E_8)^{\tau_4}$ の実現

  ▹ 補足　E_6 $\cdots$ I. Yokota and O. Shukuzawa, Automorphisms of order 4 of the simply connected compact Lie group E_6, Tsukuba J. Math.,15-2 (1991), 451-463

1. これまで - 1

■ order 2

- Berger M., Les espaces symétriques non compacts, Ann. Sci. Ecole Norm. Sup., 74 (1957), 85-177

⇒ I. Yokota, Realization of involutive automorphisms σ and G^σ of exceptional linear Lie groups G, Part I, $G = G_2, F_4$ and E_6, Part II, $G = E_7$, Part III, $G = E_8$, Tsukuba J. Math. 14(1990), 185-223, 14(1990), 379-404, 15(1991), 301-314

■ order 3

- J. A. Wolf and A. Gray, Homogeneous spaces defined by Lie group automorphisms I, J. diff. Geometry, 2 (1968), 77-114

⇒
- ◦ I. Yokota, Realization of automorphisms σ of order 3 and G^σ of compact exceptional Lie groups G, I, $G = G_2, F_4, E_6$, J. Fac. Sci. Shinshu Univ. 20(1985), 131-144
- ◦ T. Miyashita and I. Yokota, Realization of automorphisms of order 3 and G^σ of compact exceptional Lie groups G, II, $G = E_7$, Yokohama Math. J., 47 (1999), 31-44

◦ S. Gomyo, Realization of maximal subgroups of rank 8 of the simply connected compact simple Lie group of type E_8, Tsukuba J. Math. 21(1997), 595-616

■ order 4

- J. A. Jiménez, Riemannian 4-symmetric spaces, Trans. Amer. Math. Soc. 306 (1988) 715-734

⇒ ◦ T. Miyashita, Realizations of inner automorphisms of order 4 and fixed points subgroups by them on the connected compact exceptional Lie group E_8, Part I, Part II, Tsukuba J. Math. 41-1(2017), 91-166, 43-1(2019), 1-22

◦ Part III, in preparation

■■ order 5

- J. A. Wolf and A. Gray, Homogeneous spaces defined by Lie group automorphisms I, J. diff. Geometry, 2 (1968), 77-114 (次のスライド参照)

⇒ ◦ 位数 5 の内部自己同型写像の分類？

◦ 位数 5 の内部自己同型写像を与え, それによる例外群の固定点部分群を決定したい.

Proposition 3.4(p.90) *Let $\tilde{\tau}_5$ be an automorphism of order 5 on compact or complex simple Lie algebras $\mathfrak{g}$, and $\mathfrak{g}^{\tau_5}$ the fixed point set of $\tilde{\tau}_5$. Then $\tilde{\tau}_5$ is conjugate, by an inner automorphism of $\mathfrak{g}$, to* $\mathrm{Ad}(\exp 2\pi i X)$ *where*

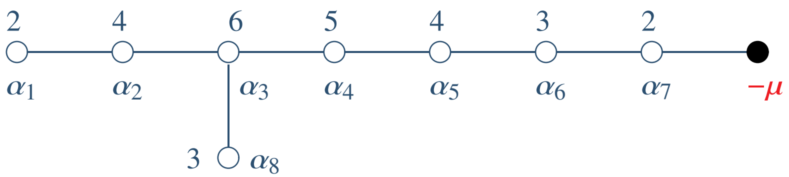
(i) $X = (m_i/5)V_i$ *with* $1 \leq m_i \leq 5$, *or* $X = (4/5)V_i$ *with* $m_i = 2$, *or* $x = (2/5)V_i$ *with* $m_i = 1$; *or*

(ii) $X = (1/5)(V_i + V_j)$ *with* $m_i = m_j = 1$, *or* $X = (1/5)(2V_i + V_j)$ *with* $1 = m_j \leq m_i \leq 2$, *or* $X = (1/5)(3V_i + V_j)$ *with* $m_j = 1, m_i = 3$, *or* $X = (2/5)(V_i + V_j)$ *with* $1 \leq m_j \leq m_i \leq 2$, *or* $X = (1/5)(3V_i + 2V_j)$ *with* $m_i = 3, m_j = 2$; *or*

(iii) $X = (1/5)(V_i + V_j + V_k)$ *with* $m_i = m_j = m_k = 1$, *or* $X = (1/5)(2V_i + V_j + V_k)$ *with* $1 = m_k = m_j \leq m_i \leq 2$; *or*

(iv) $X = (1/5)(V_i + V_j + V_k + V_l)$ *with* $m_i = m_j = m_k = m_l = 1$

In particular, if $\mathfrak{g}^{\tau_5}$ is not the centralizer of a torus, then $\mathfrak{g}$ is of type E_8

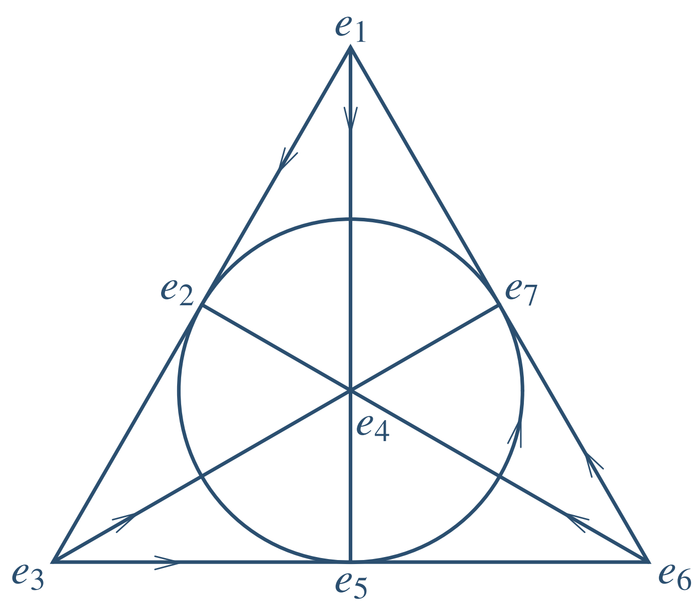


with $X = V_i$, $m_i = 5$, and $\mathfrak{g}^{\tau_5}$ of type $A_4 \oplus A_4$.

(The author said that this was a complete list of the possibilities of X.)

4. 例外 Lie 群 G_2

- Cayley 代数: $\mathfrak{C} = \{e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}_{span}$



- G_2 の定義: $G_2 = \mathrm{Aut}(\mathfrak{C}) = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}) \mid \alpha(xy) = (\alpha x)(\alpha y)\}$

- Involution $\tilde{\gamma}$: $(G_2)^\gamma \cong (Sp(1) \times Sp(1))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1,1),(-1,-1)\}$

(参考) 線形変換 $\gamma : \mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H}e_4 \to \mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H}e_4,$

$$\gamma(m + ae_4) = m - ae_4 \Rightarrow \gamma \in G_2$$

5. 例外 Lie 群 F_4 -1

- 例外 Jordan 代数: $\mathfrak{J}(3,\mathfrak{C}) = \{X \in M(3,\mathfrak{C}) \mid X^* = X\}$

$$= \left\{ \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} \middle| \, \xi_i \in \boldsymbol{R}, x_i \in \mathfrak{C} \right\}$$

- Jordan 積, 内積: $X \circ Y = \dfrac{1}{2}(XY + YX),\ (X,Y) = \mathrm{tr}(X \circ Y)$

- Freudenthal 積: $X \times Y = \dfrac{1}{2}(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - (X,Y))E)$

- F_4 の定義: $F_4 = \mathrm{Aut}(\mathfrak{J}(3,\mathfrak{C})) = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}(3,\mathfrak{C})) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}$

$$= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}(3,\mathfrak{C})) \mid \alpha(X \times Y) = \alpha X \times \alpha Y\}$$

- ${F_4}^C$ の定義: ${F_4}^C = \mathrm{Aut}(\mathfrak{J}(3,\mathfrak{C})^C) = \left\{\alpha \in \mathrm{Iso}_C(\mathfrak{J}(3,\mathfrak{C})^C) \,\middle|\, \alpha(X \circ Y) = \alpha X \circ \alpha Y\right\}$

$$= \left\{\alpha \in \mathrm{Iso}_C(\mathfrak{J}(3,\mathfrak{C})^C) \,\middle|\, \alpha(X \times Y) = \alpha X \times \alpha Y\right\}$$

- Involution $\tilde{\gamma}$: $(F_4)^{\gamma} \cong (Sp(1)\times Sp(3))/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 = \{(1,E),(-1,-E)\}$

  (参考) 線形変換 $\gamma : \mathfrak{J} = \mathfrak{J}_{\boldsymbol{H}} \oplus \boldsymbol{H}^3 \to \mathfrak{J} = \mathfrak{J}_{\boldsymbol{H}} \oplus \boldsymbol{H}^3,$

  $$\gamma(M + \boldsymbol{a}) = M - \boldsymbol{a} \Rightarrow \gamma \in G_2 \subset F_4$$

- Involution $\tilde{\sigma}$: $(F_4)^{\sigma} \cong Spin(9)$

  (参考) 線形変換 $\sigma : \mathfrak{J} \to \mathfrak{J},$

  $$\sigma\begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\overline{x}_2 \\ -\overline{x}_3 & \xi_2 & x_1 \\ -x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} \Rightarrow \sigma \in F_4$$

  $$(F_4)_{E_1} = (F_4)^{\sigma} \implies F_4/Spin(9) \simeq \mathfrak{C}P_2$$

▶ 記号

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$F_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix}, \quad F_2(x) = \begin{pmatrix} 0 & 0 & \overline{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}, \quad F_3(x) = \begin{pmatrix} 0 & x & 0 \\ \overline{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

7. 例外 Lie 群 E_6 -1

- 複素例外 Jordan 代数: $\mathfrak{J}(3,\mathfrak{C})^C = \left\{X \in M(3,\mathfrak{C})^C \,\middle|\, X^* = X\right\}$

  Jordan 積, 内積, Freudenthal 積等が $\mathfrak{J}(3,\mathfrak{C})$ の場合と同様に定義できる.

- 行列式: $\det X = \xi_1\xi_2\xi_3 + 2\mathrm{Re}(x_1x_2x_3) - \xi_1x_1\overline{x}_1 - \xi_2x_2\overline{x}_2 - \xi_3x_3\overline{x}_3,$

  ここに，$X = \xi_1E_1 + \xi_2E_2 + \xi_3E_3 + F_1(x_1) + F_2(x_2) + F_3(x_3)$

- Hermite 内積: $\langle X, Y\rangle = (\tau X, Y),\quad \tau$: 複素共役

- E_6 の定義: $E_6 = \left\{\alpha \in \mathrm{Iso}_C(\mathfrak{J}(3,\mathfrak{C})^C) \,\middle|\, \det\alpha X = \det X, \langle\alpha X, \alpha Y\rangle = \langle X, Y\rangle\right\}$

$$= \left\{\alpha \in \mathrm{Iso}_C(\mathfrak{J}(3,\mathfrak{C})^C) \,\middle|\, \begin{array}{l}\alpha X \times \alpha Y = \tau\alpha\tau(X \times Y),\\ \langle\alpha X, \alpha Y\rangle = \langle X, Y\rangle\end{array}\right\}$$

- ${E_6}^C$ の定義: ${E_6}^C = \left\{\alpha \in \mathrm{Iso}_C(\mathfrak{J}(3,\mathfrak{C})^C) \,\middle|\, \det\alpha X = \det X\right\}$

$$= \left\{\alpha \in \mathrm{Iso}_C(\mathfrak{J}(3,\mathfrak{C})^C) \,\middle|\, \alpha X \times \alpha Y = {}^t\alpha^{-1}(X \times Y)\right\}$$

(参考) $z(E_6) = z({E_6}^C) = \{1, \omega 1, \omega^2 1\} \cong \boldsymbol{Z}_3$

- Involution λ: $(E_6)^\lambda \cong F_4$

(参考) 外部自己同型写像 $\lambda : E_6 \to E_6, \lambda(\alpha) = {}^t\alpha^{-1}(= \tau\alpha\tau)$

- Involution $\tilde{\sigma}$: $(E_6)^\sigma \cong (U(1) \times Spin(10))/\boldsymbol{Z}_4,$

$$\boldsymbol{Z}_4 = \{(1,1),(-1,\sigma),(i,\phi(-i)),(-i,\phi(i))\}$$

(参考) 線形変換 $\sigma : \mathfrak{J}(3,\mathfrak{C})^C \to \mathfrak{J}(3,\mathfrak{C})^C \Rightarrow \sigma \in F_4 \subset E_6$

- Involution $\tilde{\gamma}$: $(E_6)^\gamma \cong (Sp(1) \times SU(6))/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 = \{(1,E),(-1,-E)\}$

(参考) 線形変換 $\gamma : \mathfrak{J}(3,\mathfrak{C})^C \to \mathfrak{J}(3,\mathfrak{C})^C \Rightarrow \gamma \in G_2 \subset F_4 \subset E_6$

- Involution $\lambda\tilde{\gamma}$: $(E_6)^{\lambda\gamma} \cong Sp(4)/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 = \{E,-E\}$

(参考) 外部自己同型写像 $\lambda\tilde{\gamma} : E_6 \to E_6, \lambda\tilde{\gamma}(\alpha) = \gamma\,{}^t\alpha^{-1}\gamma(= \gamma(\tau\alpha\tau)\gamma)$

▶ 元 $A \vee B$ の定義

$$A \vee B := [\tilde{A},\tilde{B}] + \left(A \circ B - \frac{1}{3}(A,B)E\right)^{\sim} \in {\mathfrak{e}_6}^C,\ A,B \in \mathfrak{J}(3,\mathfrak{C})^C$$

9. 例外 Lie 群 E_7 -1

- Freudenthal ベクトル空間: $\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C$

  $\mathfrak{P}^C$ の元を $P := (X, Y, \xi, \eta), Q := (Z, W, \zeta, \omega)$ 等で表す.

- 内積: $(P, Q) = (X, Z) + (Y, W) + \xi\zeta + \eta\omega$
- Hermite 内積: $\langle P, Q \rangle = \langle X, Z \rangle + \langle Y, W \rangle + (\tau\xi)\zeta + (\tau\eta)\omega$
- 交代内積: $\{P, Q\} = (X, W) - (Z, Y) + \xi\omega - \zeta\eta$
- C-線形写像 $\Phi(\phi, A, B, \nu) : \mathfrak{P}^C \to \mathfrak{P}^C$:

$$\Phi(\phi, A, B, \nu)\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} := \begin{pmatrix} \phi X - \dfrac{1}{3}\nu X + 2B \times Y + \eta Y \\ 2A \times X - {}^t\phi Y + \dfrac{1}{3}Y + \xi B \\ (A, Y) + \nu\xi \\ (B, X) - \nu\eta \end{pmatrix}, \ \phi \in \mathfrak{e}_6{}^C, A \in \mathfrak{J}^C, \nu \in C$$

(参考) $\phi \in \mathfrak{e}_6{}^C \underset{\text{def}}{\Longleftrightarrow} \phi = (D_1, D_2, D_3) + \tilde{A} + \tilde{T}, D_i \in \mathfrak{so}(8), A \in \mathfrak{M}^- \ (A^* = -A), T \in (\mathfrak{J}^C)_0$

- C-線形写像 $P \times Q : \mathfrak{P}^C \to \mathfrak{P}^C$:

$$P \times Q := \Phi(\phi, A, B, \nu), \quad \begin{cases} \phi = -\dfrac{1}{2}(X \vee W + Z \vee Y) \\ A = -\dfrac{1}{4}(2Y \times W - \xi Z - \zeta X) \\ B = \dfrac{1}{4}(2X \times Z - \eta W - \omega Y) \\ \nu = \dfrac{1}{8}((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta)) \end{cases}$$

ここに， $P := (X, Y, \xi, \eta), Q := (Z, W, \zeta, \omega)$

- E_7 の定義: $E_7 = \left\{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \,\middle|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\right\}$
- ${E_7}^C$ の定義: ${E_7}^C = \left\{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \,\middle|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\right\}$

(参考) · $\alpha \in {E_7}^C \Rightarrow \{\alpha P, \alpha Q\} = \{P, Q\}$

· $z(E_7) = z({E_7}^C) = \{1, -1\} \cong \boldsymbol{Z}_2$

11. 例外 Lie 群 E_7 -3

- Involution $\tilde{\iota}$: $(E_7)^\iota \cong (U(1) \times E_6)/\mathbf{Z}_3,\ \mathbf{Z}_3 = \left\{(1,1), (\omega, \phi(\omega^2)), (\omega^2, \phi(\omega))\right\}$

  (参考) 線形変換 $\iota : \mathfrak{P}^C \to \mathfrak{P}^C,\ \iota(X, Y, \xi, \eta) = (iX, iY, -i\xi, i\eta)$

  $\Rightarrow \iota \in E_7,\ E_6 \cong (E_7)_{(0,0,1,0)}$

- Involution $\tilde{\sigma}$: $(E_7)^\sigma \cong (SU(2) \times Spin(12))/\mathbf{Z}_2,\ \ \mathbf{Z}_2 = \{(E, 1), (-E, \sigma)\}$

  (参考) 線形変換 $\sigma : \mathfrak{P}^C \to \mathfrak{P}^C,\ \sigma(X, Y, \xi, \eta) = (\sigma X, \sigma Y, \xi, \eta)$

  $\Rightarrow \sigma \in F_4 \subset E_6 \subset E_7,\ Spin(12) \cong (E_7)^{\kappa,\mu}, (E_7)^\sigma \cong (E_7)^{-\gamma}$

  $*\ (E_7)^\gamma \cong (Sp(1) \times Spin(12))/\mathbf{Z}_2,\ \ \mathbf{Z}_2 = \{(E, 1), (-E, \gamma)\}$

- Involution $\tilde{\lambda\gamma}$: $(E_7)^{\lambda\gamma} \cong SU(8)/\mathbf{Z}_2,\ \ \mathbf{Z}_2 = \{E, -E\}$

  (参考) 線形変換 $\lambda\gamma : \mathfrak{P}^C \to \mathfrak{P}^C,\ \lambda\gamma(X, Y, \xi, \eta) = (\gamma Y, -\gamma X, \eta, -\xi)$

  $\Rightarrow \lambda\gamma \in E_7.$

  C-同型写像 $\chi : \mathfrak{P}^C \to \mathfrak{S}(8, \mathbf{C})^C$,

  $$\chi(X, Y, \xi, \eta) = k_J\left(gX - \frac{\xi}{2}E\right) + e_1 k_J\left(g(\gamma Y) - \frac{\eta}{2}E)\right)$$

いよいよ次は，type-E_8 です！

- 248 次元 C-ベクトル空間: $\mathfrak{e}_8{}^C = \mathfrak{e}_7{}^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C$

  (参考) $\tilde{\mathfrak{e}_8}{}^C = \mathfrak{sl}(9, C) \oplus \Lambda^3(C^9) \oplus \Lambda^3(C^9)$ etc. (S. Gomyo)

  $\mathfrak{e}_8{}^C$ の元を $R_i := (\Phi_i, P_i, Q_i, r_i, s_i, t_i)$ 等で表す.

- Lie 積:

$$[R_1, R_2] := (\Phi, P, Q, r, s, t), \quad \begin{cases} \Phi = [\Phi_1, \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1 \\ P = \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + s_1 Q_2 - s_2 Q_1 \\ Q = \Phi_1 Q_2 - \Phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + t_1 P_2 - t_2 P_1 \\ r = -\dfrac{1}{8}\{P_1, Q_2\} + \dfrac{1}{8}\{P_2, Q_1\} + s_1 t_2 - s_2 t_1 \\ s = \dfrac{1}{4}\{P_1, P_2\} + 2r_1 s_2 - 2r_2 s_1 \\ t = -\dfrac{1}{4}\{Q_1, Q_2\} - 2r_1 t_2 + 2r_2 t_1 \end{cases}$$

$\Rightarrow \mathfrak{e}_8{}^C$: Lie 環

(参考) $(\Phi, \tau\lambda Q, Q, r, s, -\tau s) \in \mathfrak{e}_8 \quad \leftarrow$ コンパクト Lie 環

- Hermite 内積: $\langle R_1, R_2\rangle = -\dfrac{1}{15}B_8(\tau\lambda_\omega R_1, R_2)$

  ここに，B_8: ${\mathfrak e_8}^C$ の Killing 形式．線形変換 $\lambda_\omega : {\mathfrak e_8}^C \to {\mathfrak e_8}^C$,

  $\lambda_\omega(\Phi, P, Q, r, s, t) = (\lambda\Phi\lambda^{-1}, \lambda Q, \lambda^{-1}P, -r, -t, -s)$

- ${E_8}^C$ の定義: ${E_8}^C = \mathrm{Aut}({\mathfrak e_8}^C) = \left\{\alpha \in \mathrm{Iso}_C({\mathfrak e_8}^C) \,\middle|\, \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\right\}$

- E_8 の定義: $E_8 = \left\{\alpha \in {E_8}^C \,\middle|\, \langle\alpha R_1, \alpha R_2\rangle = \langle R_1, R_2\rangle\right\}$

  $= \left\{\alpha \in {E_8}^C \,\middle|\, \tau\lambda_\omega\alpha\lambda_\omega\tau = \alpha\right\} = ({E_8}^C)^{\tau\lambda_\omega}$

- Involution $\tilde{\lambda_\omega\gamma}$: $(E_8)^{\lambda_\omega\gamma} \cong Ss(16)(= Spin(16)/\boldsymbol{Z}_2)$

  (参考) 線形変換 $\lambda_\omega\gamma : {\mathfrak e_8}^C \to {\mathfrak e_8}^C$,

  $\lambda_\omega\gamma(\Phi, P, Q, r, s, t) = (\lambda\gamma\Phi\gamma\lambda^{-1}, \lambda\gamma Q, \lambda^{-1}\gamma P, -r, -t, -s)$

  $\Rightarrow \lambda_\omega\gamma \in E_8$

  $*\quad \tilde{\mathfrak e}_8^C := \mathfrak{so}(16, C) \oplus (\mathfrak C^C \otimes \mathfrak C^C) \oplus (\mathfrak C^C \otimes \mathfrak C^C)$

  $\Rightarrow\quad \tilde E_8^C := \mathrm{Aut}(\tilde{\mathfrak e}_8^C)$

  $\Rightarrow (\tilde E_8^C)^\varepsilon \cong Ss(16, C)$ (S. Gomyo)

- Involution $\tilde{\upsilon}$: $(E_8)^{\upsilon} \cong (SU(2) \times E_7)/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(E,1),(-E,-1)\}$

(参考) 線形変換 $\lambda_\omega \gamma : \mathfrak{e}_8{}^C \to \mathfrak{e}_8{}^C$,

$$\upsilon(\Phi, P, Q, r, s, t) = (\Phi, -P, -Q, r, s, t)$$

$$\Rightarrow \upsilon \in E_8, \quad E_7 \cong (E_8)_{(0,0,0,0,0,1)}$$

■ 例外群 E_6, E_7, E_8-type の実現の記録

type	group	year	Journal
E_6	$E_{6(-78)}$	1980	Kyoto
	$E_{6(6)}$	1979	Shinshu
	$E_{6(2)}$	1979	Shinshu
	$E_{6(-14)}$	1979	Shinshu
	$E_{6(-26)}$	?	

type	group	year	Journal
E_7	$E_{7(-133)}$	1981	Kyoto
	$E_{7(7)}$	1982	Okayama
	$E_{7(-5)}$	1982	Hiroshima
	$E_{7(-25)}$	1980	Shinshu
E_8	$E_{8(-248)}$	1981	Kyoto
	$E_{8(8)}$	1986	Tsukuba
	$E_{8(-24)}$	1980	Shinshu

15. 横田流とよく利用する定理等

○　出来るだけ一般論に依らない直接的な証明をする．

- 準同型写像を定義する (群の埋め込み方)
  - ∗　群の埋め込み方が分からない場合は，群に付随する Lie 環のタイプの決定
- 群の連結性

◦　G: 単連結 Lie 群, $\tilde{\sigma} \in \mathrm{Aut}(G)$(有限位数) $\Rightarrow$ G^{σ}: 連結 (E. Cartan-P.K.Rasevskii)

◦　G, G': Lie 群，$\varphi : G \to G'$: 連続な準同同型写像.

このとき，G': 連結，Ker φ: 離散，dim G = dim G' $\Rightarrow$ φ: 全射

◦　G: 連結 Lie 群，N: 離散な G の正規部分群 $\Rightarrow$ $N \subset z(G)$

◦　G: Lie 群，G が空間 M に推移的かつ連続に作用する．また，G_{x_0}: $x_0 \in M$ における等方部分群とする．このとき，

$$G/G_{x_0} \simeq M$$

特に，G_{x_0}, M: 連結 $\Rightarrow$ G: 連結

16. 位数 4 の自己同型写像

- 結果

Case	$\mathfrak{h}$	$\tilde{\tau}_4$	$H = G^{\tau_4}$
1	$\mathfrak{so}(6) \oplus \mathfrak{so}(10)$	$\tilde{\sigma}'_4$	$(Spin(6) \times Spin(10))/\boldsymbol{Z}_4$
2	$i\boldsymbol{R} \oplus \mathfrak{su}(8)$	$\tilde{w}_4$	$(U(1) \times SU(8))/\boldsymbol{Z}_{24}$
3	$i\boldsymbol{R} \oplus \mathfrak{e}_7$	$\tilde{\upsilon}_4$	$(U(1) \times E_7)/\boldsymbol{Z}_2$
4	$\mathfrak{su}(2) \oplus \mathfrak{su}(8)$	$\tilde{\mu}_4$	$(SU(2) \times SU(8))/\boldsymbol{Z}_4$
5	$\mathfrak{su}(2) \oplus i\boldsymbol{R} \oplus \mathfrak{e}_6$	$\tilde{\omega}_4$	$(SU(2) \times U(1) \times E_6)/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_3)$
6	$i\boldsymbol{R} \oplus \mathfrak{so}(14)$	$\tilde{\kappa}_4$	$(U(1) \times Spin(14))/\boldsymbol{Z}_4$
7	$\mathfrak{su}(2) \oplus i\boldsymbol{R} \oplus \mathfrak{so}(12)$	$\tilde{\varepsilon}_4$	$(SU(2) \times U(1) \times Spin(12))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$

Case 1: Tsukuba J. Math. Vol. 41-1 (2017), 91-166

Cases 2-4: Tsukuba J. Math. Vol. 43-1 (2019), 1-22

17. 部分群 $(E_8)^{\sigma'_4}$ -1

- 線形変換 σ'_4 の定義　$\sigma'_4 : \mathfrak{e}_8{}^C \to \mathfrak{e}_8{}^C$,

$$\sigma'_4(\Phi, P, Q, r, s, t) = (\sigma'_4 \Phi \sigma'^{-1}_4, \sigma'_4 P, \sigma'_4 Q, r, s, t),$$

ここに，$\sigma'_4 P = \sigma'_4(X, Y, \xi, \eta) = (\sigma'_4 X, \sigma'_4 Y, \xi, \eta),\ P \in \mathfrak{P}^C$,

さらに，$\sigma'_4 X = \begin{pmatrix} \xi_1 & -x_3 e_1 & \overline{e_1 x_2} \\ -\overline{x_3 e_1} & \xi_2 & -e_1 x_1 e_1 \\ e_1 x_2 & -\overline{e_1 x_1 e_1} & \xi_3 \end{pmatrix}, X \in \mathfrak{J}^C.$

$\Rightarrow\ \sigma'_4 \in Spin(8) \subset F_4 \subset E_6 \subset E_7 \subset E_8.$

(参考)　$(F_4)^{\sigma'_4} \cong (Spin(3) \times Spin(6))/\boldsymbol{Z}_2$

$(E_6)^{\sigma'_4} \cong (U(1) \times Spin(4) \times Spin(6))/\boldsymbol{Z}_2$　$(\triangle)$

$(E_7)^{\sigma'_4} \cong (SU(2) \times Spin(6) \times Spin(6))/\boldsymbol{Z}_2$

▶ まずは，複素から!

- 定理

$$(E_8{}^C)^{\sigma'_4} \cong (Spin(6, C) \times Spin(10, C))/\boldsymbol{Z}_4,$$

$$\boldsymbol{Z}_4 = \{(1,1), (\sigma_4, \sigma\sigma'_4), (\sigma, \sigma), (\sigma\sigma'_4, \sigma'_4)\}$$

- 証明の概要

  ∗ $Spin(6, C)$ の構成

$$
\begin{array}{c}
({F_4}^C)_{E_1,E_2,E_3,F_1(e_i),i=0,1} \cong Spin(6,C)\ (*) \\
\cup \\
({F_4}^C)_{E_1,E_2,E_3,F_1(e_i),i=0,1,2} \cong Spin(5,C) \\
\cup \\
({F_4}^C)_{E_1,E_2,E_3,F_1(e_i),i=0,\ldots,3} \cong Spin(4,C) \\
\cup \\
({F_4}^C)_{E_1,E_2,E_3,F_1(e_i),i=0,\ldots,4} \cong Spin(3,C) \\
\cup \\
({F_4}^C)_{E_1,E_2,E_3,F_1(e_i),i=0,\ldots,5} \cong Spin(2,C) \cong U(1,\boldsymbol{C}^C)
\end{array}
$$

例えば，$(*)$ に関して，次の C 上 6 次元のベクトル空間を考える．

$$
(V^C)^6 := \left\{ X \in \mathfrak{J}^C \;\middle|\; \begin{array}{l} E_1 \circ X = 0, (E_2, X) = (E_3, X) = 0, \\ (F_1(e_i), X) = 0, i = 0, 1 \end{array} \right\}
$$

$$
= \{ X = F_1(t) \,|\, t = t_2 e_2 + t_3 e_3 + t_4 e_4 + t_5 e_5 + t_6 e_6 + t_7 e_7, t_k \in C \},
$$

ノルム $(X, X) = 2({t_2}^2 + {t_3}^2 + {t_4}^2 + {t_5}^2 + {t_6}^2 + {t_7}^2)$.

19. 部分群 $(E_8)^{\sigma_4'}$ -3

- $(S^C)^5 := \{X \in (V^C)^6 \mid (X, X) = 2\}$

$$= \left\{ X = F_1(t) \;\middle|\; \begin{array}{l} t = t_2e_2 + t_3e_3 + t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7, \\ {t_2}^2 + {t_3}^2 + {t_4}^2 + {t_5}^2 + {t_6}^2 + {t_7}^2 = 1, t_k \in C \end{array} \right\}: \text{5 次元複素球面}$$

- $\boxed{({F_4}^C)_{E_1,E_2,E_3,F_1(e_i),i=0,1}/Spin(5, C) \simeq (S^C)^5} \;\Rightarrow\; ({F_4}^C)_{E_1,E_2,E_3,F_1(e_i),i=0,1}$: 連結

次,

- $\boxed{({F_4}^C)_{E_1,E_2,E_3,F_1(e_i),i=0,1} \cong Spin(6, C)}$

[略証] • $O(6, C) = O((V^C)^6) = \{\beta \in \mathrm{Iso}_C((V^C)^6) \mid (\beta X, \beta Y) = (X, Y)\}$

• $({F_4}^C)_{E_1,E_2,E_3,F_1(e_i),i=0,1}$: 連結

$\Rightarrow\;\; p : ({F_4}^C)_{E_1,E_2,E_3,F_1(e_i),i=0,1} \to SO((V^C)^6) = SO(6, C)$

$$p(\alpha) = \alpha\,\big|_{(V^C)^6}\,.$$

$\Rightarrow$ 準同型定理により $({F_4}^C)_{E_1,E_2,E_3,F_1(e_i),i=0,1}/\boldsymbol{Z}_2 \cong SO(6, C)$

$\Rightarrow$ $(F_4)^C{}_{E_1,E_2,E_3,F_1(e_i),i=0,1}$ は，$SO(6, C)$ の普遍被覆群として，$Spin(6, C)$ に同型である:

$$({F_4}^C)_{E_1,E_2,E_3,F_1(e_i),i=0,1} \cong Spin(6, C).$$

□

- $\boxed{Spin(6,C) \cong (F_4{}^C)_{E_1,E_2,E_3,F_1(e_i),i=0,1} \subset (F_4{}^C)^{\sigma'_4}}$
  - ◇ $\mathfrak{J}^C = (\mathfrak{J}^C)_{\sigma'_4} \oplus (\mathfrak{J}^C)_{-\sigma'_4}$ を使って，直接証明可.

∗ $Spin(10,C)$ の構成

$$
\begin{array}{rl}
((E_7{}^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\ldots,7} & \cong Spin(6,C) \\
\cup & \\
((E_7{}^C)^{\kappa,\mu})_{\tilde{E}_1,\dot{F}_1(e_k),k=2,\ldots,7} & \cong Spin(5,C) \\
\cup & \\
((E_6{}^C)^{\sigma})_{E_1,F_1(e_k),k=2,\ldots,7} & \cong Spin(4,C) \\
\cup & \\
(F_4{}^C)_{E_1,F_1(e_k),k=2,\ldots,7} & \cong Spin(3,C) \\
\cup & \\
(F_4{}^C)_{E_1,E_2,E_3,F_1(e_k),k=2,\ldots,7} & \cong Spin(2,C) \cong U(1,\boldsymbol{C}^C)
\end{array}
$$

(参考) $(E_7{}^C)^{\kappa,\mu} = \left\{\alpha \in E_7{}^C \,\middle|\, \kappa\alpha = \alpha\kappa, \mu\alpha = \alpha\mu\right\}$

$\Rightarrow \ (E_7{}^C)^{\kappa,\mu} \cong Spin(12,C)$

$\kappa := \varPhi(-2E_1 \vee E_1, 0, 0, -1) \in \mathfrak{e}_7{}^C, \mu := \varPhi(0, E_1, E_1, 0) \in \mathfrak{e}_7{}^C$

21. 部分群 $(E_8)^{\sigma'_4}$ -5

■ $((E_7{}^C)^{\kappa,\mu})^{\sigma'_4} \cong (Spin(6,C) \times Spin(6,C))/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 = \{(1,1),(\sigma,\sigma)\}$

$$\Downarrow$$

■ $$(E_7{}^C)^{\sigma'_4} \cong (SL(2,C) \times Spin(6,C) \times Spin(6,C))/\boldsymbol{Z}_4\,,$$
$$\boldsymbol{Z}_4 = \big\{(E,1,1),(E,\sigma,\sigma),(-E,\sigma'_4,-\sigma'_4),(-E,\sigma\sigma'_4,-\sigma\sigma'_4)\big\}$$

[略証] • $\varphi : SL(2,C) \times Spin(6,C) \times Spin(6,C) \to (E_7{}^C)^{\sigma'_4}$,

$$\varphi(A,\beta_1,\beta_2) = \psi(A)\beta_1\beta_2$$

• well-defined, homomorphism (略)

• surjective $\quad \alpha \in (E_7{}^C)^{\sigma'_4} \subset (E_7{}^C)^{\sigma}\ (\cong (SL(2,C) \times Spin(12,C))/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 = \{(E,1),(-E,-\sigma)\})$

$$\Rightarrow \left\{\begin{array}{l} A = A \\ \underset{\sim\sim\sim\sim\sim\sim\sim}{\sigma'_4\beta{\sigma'_4}^{-1} = \beta} \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} A = -A \\ \sigma'_4\beta{\sigma'_4}^{-1} = -\sigma\beta. \end{array}\right.$$

• kernel

$$\mathrm{Ker}\,\varphi = \{(A,\beta_1,\beta_2) \in SL(2,C) \times Spin(6,C) \times Spin(6,C) \mid A = E,\ \underset{\sim\sim\sim}{\beta_1\beta_2 = 1}\}$$
$$\cup\ \{(A,\beta_1,\beta_2) \in SL(2,C) \times Spin(6,C) \times Spin(6,C) \mid A = -E,\ \underset{\sim\sim\sim}{\beta_1\beta_2 = -\sigma}\}\ \square$$

22. 部分群 $(E_8)^{\sigma'_4}$ -6

▶ $(E_7{}^C)^{\sigma'_4, \mathfrak{so}(6,C)} := \left\{\alpha \in (E_7{}^C)^{\sigma'_4} \,\middle|\, \varPhi_D \alpha = \alpha \varPhi_D \text{ for all } D \in \mathfrak{so}(6,C)\right\}$

ここに，$\varPhi_D := (D,0,0,0) \in \mathfrak{e}_7{}^C$, $D \in \mathfrak{so}(6,C) \cong (\mathfrak{f}_4{}^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1}$

このとき，次の結果を得る．

$$(E_7{}^C)^{\sigma'_4, \mathfrak{so}(6,C)} \cong SL(2,C) \times Spin(6,C)$$
$$\Rightarrow \quad (E_7{}^C)^{\sigma'_4, \mathfrak{so}(6,C)}\text{: 連結}$$

[略証] • $\varphi : SL(2,C) \times Spin(6,C) \to (E_7{}^C)^{\sigma'_4, \mathfrak{so}(6,C)}$, $\varphi(A, \beta_2) = \psi(A)\beta_2$

• well-defined $\psi(SL(2,C)), Spin(6,C)$: 連結 $\Rightarrow$ Lie 環の計算から

• homomorphism $\varphi : SL(2,C) \times Spin(6,C)^{\times 2} \to (E_7{}^C)^{\sigma'_4}$ の制限写像から

• injective $\operatorname{Ker}\varphi_* = \{0\} \Rightarrow \operatorname{Ker}\varphi$: 離散 $\Rightarrow \operatorname{Ker}\varphi \subset z(SL(2,C) \times Spin(6,C))$

$= \{(E,1), (E,\sigma), (E,-\sigma'_4), (E,-\sigma\sigma'_4), (-E,1), (-E,\sigma), (-E,\sigma'_4), (-E,-\sigma\sigma'_4)\}$

$\Rightarrow \operatorname{Ker}\varphi = \{(E,1)\}$

• surjective $\alpha \in (E_7{}^C)^{\sigma'_4, \mathfrak{so}(6,C)} \subset (E_7{}^C)^{\sigma'_4}$

$\Rightarrow \exists A \in SL(2,C), \beta_1 \in Spin(6,C), \beta_2 \in Spin(6,C)$ s.t $\alpha = \varphi(A, \beta_1, \beta_2)$

$\Rightarrow \varPhi_D \beta_1 = \beta_1 \varPhi_D \Rightarrow \beta_1 = 1$ □

23. 部分群 $(E_8)^{\sigma'_4}$ -7

- $({E_8}^C)^{\sigma'_4, \mathfrak{so}(6,C)} := \left\{ \alpha \in ({E_8}^C)^{\sigma'_4} \,\middle|\, (\mathrm{ad}R_D)\alpha = \alpha(\mathrm{ad}R_D) \text{ for all } D \in \mathfrak{so}(6,C) \right\}$
- $(({E_8}^C)^{\sigma'_4, \mathfrak{so}(6,C)})_{1_-} := \left\{ \alpha \in ({E_8}^C)^{\sigma'_4, \mathfrak{so}(6,C)} \,\middle|\, \alpha 1_- = 1_- \right\}, \quad 1_- := (0,0,0,0,0,1)$

ここに，$R_D = (\Phi_D, 0,0,0,0,0) \in {\mathfrak{e}_8}^C,\ D \in \mathfrak{so}(6,C) \cong ({\mathfrak{f}_4}^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1}$.
このとき，次の結果を得る．

$$(({E_8}^C)^{\sigma'_4, \mathfrak{so}(6,C)})_{1_-} = \exp(\mathrm{ad}((({\mathfrak{P}}^C)_{\sigma'_4})_- \oplus C_-)) \rtimes ({E_7}^C)^{\sigma'_4, \mathfrak{so}(6,C)}$$
$$\Rightarrow \quad (({E_8}^C)^{\sigma'_4, \mathfrak{so}(6,C)})_{1_-}\text{: 連結}$$

[証明のスケッチ] • $((\mathfrak{P}^C)_{\sigma'_4})_- \oplus C_- := \{(0,0,Q,0,0,t) \,|\, Q \in (\mathfrak{P}^C)_{\sigma'_4}, t \in C\} \subset (({\mathfrak{e}_8}^C)^{\sigma'_4, \mathfrak{so}(6,C)})_{1_-}$
$\Rightarrow \exp(\mathrm{ad}(((\mathfrak{P}^C)_{\sigma'_4})_- \oplus C_-)) \subset (({E_8}^C)^{\sigma'_4, \mathfrak{so}(6,C)})_{1_-}$

• $\tilde{1} := (0,0,0,1,0,0),\ 1^- := (0,0,0,0,1,0),\ ({E_8}^C)_{\tilde{1},1^-,1_-} = {E_7}^C$
$\Rightarrow (({E_8}^C)^{\sigma'_4, \mathfrak{so}(6,C)})_{1_-} = \exp(\mathrm{ad}(((\mathfrak{P}^C)_{\sigma'_4})_- \oplus C_-))({E_7}^C)^{\sigma'_4, \mathfrak{so}(6,C)}$

• $1 \to \exp(\Theta(((\mathfrak{P}^C)_{\sigma'_4})_- \oplus C_-)) \to (({E_8}^C)^{\sigma'_4, \mathfrak{so}(6,C)})_{1_-} \underset{s}{\overset{p}{\rightleftarrows}} ({E_7}^C)^{\sigma'_4, \mathfrak{so}(6,C)} \to 1$: split exact sequence
$\Rightarrow (({E_8}^C)^{\sigma'_4, \mathfrak{so}(6,C)})_{1_-} = \exp(\mathrm{ad}(((\mathfrak{P}^C)_{\sigma'_4})_- \oplus C_-)) \rtimes ({E_7}^C)^{\sigma'_4, \mathfrak{so}(6,C)}$

• $\exp(\mathrm{ad}(((\mathfrak{P}^C)_{\sigma'_4})_- \oplus C_-)),\ ({E_7}^C)^{\sigma'_4, \mathfrak{so}(6,C)}$: 連結 $\Rightarrow (({E_8}^C)^{\sigma'_4, \mathfrak{so}(6,C)})_{1_-}$: 連結 □

24. 部分群 $(E_8)^{\sigma'_4}$ -8

▶ $(\mathfrak{W}^C)_{\sigma'_4,\mathfrak{so}(6,C)} := \left\{ R \in {\mathfrak{e}_8}^C \;\middle|\; \begin{array}{l} R \times R = 0, R \neq 0, \\ \sigma'_4 R = R, [R_D, R] = 0 \text{ for all } D \in \mathfrak{so}(6,C) \end{array} \right\}$

ここに，$R \times R : {\mathfrak{e}_8}^C \to {\mathfrak{e}_8}^C, (R \times R)R_1 = [R,[R,R_1]] + \dfrac{1}{30} B_8(R,R_1)R, R_1 \in {\mathfrak{e}_8}^C.$

■ $(({E_8}^C)^{\sigma'_4,\mathfrak{so}(6,C)})_0 \underset{\text{transitive}}{\curvearrowright} (\mathfrak{W}^C)_{\sigma'_4,\mathfrak{so}(6,C)} \ (\forall R \mapsto 1_-)$

$$\Downarrow$$

$$({E_8}^C)^{\sigma'_4,\mathfrak{so}(6,C)} / (({E_8}^C)^{\sigma'_4,\mathfrak{so}(6,C)})_{1_-} \simeq (\mathfrak{W}^C)_{\sigma'_4,\mathfrak{so}(6,C)}$$

$$\Rightarrow \ ({E_8}^C)^{\sigma'_4,\mathfrak{so}(6,C)}: \text{連結}$$

(参考) transitive

- $\alpha \in ({E_8}^C)^{\sigma'_4,\mathfrak{so}(6,C)}, R \in (\mathfrak{W}^C)_{\sigma'_4,\mathfrak{so}(6,C)} \Rightarrow \alpha R \in (\mathfrak{W}^C)_{\sigma'_4,\mathfrak{so}(6,C)}$
- 例えば，$R = (\Phi, P, Q, r, s, t) \in (\mathfrak{W}^C)_{\sigma'_4,\mathfrak{so}(6,C)}, t \neq 0$ に対して，$R \in (\mathfrak{W}^C)_{\sigma'_4,\mathfrak{so}(6,C)}$ である為の必要十分条件から

$$\Phi = -\frac{1}{2t} Q \times Q, \ \ P = \frac{r}{t} Q - \frac{1}{6t^2}(Q \times Q)Q, \ \ s = -\frac{r^2}{t} + \frac{1}{96t^3}\{Q, (Q \times Q)Q\}$$

を得て，$\Theta := \mathrm{ad}(0, P, 0, r, s, 0) \in \mathrm{ad}(({\mathfrak{e}_8}^C)^{\sigma'_4,\mathfrak{so}(6,C)}) \Rightarrow (\exp \Theta)1_- = \forall R \in (\mathfrak{W}^C)_{\sigma'_4,\mathfrak{so}(6,C)}$

25. 部分群 $(E_8)^{\sigma_4'}$ -9

▶ $(E_8)^{\sigma_4', \mathfrak{so}(6)} := \left\{\alpha \in (E_8)^{\sigma_4'} \,\middle|\, \Theta(R_D)\alpha = \alpha\Theta(R_D) \text{ for all } D \in \mathfrak{so}(6)\right\}$ (コンパクト-)

ここで，$(\mathfrak{e}_8)^{\sigma_4', \mathfrak{so}(6)}$: $(E_8)^{\sigma_4', \mathfrak{so}(6)}$ の Lie 環

■ $(\mathfrak{e}_8)^{\sigma_4', \mathfrak{so}(6)} \cong \mathfrak{so}(10)$，$({\mathfrak{e}_8}^C)^{\sigma_4', \mathfrak{so}(6,C)} \cong \mathfrak{so}(10, C)$

[証明のスケッチ] • $(\mathfrak{e}_8)^{\sigma_4', \mathfrak{so}(6)}$ の決定 (具体的な型).

• $\varphi_* : \mathfrak{so}(10) \longrightarrow (\mathfrak{e}_8)^{\sigma_4', \mathfrak{so}(6)}$，$\varphi_*(G_{ij}) = R_{ij}, 0 \le i < j \le 9$

例えば，次の様に Lie-homomorphism を基底の対応毎に確認する.

· G_{23} := (3, 4)-成分 1, (4, 3)-成分 -1$\in M(10, \boldsymbol{R})$, G_{45} := (5, 6)-成分 1, (6, 5)-成分 -1 $\in M(10, \boldsymbol{R})$

· $R_{23} := (\Phi(-i(E_1 \vee E_1), 0, 0, i), 0, 0, 0, 0, 0)$

· $R_{45} := \left(\Phi\left(i(E_1 \vee E_1), 0, 0, \dfrac{i}{2}\right), 0, 0, -\dfrac{i}{2}, 0, 0\right)$

$\Rightarrow \varphi_*([G_{23}, G_{45}]) = [\varphi_*(G_{23}, \varphi_*(G_{45}))]$

$\Rightarrow (\mathfrak{e}_8)^{\sigma_4', \mathfrak{so}(6)} \cong \mathfrak{so}(10)$

• $({\mathfrak{e}_8}^C)^{\sigma_4', \mathfrak{so}(6,C)}$: $(\mathfrak{e}_8)^{\sigma_4', \mathfrak{so}(6)}$ の複素化, $\mathfrak{so}(10, C)$: $\mathfrak{so}(10)$ の複素化

$\Rightarrow {\varphi_*}^C : \mathfrak{so}(10, C) \to ({\mathfrak{e}_8}^C)^{\sigma_4', \mathfrak{so}(6,C)}$，${\varphi_*}^C(D + iD') = \varphi_*(D) + i\varphi_*(D')$ □

⇓

$$({E_8}^C)^{\sigma_4', \mathfrak{so}(6,C)} \cong Spin(10, C)$$

[略証] • $({E_8}^C)^{\sigma_4', \mathfrak{so}(6,C)}$: 連結

• $({E_8}^C)^{\sigma_4', \mathfrak{so}(6,C)}$ の Lie 環の type: $\mathfrak{so}(10, C)$

• $Spin(10, C)$: 単連結, $z(Spin(10, C)) \cong \boldsymbol{Z}_4$

$\Rightarrow ({E_8}^C)^{\sigma_4', \mathfrak{so}(6,C)}$ は，次のどれかに同型である．

$$Spin(10, C), \quad SO(10, C), \quad Spin(10, C)/\boldsymbol{Z}_4$$

• $z\left(({E_8}^C)^{\sigma_4', \mathfrak{so}(6,C)}\right) \supset \{1, \sigma, \sigma_4', \sigma\sigma_4'\} \cong \boldsymbol{Z}_4$

$\Rightarrow ({E_8}^C)^{\sigma_4', \mathfrak{so}(6,C)} \cong Spin(10, C)$ □

これまでの結果を整理する．

★ $Spin(6, C) \cong ({F_4}^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1} \subset ({F_4}^C)^{\sigma_4'} \subset \cdots \subset ({E_8}^C)^{\sigma_4'}$

★ $Spin(10, C) \cong ({E_8}^C)^{\sigma_4', \mathfrak{so}(6,C)} \subset ({E_8}^C)^{\sigma_4'}$

- <u>証明</u>　• $\varphi : Spin(6, C) \times Spin(10, C) \to ({E_8}^C)^{\sigma'_4},\ \varphi(\alpha, \beta) = \alpha\beta$
  - well-defined 明らか
  - homomorphism　$[R_D, R_{10}] = 0, R_D \in \mathfrak{spin}(6, C), R_{10} \in \mathfrak{spin}(10, C) \implies \alpha\beta = \beta\alpha$
  - kernel　$\mathrm{Ker}\,\varphi_* = \{0\} \Rightarrow \mathrm{Ker}\,\varphi$: 離散 $\Rightarrow \mathrm{Ker}\,\varphi \subset z(Spin(6, C) \times Spin(10, C))$
    $= \{1, \sigma, \sigma'_4, \sigma\sigma'_4\} \times \{1, \sigma, \sigma'_4, \sigma\sigma'_4\}$
    $\Rightarrow \mathrm{Ker}\,\varphi = \{(1, 1), (\sigma'_4, \sigma\sigma'_4), (\sigma, \sigma), (\sigma\sigma'_4, \sigma'_4)\} \cong \boldsymbol{Z}_4$
  - surjective　$({E_8}^C)^{\sigma'_4}$: 連結, $\mathrm{Ker}\,\varphi$: 離散,
    $\dim_C(\mathfrak{so}(6, C) \oplus \mathfrak{so}(10, C)) = 15 + 45 = 60 = \dim_C(({\mathfrak{e}_8}^C)^{\sigma'_4})$

□

$\Downarrow$

- <u>主定理</u>

$$(E_8)^{\sigma'_4} \cong (Spin(6) \times Spin(10))/\boldsymbol{Z}_4,$$
$$\boldsymbol{Z}_4 = \left\{(1, 1), (\sigma_4, \sigma\sigma'_4), (\sigma, \sigma), (\sigma\sigma'_4, \sigma'_4)\right\}$$

(参考) $E_8 = ({E_8}^C)^{\tau\lambda\omega}$