代数曲線のアーベル函論の再構築と巡回型３次曲線の退化について

第２６回沼津改め静岡研究会ー幾何，数理物そして量子論ー

静岡大学　理学部　数学教室

松谷茂樹

National Institute of Technology, Sasebo College
Kanazawa University

joint work with Kazuhiko Aomoto, Yuri Fedorov, Jiryo Komeda and Emma Previato

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Menu

1. σ 函数の研究の目的
2. σ 函数と Weierstrass 正規形式の歴史
3. 楕円曲線の σ 函数
4. General affine curves：本日の対象
5. Sigma function of $y^3 = x(x-b_1)(x-b_2)(x-b_3)$ of $g=3$,
6. Sigma function of $y^3 = x^2(x-b_1)(x-b_2)$ of $g=2$,
7. Sigma function of $\lim_{s\to 0}\{y^3 = x(x-s)(x-b_2)(x-b_3)\}$

Purpose of study of sigma functions

目的

本研究の目的

種数 1 の代数曲線である楕円曲線のアーベル関数（楕円関数）が工学・理学の様々な分野で活躍し科学・科学技術を発展させたように，高次種数の複素代数曲線のアーベル関数の関数論を応用に向けて再構築し，諸科学・諸科学技術の発展につなげる

XIX		XX		XXI
Concrete	→	**Abstract**	→	**Unification**

History of sigma function & Weierstrass normal form

1. Abel 1829 (Parisian memoir 1826 (1841)):
   A curve X,
   $$y^n + p_{n-1}y^{n-1} + \cdots + p_3y^3 + p_2y^2 + p_1y + p_0 = 0, \qquad (1)$$
   where p_i is an entire function of x. and for its intersections with
   $$y^{n-1} + q_{n-2}y^{n-2} + \cdots + q_3y^3 + q_2y^2 + q_1y + q_0 = 0, \qquad (2)$$
   where q_i is an entire function of x, the Able(-Jacobi) theorem was proposed. Then its Algebraic and transcendental properties are related.

14.

Démonstration d'une propriété générale d'une certaine classe de fonctions transcendentes.

(Par Mr. *N. H. Abel.*)

Théorème. Soit y une fonction de x qui satisfait à une équation quelconque irréductible de la forme:

1. $0 = p_0 + p_1 \cdot y + p_2 \cdot y^2 + \cdots + p_{n-1} \cdot y^{n-1} + y^n$,

où $p_0, p_1, p_2, \ldots p_{n-1}$ sont des fonctions entières de la variable x. Soit

2. $0 = q_0 + q_1 \cdot y + q_2 \cdot y^2 + \cdots + q_{n-1} \cdot y^{n-1}$,

History of sigma function

1. 1832-44 **Jacobi:(Jacobi inversion problem)**: Posing "inverse functions of Abelian integrals for curves of genus two".
2. 1840 **Weierstrass(1815-1897)**:(publication in 1889): Study on **Al (σ) functions of elliptic curves**
3. 1854 **Weierstrass**: Study on **Al (σ) functions of hyperelliptic curves of general genus** and Jacobi inversion problem.
4. 1856 **Riemann**: Construction of Abelian functions for general compact Riemann surfaces.
5. 1882 **Weierstrass**: **Renaming his Al function σ function** of genus one by refining it.
6. 1886 年:**Klein**: Refining Weierstrass' Al function of hyperelliptic curves and calling it **hyperelliptic sigma function**.
7. 1903 **Baker** (Hodge's supervisor): Discovery of **KdV and KP hierarchy** using **bilinear form**. Posing the problem: to find **whether KdV hierarchy characterizes the sigma functions**.

History of sigma function

1. **Nonlinear integrable differential equations** had been studied 1950-1990.
2. Krichever, Novikov, Mumford and so on **rediscovered** the algebro-geometric solutions of the KdV and KP hierarchies 1970's.
3. **Hirota** rediscovered the bilinear operator and bilinear equations and developed the study of the bilinear equations.
4. Sato constructed **Sato-theory of UGM** around 1980.
5. Novikov gave a conjecture that the KP hierarchy characterizes the Jacobi varieties in the Abelian varieties, which is closely related to Baker's problem.
6. **Mulase and Shiota proved Novikov's conjecture 1987**. **It means the settlement of Baker's problem;** in the settlement, Baker's theory on the differential equations plays an important role, which was prepared for his problem.

Contemporary sigma function: 1990-

1. 1990: **Grant(Number theory)**: Sigma function of genus two: Linear dependence relation of differentials of σ / σ^ℓ as $\mathcal{O}(\mathcal{J}, n\Theta)$.
2. 1995- **Yoshihiro Ônishi (Meijo Univ. Number theory)**: Structure of Jacobian and addition structures of its strata using hyperelliptic σ function.
3. 1997- **Buchsterber-Enolskii-Leykin**: Investigation on the **integrable system** using hyperelliptic σ function.
4. 2000 **Eilbeck-Enolskii-Leykin (EEL)**: Generalization of sigma function of hyperelliptic curve to (n, s)-**type plan curve** $(y^n + x^s + ...)$
5. 2008-: **Nakayashiki (Tsudajuku univ.)**: Show the expression of sigma functions of (n, s) in terms of **Fay's results and tau functions of Sato theory**.
6. 1997- **Previato, Onishi, Enolskii, Eilbeck, Gibbons, Kodama, M**, Reconstruction of the Abelian function theory
7. 2010- **Previato, Komeda, M: Its extension to space curves.**

History of Weierstrass normal form and Numerical semigroup

1. Abel 1829
2. 1860's(?) **Weierstrass**: Über Normalformen algebraischer Grebilde" Werke III,
3. 1856- Weierstrass School: **Frobenius, Schwarz, and so on** studied the numerical semigroup, and Weierstrass normal form.
4. 1888 **Hurwitz**: **Hurwitz problem: It is known that the non-gap sequence of Weierstrass point is a numerical semigroup. Hurwitz problem is a problem whether a curve exists such that is non-gap sequence is H for every numerical semigroup H.**
5. 1889 **Baker**: Review of **Weierstrass canonical form and non gap sequence**.

History of Weierstrass normal form and Numerical semigroup

1. 1980 **Buchweitz and Greuel,** : Counterexample of **Hurwitz problem**.
2. 1980- **J. Komeda**:: **Relations between numerical semigroups and curves in general**
3. 1980 **M. Kato Weierstrass normal form and Automorphism of theta function.**

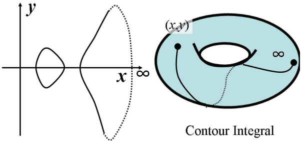
Review of elliptic sigma function

楕円の σ 関数の Review

σ of genus one

Elliptic curve

$$X_1 := \left\{ (x,y) \;\middle|\; \begin{aligned} y^2 &= x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ &= (x-e_1)(x-e_2)(x-e_3) \end{aligned} \right\} \cup \infty.$$



Commutative Ring

$$R = \mathbb{C}[x,y]/(y^2 - x^3 - \lambda_2 x^2 - \lambda_1 x - \lambda_0)$$

Differentials of the first kind (holo. one-from)

$$du := \nu^{\mathrm{I}} := \frac{dx}{2y},$$

Differential of the second kind: $\mathcal{H}^1(X_1 \setminus \{\infty\})$

$$\nu^{\mathrm{II}} := \frac{xdx}{2y},$$

σ of genus one: Abelian map & Period

Covering space of X_1

$\widetilde{X}_1$: Abelian covering of X_1 with fixed point ∞

(Abelianization of quotient space of path space of X_1)

$$\kappa_X : \widetilde{X}_1 \to X_1 \quad \iota_X : X_1 \hookrightarrow \widetilde{X}_1$$

Abel integral (Elliptic incomplete integral of 1st kind)

$$\widetilde{w} : \widetilde{X}_1 \to \mathbb{C}\text{:}\ \ u = \widetilde{w}(x, y) \equiv \int_{\gamma_{(x,y),\infty}} \nu^{\mathrm{I}},\ \ \nu^{\mathrm{I}} = \frac{dx}{2y},$$

Elliptic Jacobi variety & Legendre's relation

Branched point integrals (Elliptic complete integrals of 1st kind)

$$\omega_i := w(e_i, 0) = \int_{\infty}^{(e_i,0)} \nu^{\mathrm{I}}, \quad \nu^{\mathrm{I}} := \frac{dx}{2y},$$

where $(e_i, 0)$ $(i = 1, 2, 3)$ and ∞ are branch pints

Homology basis

$$H_1(X_1, \mathbb{Z}) = \mathbb{Z}\alpha + \mathbb{Z}\beta.$$

Period Matrices

$$\int_{\alpha} \nu^{\mathrm{I}} = 2\omega_1, \quad \int_{\beta} \nu^{\mathrm{I}} = 2\omega_3.$$

Elliptic Jacobi variety & Legendre's relation

Lattice

$$\Gamma = 2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_3 \subset \mathbb{C}. \quad \Gamma^\circ = \mathbb{Z} + \mathbb{Z}\tau, \quad (\tau = \omega_3/\omega_1)$$

Jacobi variety

$$\kappa_{\mathcal{J}} : \mathbb{C} \to \mathcal{J} = \mathbb{C}/\Gamma, \quad \kappa_{\mathcal{J}}^\circ : \mathbb{C} \to \mathcal{J}^\circ = \mathbb{C}/\Gamma^\circ,$$

Jacobi variety

$$\iota_{\mathcal{J}} : \mathcal{J} \to \mathbb{C}, \quad \iota_{\mathcal{J}}^\circ : \mathcal{J}^\circ \to \mathbb{C},$$

Elliptic Jacobi variety & Legendre's relation

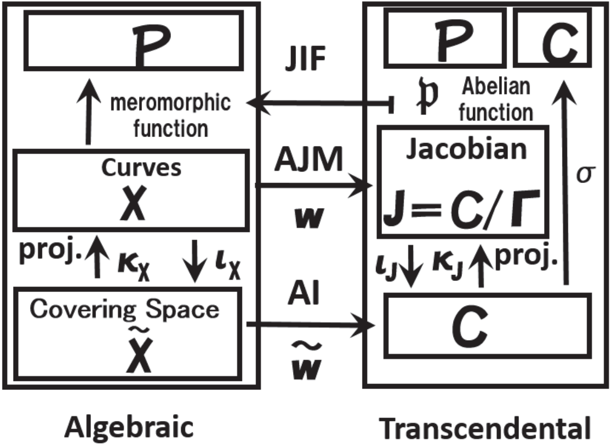
Complete Elliptic integral of 2nd kind

$$\eta_i := \int_{\infty}^{(e_i,0)} \nu^{\mathrm{II}}, \quad \nu^{\mathrm{II}} := \frac{xdx}{2y},$$

Legendre's relation: (Symplectic Str.($\approx$ Hodge Str.))

$$\omega_3\eta_1 - \omega_1\eta_3 = \frac{\pi}{2}\sqrt{-1}.$$

The Roles of Abel-Jacobi Map



$$\text{AJM} : w : X_1 \to \mathcal{J} \quad w := \kappa_{\mathcal{J}} \circ \widetilde{w} \circ \iota_X$$

The Jacobi inversion formulae

Algebraic space and Analytic space

Natural Projection

Covering space $\tilde{X}_1 =$ Abelianization of quotient space of Path space of X_1

$\kappa_X : \widetilde{X}_1 \to X_1$

$\widetilde{X}_1 \supset LX_1(\infty) \cong \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$

SL(2,$\mathbb{Z}$) $\curvearrowright$ this expression

Natural Projection

Covering space $\mathbb{C} =$ Abelianization of quotient space of Path space of $\mathcal{J}$

$\kappa_{\mathcal{J}} : \mathbb{C} \to \mathcal{J} = \mathbb{C}/\Gamma$

$\Gamma = \mathbb{Z}(2\omega') + \mathbb{Z}(2\omega'')$

SL(2,$\mathbb{Z}$) $\curvearrowright$ Γ, $\mathbb{C}$

In order to obtain Meromorphic functions over X_1 and $\mathcal{J}$, the entire function of $\mathbb{C}$ should be invariant for SL(2,$\mathbb{Z}$).

σ is modular invariant for SL(2,$\mathbb{Z}$)

Elliptic Weierstrass' σ function

Weierstrass' σ function over $\mathbb{C}$ as an entire function:

$$\sigma(u) = 2\omega_1 \exp\left(\frac{\eta_1 u^2}{2\omega_1}\right) \frac{\theta_1(\frac{u}{2\omega_1})}{\theta_1'^0}$$

Translation formula: $\Omega_{m,n} := 2m\omega_1 + 2n\omega_3$:

$$\sigma(u + \Omega_{m,n}) = (-1)^{m+n+mn} \exp\left((m\eta_1 + n\eta_3)(2u + \Omega_{m,n})\right) \sigma(u).$$

Zero of σ function:

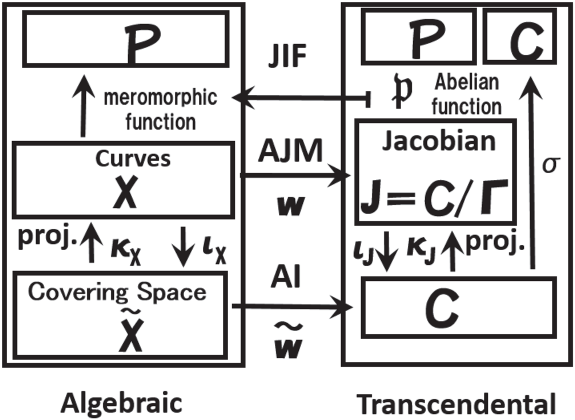
$$\{\text{zeros of } \sigma\} \equiv 0 \text{ mod. } \Gamma, \quad \sigma(u) = u + \cdots .$$

$\mathrm{SL}(2,\mathbb{Z})$

Modular invariance for $\mathrm{SL}(2,\mathbb{Z})$.

The Roles of AI, AJM and Jacobi inversion formulae

Algebraic space and Analytic space



$$\mathcal{L}_{X_1} \cong \mathbb{C}(\sigma \circ \widetilde{w} \circ \iota_X), \ \mathcal{L}_{\mathcal{J}} = \mathbb{C}\sigma \circ \iota_{\mathcal{J}}$$

Elliptic Weierstrass' $\wp$ function / $\mathcal{J} = \mathbb{C}/\Gamma$

$$\zeta(u) = \frac{d}{du}\log\sigma(u). \qquad \wp(u) = -\frac{d^2}{du^2}\log\sigma(u).$$

Theorem (Jacobi inversion formula): for $u = w(x, y)$

$$(x, y) = (\wp(u), \wp_u(u)), \quad \wp_u(u) := \frac{d}{du}\wp(u).$$

JIF recovers the governing equation $y^2 = x^3 + \cdots$

JIF recovers the governing equation $y^2 = x^3 + \cdots$, i.e.,

$$\wp_u(u)^2 = \wp(u)^3 + \lambda_2\wp(u)^2 + \lambda_1\wp(u) + \lambda_0.$$

Elliptic Weierstrass' al function

Jacobi elliptic function $u = \widetilde{w}(x, y)$

$$\mathrm{sn}(u) = \sqrt{\frac{e_1 - e_3}{\wp(u) - e_3}}, \quad \mathrm{cn}(u) = \sqrt{\frac{\wp(u) - e_1}{\wp(u) - e_3}}, \quad \mathrm{dn}(u) = \sqrt{\frac{\wp(u) - e_2}{\wp(u) - e_3}},$$

The al-functions $u = \widetilde{w}(x, y)$

$$\mathrm{al}_r(u) = c_r \sqrt{x(u) - e_r}, \quad (r = 1, 2, 3)$$

$$\boxed{\mathrm{al}_r(u) \sim t \text{ at } (e_r, 0), \ (t \text{ local parameter at } (e_r, 0))}$$

$$\mathrm{al}_r(u) = c_r' \frac{\mathrm{e}^{\eta_r u} \sigma(u + \omega_r)}{\sigma(u)}$$

Generalization
Curves of this talk

Weierstrass curves:

Weierstrass curve is an affine curve as a pointed compact Riemann surface (X, ∞) whose Weierstrass non-gap sequence at ∞ is given by numerical semigroup (NSG)
Numerical semigroup: $H(X,\infty) = \mathbb{N}_0 r_1 + \mathbb{N}_0 r_2 + \cdots + \mathbb{N}_0 r_\ell \subset \mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ $(g := \# N_0 \setminus H(X,\infty) < \infty)$
Weierstrass curve is given by the normalization of **the Weierstrass normal form (Abel, Weierstrass, Baker, Kato)** :

$$\boxed{y^r + \lambda_{r-1}(x) y^{r-1} + \cdots + \lambda_0(x) = 0}$$

$$\left(\lambda_i \in \mathbb{C}[x] \text{ of } \left\lceil \frac{(r-i)s}{r} \right\rceil \text{-th polynomial, } (r,s) = 1, r < s\right)$$

Since $X_{\rm sing}$ of WNF is singular in general,
We normalize it $X \to X_{\rm sing}$.

Curves

$$\{\text{compact Rie. surface}\}/\sim$$
$$\cong \{X : \text{WNF}\}$$
$$\sim: \text{birational}$$

Today's Curves:

1) $y^3 = x(x - s)(x - b_1)(x - b_2)$
2) $y^3 = x^2(x - b_1)(x - b_2)$

Weierstrass curves:

$$\begin{array}{ccc} X & \xrightarrow{\pi_1} & \mathbb{P} \\ \downarrow{\scriptstyle \pi_2} & & \\ \mathbb{P} & & \end{array}$$

$$\pi_1(P) = x, \quad \pi_2(P) = y$$

$$y^3 = x(x-s)(x-b_1)(x-b_2)$$

A Family of Degenerating curve: X_s

$$X_s := \{(x,y) \mid y^3 = x(x-s)(x-b_1)(x-b_2)\} \cup \{\infty\}$$

1. Let ε be $0 < \varepsilon < \min\{|b_1|, |b_2|\}$ and $b_1 \neq b_2$.
2. $D_\varepsilon := \{s \in \mathbb{C} \mid |s| < \varepsilon\}$
3. Family of degenerating curves
   $Z := \{(x,y,s) \mid (x,y) \in X_s, s \in D_\varepsilon\}$ as a fibering

$$\begin{array}{ccc} X_s & & \\ \downarrow & & \\ \{s\} & \subset & D_\varepsilon \end{array}$$

A Family of Degenerating curve: X_s

$$X_s := \{(x, y) \mid y^3 = x(x - s)(x - b_1)(x - b_2)\} \cup \{\infty\}$$

1. Let ε be $0 < \varepsilon < \min\{|b_1|, |b_2|\}$ and $b_1 \neq b_2$.
2. $D_\varepsilon := \{s \in \mathbb{C} \mid |s| < \varepsilon\}$, $\boxed{D^*_\varepsilon := D_\varepsilon \setminus \{0\}}$,
3. A non-degenerating curve X_s
   $Z := \{(x, y, s) \,|(x, y) \in X_s, s \in D_\varepsilon\}$ as a fibering

$$\begin{array}{ccccc} X_s & & & & \\ \downarrow & & & & \\ \{s\} & \subset & D^*_\varepsilon & \subset & D_\varepsilon \end{array}$$

A Family of Degenerating curve: X_s

X_s of $s \in D_\varepsilon^*$ $(s \neq 0)$

$$X_s := \{(x,y) \mid y^3 = x(x-s)(x-b_1)(x-b_2)\} \cup \{\infty\}$$

∞ as a branch point: $x = \dfrac{1}{t^3}, \quad y = \dfrac{1}{t^4}(1 + t^{\geq 1})$

t : local parameter at ∞,

Weight $\mathrm{wt}(x) = 3$, $\mathrm{wt}(y) = 4$

Affine ring: $R_s := \mathbb{C}[x,y]/(y^3 - x(x-s)(x-b_1)(x-b_2))$

A Family of Degenerating curve: X_s

R_s of $s \in D_\varepsilon^*$ $(s \neq 0)$, $(3,4)$

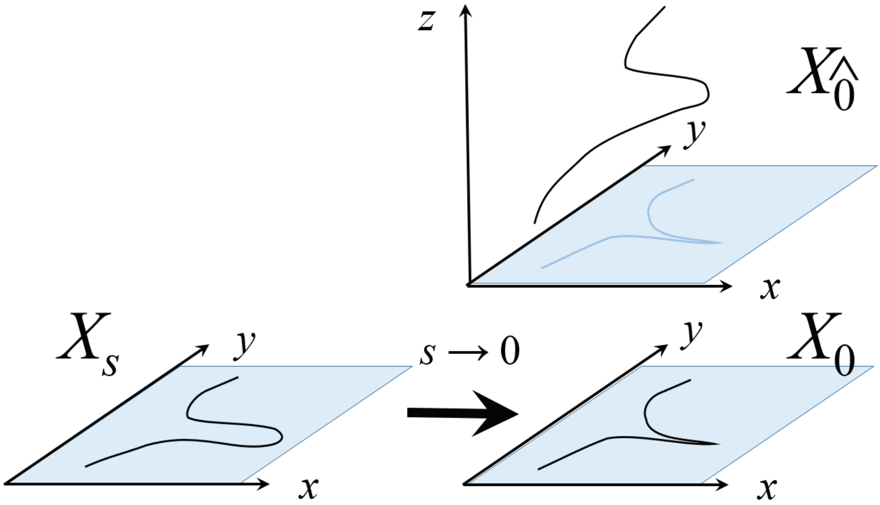
$R_s := \mathbb{C}[x,y]/(y^3 - x(x-s)(x-b_1)(x-b_2))$,
$R_s = \mathcal{H}^0(X_s, \mathcal{O}(*\infty))$ with $\mathrm{wt}(x) = 3$, $\mathrm{wt}(y) = 4$

Numerical semigroup:
$H(X,\infty) = \langle 3,4 \rangle = \mathbb{N}_0 3 + \mathbb{N}_0 4$
$R_s = \bigoplus_{n=0}^{\infty} \mathbb{C}\phi_i$, as $\mathbb{C}$-vector space
(Non-gap sequennce = order of the weight)

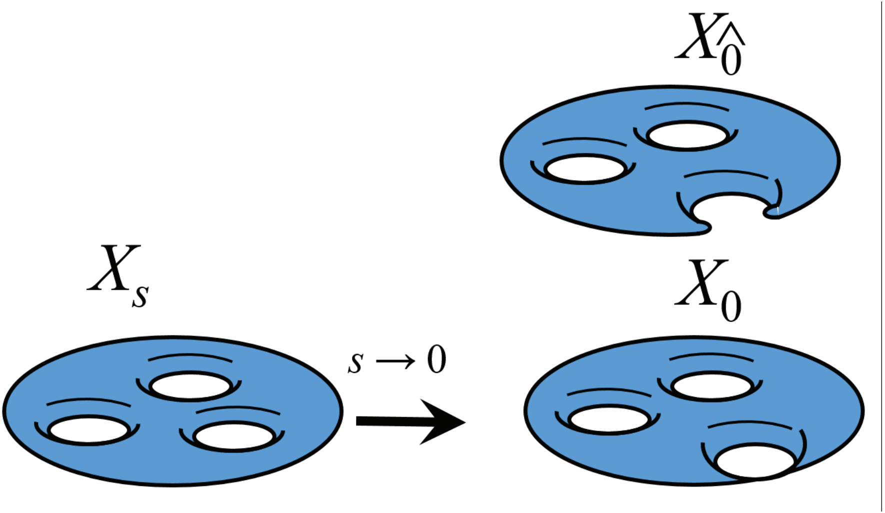
ϕ	0	1	2	3	4	5	6	7	8	9	10	11
X_s	1	-	-	x	y	-	x^2	xy	y^2	x^3	x^2y	xy^2
$g=1$	1	-	x	y	x^2	xy	x^3	x^2y	x^4	x^3y	x^5	x^4y

$(\to$ genus $g = 3)$

A Family of Degenerating curves: X_s



A Family of Degenerating curves: X_s



A Family of Degenerating curves: X_s

X_s of $(s=0)$: a singular curve

$$X_0 := \{(x,y) \mid y^3 = x^2(x-b_1)(x-b_2)\} \cup \{\infty\}$$

X_0 is singular $\Rightarrow$ $X_{\hat{0}}$ normalization of X_0:

$$X_{\hat{0}} := \{(x,y,z) \mid (y^2 - xz, zy - x, z^2 - (x-b_1)(x-b_2)y\} \cup \{\infty\}$$

$$X_{\hat{0}}:\ R_{\hat{0}} := \mathbb{C}[x,y,z] \Big/ (y^2 - xz, zy - x, z^2 - (x-b_1)(x-b_2)y)$$

At ∞: $x = \dfrac{1}{t^3}$, $y = \dfrac{1}{t^4}(1+t^{\geq 1})$, $z = \dfrac{1}{t^5}(1+t^{\geq 1})$,

Weight $\mathrm{wt}(x) = 3$, $\mathrm{wt}(y) = 4$, $\mathrm{wt}(z) = 5$.

A Family of Degenerating curves: X_s

X_s of $(s = 0)$: $(3, 4, 5)$-curve

$R_{\hat{0}} = H^0(X_{\hat{0}}, \mathcal{O}(*\infty))$, commutative ring

Numerical semigroup:

$$H(X_{\hat{0}}, \infty) = \langle 3, 4, 5 \rangle = \mathbb{N}_0 3 + \mathbb{N}_0 4 + \mathbb{N}_0 5$$

$$R_{\hat{0}} = H^0(X_{\hat{0}}, \mathcal{O}(*\infty)), \qquad R_{\hat{0}} = \oplus_{i=0} \mathbb{C}\phi_i$$

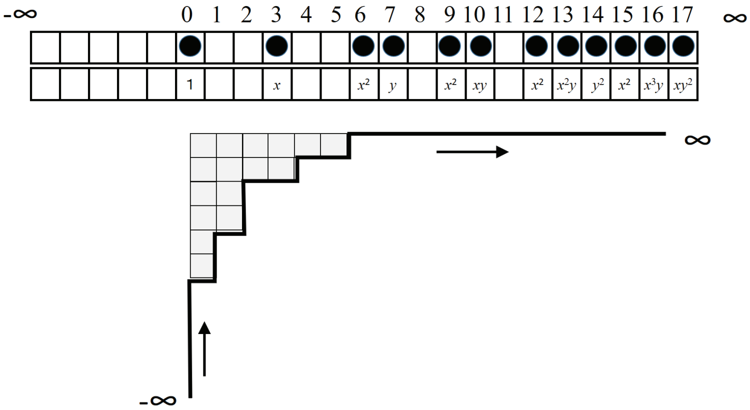
ϕ_i	0	1	2	3	4	5	6	7	8	9	10
$X_{\hat{0}}$	1	-	-	x	y	z	x^2	xy	xz	yz	x^2y

$\to g = 2$

Pointed compact Riemannian surface

1) For a gap, upward by a box,
2) For a non-gap, step to the right by a box:

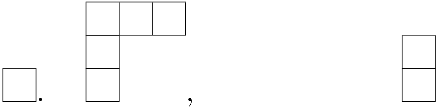
(3,7)-curve



Pointed compact Riemannian surface

Weierstrass curves (X, P)

1) For a gap, upward by a box,
2) For a non-gap, step to the right by a box:



$g = 1$ $\quad X_s \quad\; {}^tY = Y$ $\quad X_{\hat{0}} \quad\; {}^tY \neq Y$

NSG $H(X, \infty)$ **of** (n, s)**-curves are symmetric, whereas** $(3, 4, 5)$ **curve is not.**

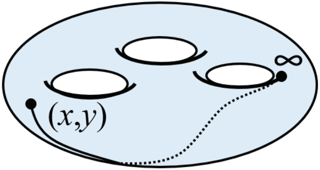
Sigma function of

$$y^3 = x(x-s)(x-b_1)(x-b_2):$$

$$s \in D_\varepsilon^*, \ (s \neq 0)$$

Cyclic trigonal curve (A non-singular plane curve of $g = 3$)

$$X_s := \left\{ (x, y) \;\middle|\; \begin{aligned} y^3 &= x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ &= x(x-s)(x-b_1)(x-b_2) \end{aligned} \right\} \cup \{\infty\}.$$



$$R_s = \mathbb{C}[x, y]/(y^3 - x^4 - \cdots - \lambda_0).$$

Weierstrass gap at ∞

Weierstrass gap at ∞

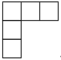
$$wt(x) = 3, \ wt(y) = 4,$$
$$\phi_0 = 1, \ \phi_1 = x, \ \phi_2 = y, \ \phi_3 = x^2, \ \cdots$$

wt	0	1	2	3	4	5	6	7	8	9	10
ϕ	1	-	-	x	y	-	x^2	xy	y^2	x^3	x^2y

$R_s = \oplus_{i=0}\mathbb{C}\phi_i$ weighted $\mathbb{C}$-vector space,

Numerical Semigroup $H := \{3a + 4b\}_{a,b\in\mathbb{Z}_{\geq 0}} := \langle 3, 4\rangle$

$$H = \{0, 3, 4, 6, 7, \cdots\}, \quad L = \mathbb{Z} \setminus H = \{1, 2, 5\}.$$



Differentials

differentials of the 1st kind(= canonical form)

$$\nu^I_{X_s} := \begin{pmatrix} \nu^I_{X_s1} \\ \nu^I_{X_s2} \\ \nu^I_{X_s3} \end{pmatrix} := \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} := {}^t\left(\frac{dx}{3y^2}, \frac{xdx}{3y^2}, \frac{dx}{3y}\right) = {}^t\left(\frac{\phi_0 dx}{3y^2}, \frac{\phi_1 dx}{3y^2}, \frac{\phi_2 dx}{3y^2}\right)$$

$\mathrm{Div}(\nu^I_{X_s i}) \sim (2g-2)\infty$ as a linear equivalent ($g = 3$).

differentials of the 2nd kind(holo. except ∞)

$$\nu^{II}_{X_s} := \begin{pmatrix} \nu^{II}_{X_s1} \\ \nu^{II}_{X_s2} \\ \nu^{II}_{X_s3} \end{pmatrix},$$

$$\nu^{II}_{X_s1} = -\frac{5x^2 + 3\lambda_3 x + \lambda_2}{3y}, \quad \nu^{II}_{X_s2} = -\frac{2x}{3y}, \quad , \nu^{II}_{X_s3} = -\frac{x^2}{3y^2},$$

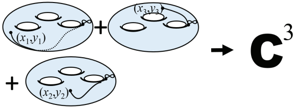
AI (Abelian Integral)

$\widetilde{X}_s$: Abelianization of quotient space of Path(X_s)

$$\widetilde{w}_{X_s} : \widetilde{X}_s \to \mathbb{C}^3; \quad \left(w_{X_s}(P) = \int_\infty^P \nu^I_{X_s} = \int_\infty^P \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} \right)$$

$$\widetilde{w}_{X_s} : S^3(\widetilde{X}_s) \to \mathbb{C}^3; \quad \widetilde{w}_{X_s}(P_1, P_2, P_3) := \widetilde{w}_{X_s}(P_1) + \widetilde{w}_{X_s}(P_2) + \widetilde{w}_{X_s}(P_3).$$

$S^3(X_s) := X_s \times X_s \times X_s / \sim$:
Symmetric product



Legendre relation of cyclic trigonal curve

Homology basis: $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$

$$\langle \alpha_i, \beta_j \rangle = \delta_{ij}, \quad \langle \alpha_i, \alpha_j \rangle = 0, \quad \langle \beta_i, \beta_j \rangle = 0,$$

Half period

$$(\omega'_{X_s ij}) := \frac{1}{2}\left(\int_{\alpha_i} \nu^{I}_{X_s j}\right),$$

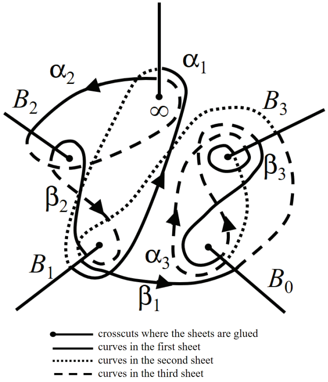
$$(\omega''_{X_s ij}) := \frac{1}{2}\left(\int_{\beta_i} \nu^{I}_{X_s j}\right)$$

Abelian integrals of the 2nd kind

$$(\eta'_{X_s ij}) := \frac{1}{2}\left(\int_{\alpha_i} \nu^{II}_{X_s j}\right),$$

$$(\eta''_{X_s ij}) := \frac{1}{2}\left(\int_{\beta_i} \nu^{II}_{X_s j}\right)$$

$y^3 = x^4 + \lambda_3 x^3 + \cdots + \lambda_0$



Legendre relation of non-hyperelliptic curve

For $\omega_{X_s a} := \int_{\infty}^{B_a} \nu^I_{X_s}, (a = 0, 1, 2, 3)$

$$\omega'_{X_s 1} = \frac{1}{2}\hat{\zeta}_3(1 - \hat{\zeta}_3^2)\omega_{X_s 1}, \quad \omega'_{X_s 2} = \frac{1}{2}(1 - \hat{\zeta}_3^2)\omega_{X_s 2},$$

$$\omega'_{X_s 3} = \frac{1}{2}\Big((1 - \hat{\zeta}_3^2)\omega_{X_s 0} + \hat{\zeta}_3^2(1 - \hat{\zeta}_3)\omega_{X_s 3}\Big)$$

$$\omega''_{X_s 1} = \frac{1}{2}(\hat{\zeta}_3(1 - \hat{\zeta}_3^2)\omega_{X_s 1} + (1 - \hat{\zeta}_3^2)\omega_{X_s 0} + \hat{\zeta}_3^2(1 - \hat{\zeta}_3^2)\omega_{X_s 3})$$

$$\omega''_{X_s 2} = \frac{1}{2}(\hat{\zeta}_3 - 1)(\omega_{X_s 1} - \omega_{X_s 2}), \quad \omega''_{X_s 3} = \frac{1}{2}(\hat{\zeta}_3 - 1)(\omega_{X_s 0} - \omega_{X_s 3})$$

Lattice

$$\Gamma_s := \langle \omega'_{X_s}, \omega''_{X_s} \rangle_{\mathbb{Z}} = \left\{ \omega'_{X_s} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \omega''_{X_s} \begin{pmatrix} a_4 \\ a_5 \\ a_6 \end{pmatrix} \middle| a_i \in 2\mathbb{Z} \right\} \subset \mathbb{C}^3$$

Jacobi variety

$$\kappa_{\mathcal{J}} : \mathbb{C}^3 \to \mathcal{J}_s = \mathbb{C}^3/\Gamma_s, \quad \iota_{\mathcal{J}} : \mathcal{J}_s \to \mathbb{C}^3$$

Legendre relation (Symplectic Str.($\approx$ Hodge Str.))

$${}^t\omega'_{X_s}\eta''_{X_s} - {}^t\omega''_{X_s}\eta'_{X_s} = \frac{\pi}{2} I_3.$$

Abel-Jacobi map

$$w = \kappa_{\mathcal{J}} \circ \widetilde{w} \circ \iota_X : X_s \to \mathcal{J}_s$$

The σ function of X_s

σ function $u \in \mathbb{C}^3$

$$\sigma_{X_s}(u) := c_0 \mathrm{e}^{-\frac{1}{2} u^t {\omega'_{X_s}}^{-1} {}^t\eta'_{X_s} u} \theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \left(\frac{1}{2} {\omega'_{X_s}}^{-1} u ; {\omega'_{X_s}}^{-1} \omega''_{X_s} \right).$$

$$c_0 := \left(\frac{(2\pi)^g}{\det \omega'_{X_s}} \right)^{1/2} \Delta^{-1/8} \quad \Delta : \text{discriminant}$$

$$\delta' \in (\mathbb{Z}/2)^3, \quad \delta'' \in (\mathbb{Z}/2)^3, \quad \Leftrightarrow \quad \textbf{Riemann const. } \xi$$

θ is **the Riemann theta function**,

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z;\tau) = \sum_{n \in \mathbb{Z}^2} \exp \left(\pi \sqrt{-1} ((n+a)^t \tau (n+a) - (n+a)^t (z+b)) \right).$$

σ_{X_s} function

Translational formula: $u,\ v \in \mathbb{C}^3$, and $\ell\ (= 2\omega'_{X_s}\ell' + 2\omega''_{X_s}\ell'') \in \Lambda$

$$L(u,v) := 2\ {}^t u(\eta'_{X_s} v' + \eta''_{X_s} v''),$$

$$\chi(\ell) := \exp[\pi\sqrt{-1}\big(2({}^t\ell'\delta'' - {}^t\ell''\delta') + {}^t\ell'\ell''\big)]\ (\in \{1, -1\})$$

$$\Rightarrow \quad \sigma_{X_s}(u+\ell) = \sigma_{X_s}(u)\exp(L(u+\frac{1}{2}\ell,\ell))\chi(\ell).$$

σ_{X_s} function

Properties of σ_{X_s}

1. Entire function over $\mathbb{C}^3$
2. Zeros: $\{\mathrm{div}\sigma_{X_s}\} \equiv \Theta = w_{X_s}(\mathcal{S}^2 X_s)$
3. Modular invariant for $\mathrm{Sp}(3,\mathbb{Z})$
4. Expansion $\sigma_{X_s}(u) = s_{\yng(3,1,1)} + \cdots$, $s_{\yng(3,1,1)}$: Schur polynomial of $\yng(3,1,1)$: $\sigma_{X_s}(u) = u_1 - u_2^2 u_3 + \frac{6}{5!} u_3^5 + \cdots$.
5. Expansion of $\sigma_{X_s}(u)$ is explicitly determined by UGM.

Theorem: (JIF): $S^3X_s \supset S^2X_s \supset X_s \iff \mathcal{J} \supset \Theta^{[2]} \supset \Theta^{[1]}$

1) For $u = w((x_1, y_1), (x_2, y_2), (x_3, y_3))$ and $\wp_{ij} := \dfrac{\partial^2}{\partial u_i \partial u_j} \log \sigma_{X_s}$,

$$\frac{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}} = \wp_{33}(u), \quad \frac{\begin{vmatrix} 1 & y_1 & x_1^2 \\ 1 & y_2 & x_2^2 \\ 1 & y_3 & x_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}} = \wp_{23}(u), \quad \frac{\begin{vmatrix} x_1 & y_1 & x_1^2 \\ x_2 & y_2 & x_2^2 \\ x_3 & y_3 & x_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}} = \wp_{13}(u),$$

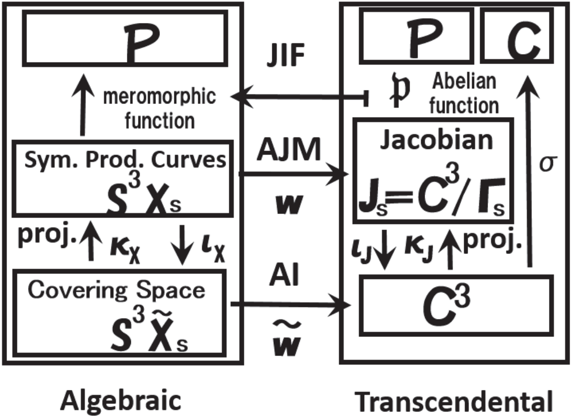
2) For $u = w((x_1, y_1), (x_2, y_2))$

$$\frac{\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}} = \frac{\sigma_2(u)}{\sigma_3(u)}, \quad \frac{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}} = \frac{\sigma_1(u)}{\sigma_3(u)},$$

3) For $u = w((x, y))$

$$x = \frac{\sigma_1(u)}{\sigma_2(u)}$$

The Roles of AI, AJM and Jacobi inversion formulae



$$\mathcal{L}_{\mathcal{S}^3 X_s} \cong \mathbb{C}(\sigma_{X_s} \circ \widetilde{w} \circ \iota_X), \ \mathcal{L}_{\mathcal{J}_s} = \mathbb{C}\sigma_{X_s} \circ \iota_{\mathcal{J}}$$

Geometric roles of AJM & JIF

Generalization of Jacobi's sn function (M-Previato 2016)

For B_a and $P_i = (x_i, y_i)$ $(i = 1, 2, 3)$ of X and $\varphi_{a;c} \in \Gamma$

$$\mathrm{al}_a^{(c)}(u) = \frac{\mathrm{e}^{-{}^t u \varphi_{a;0}} \sigma(u + \omega_a)}{\sigma(u)\sigma_{33}(\omega_a)} = \frac{\zeta_3^c A_a}{\sqrt[3]{(b_a - x_1)(b_a - x_2)(b_a - x_3)}}$$

$$A_a(P_1, P_2, P_3) := \frac{\begin{vmatrix} 1 & x_1 & y_1 & {x_1}^2 \\ 1 & x_2 & y_2 & {x_2}^2 \\ 1 & x_3 & y_3 & {x_3}^2 \\ 1 & b_a & 0 & b_a^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}, \quad \sigma_{33}(\omega_a) = \sqrt{2}/\sqrt[3]{f'(b_a)}$$

$\mathrm{al}_a^{(c)}(u) \sim t_1 t_2 t_3 + \ldots$ t_a local param. at B_a

sigma function of $y^3 = x^2(x-b_1)(x-b_2)$

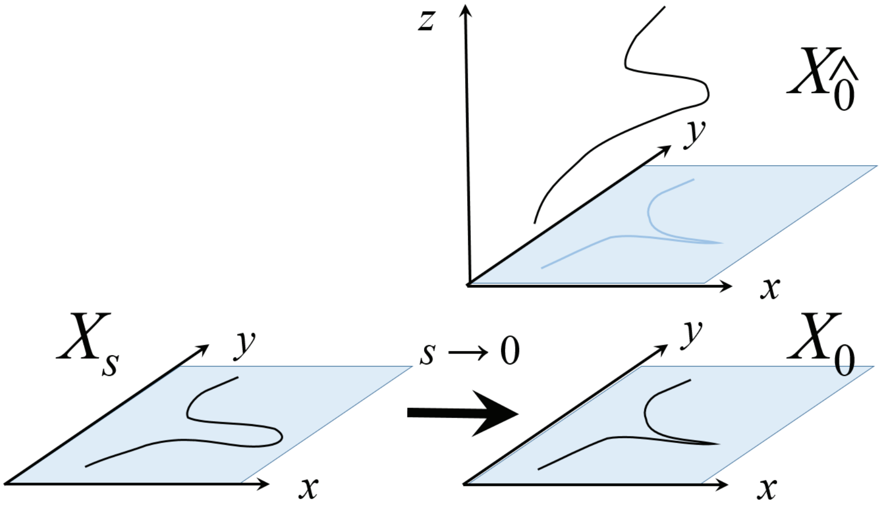
Sigma function of

$$y^3 = x^2(x-b_1)(x-b_2)$$

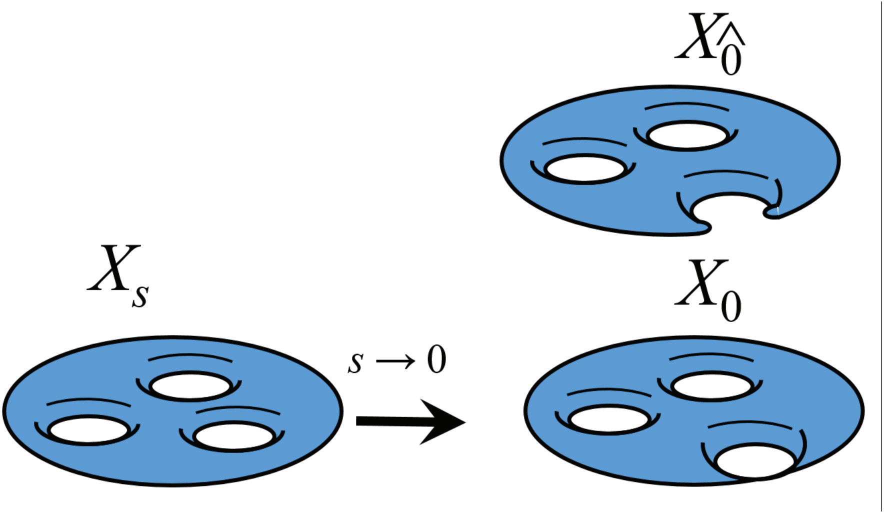
a normalized trigonal space curve of genus two

$$\begin{array}{ccc} X_0 & & \\ \downarrow & & \\ \{0\} & \subset & D_\varepsilon \end{array}$$

A Family of Degenerating curve: X_s



A Family of Degenerating curve: X_s



Singular curve $X_{s=0}$

Let us consider the singular curve

$$X_{s=0} \quad : \quad y^3 = x^2 k(x), \;\; k(x) = (x - b_1)(x - b_2)$$

and its commutative ring,

$$R_{s=0} = \mathbb{C}[x, y]/(y^3 - x^2 k(x))$$

Normalization

Normalization of the ring $R_{s=0}$ provides the ring

$$R_{\hat{0}} = \mathbb{C}[x, y, z]/(y^2 - xz, zy - xk(x), z^2 - k(x)y).$$

and a space curve, $X_{\hat{0}} \to X_{s=0}$,

$$X_{\hat{0}} = \{(x, y, z) \mid y^2 = xz, zy = xk(x), z^2 = k(x)y\} \cup \{\infty\}$$

Each branch point is given by

$$B_0 = (x = 0, y = 0, z = 0), B_1, B_2.$$

Weierstrass Sequence at ∞

wt	0	1	2	3	4	5	6	7	8	9	10	11
ϕ_i	1	-	-	x	y	z	x^2	xy	xz	yz	x^2y	x^2z

$$R = \bigoplus_{i=0} \mathbb{C}\phi_i, \quad \text{as a } \mathbb{C}\text{-vector space.}$$

wt: weight given as the order of singularity at ∞

$$\mathrm{wt}(x) = 3, \mathrm{wt}(y) = 4, \mathrm{wt}(z) = 5$$

genus $g = 2$, $\mathcal{K}_{X_{\hat{0}}} = (2g)\infty - 2(B_0) \neq (2g-2)\infty$

The fact $\mathcal{K}_{X_{\hat{0}}} \neq (2g-2)\infty$ causes the difficulty of the construction of σ!

Truncated sequence

Truncated sequence $\{\widehat{\phi}\} \subset R$

For a sheaf of the holomorphic one-from of $X_{\widehat{0}}$, $\mathcal{A}_{X_{\widehat{0}}}$,

$$H^0(X_{\widehat{0}}, \mathcal{A}_{X_{\widehat{0}}}(*\infty)) = \bigoplus_{i=0} \mathbb{C}\widehat{\phi}_i \frac{dx}{3yz}.$$

$$\{f \in R \mid \exists \ell, \text{ such that } (f) - (B_0 + B_1 + B_2) + \ell\infty > 0\} = \bigoplus_{i=0} \mathbb{C}\hat{\phi}_i$$

Weierstrass sequence at ∞

$s \backslash i$	wt	0	1	2	3	4	5	6	7	8	9	10	11
ϕ_i	1	-	-	x	y	z	x^2	xy	xz	yz	x^2y	x^2z	
$\hat{\phi}_i$	-	-	-	-	y	z	-	xy	xz	yz	x^2y	x^2z	

Differential form

differentials of 1st and 2nd kinds

1. Differentials of 1st kind (holomorphic 1-form)

$$\nu^{I}_{X_{\hat{0}}1} = \frac{\hat{\phi}_0 dx}{3yz} = \frac{dx}{3z}, \quad \nu^{I}_{X_{\hat{0}}2} = \frac{\hat{\phi}_1 dx}{3yz} = \frac{dx}{3y}$$

$$H^0(X_{\hat{0}}, \mathcal{A}_{X_{\hat{0}}}) = \mathbb{C}\nu^{I}_{X_{\hat{0}}1} + \mathbb{C}\nu^{I}_{X_{\hat{0}}2}$$

$((\hat{\phi}_0 : \hat{\phi}_1)$ **Canonical embedding of** $X_{\hat{0}})$

$$\mathcal{K}_{X_{\hat{0}}} = 2g\infty - 2B_0 \sim 2(g-1)\infty + (B_1 + B_2)$$

2. Differentials of 2nd kind

$$\nu^{II}_{X_{\hat{0}}1} = \frac{-2xdx}{3y}, \quad \nu^{II}_{X_{\hat{0}}2} = \frac{-xdx}{3z}$$

half Periods

1. half Periods:

$$\omega'_{X_{\hat{0}}} := \frac{1}{2}\left(\int_{\alpha_i} \nu^{I}_{X_{\hat{0}} i}\right), \quad \omega''_{X_{\hat{0}}} := \frac{1}{2}\left(\int_{\beta_i} \nu^{I}_{X_{\hat{0}} j}\right),$$

$$\eta'_{X_{\hat{0}}} := \frac{1}{2}\left(\int_{\alpha_i} \nu^{II}_{X_{\hat{0}} j}\right), \quad \eta''_{X_{\hat{0}}} := \frac{1}{2}\left(\int_{\beta_i} \nu^{II}_{X_{\hat{0}} j}\right),$$

2. Lattice $\Gamma_{\hat{0}} := \langle \omega'_{X_{\hat{0}}}, \omega''_{X_{\hat{0}}} \rangle_{\mathbb{Z}}$
3. Jacobi variety: $\kappa_{\mathcal{J}} : \mathbb{C}^2 \to \mathcal{J}_{\hat{0}} := \mathbb{C}^2/\Gamma_{\hat{0}}$, $\iota_{\mathcal{J}} : \mathcal{J}_{\hat{0}} \to \mathbb{C}^2$.
4. Legendre relation (Symplectic Str.)
$^t\omega'_{X_{\hat{0}}} \eta''_{X_{\hat{0}}} - {}^t\omega''_{X_{\hat{0}}} \eta'_{X_{\hat{0}}} = \dfrac{\pi}{2} I_2.$

Abelian Integral

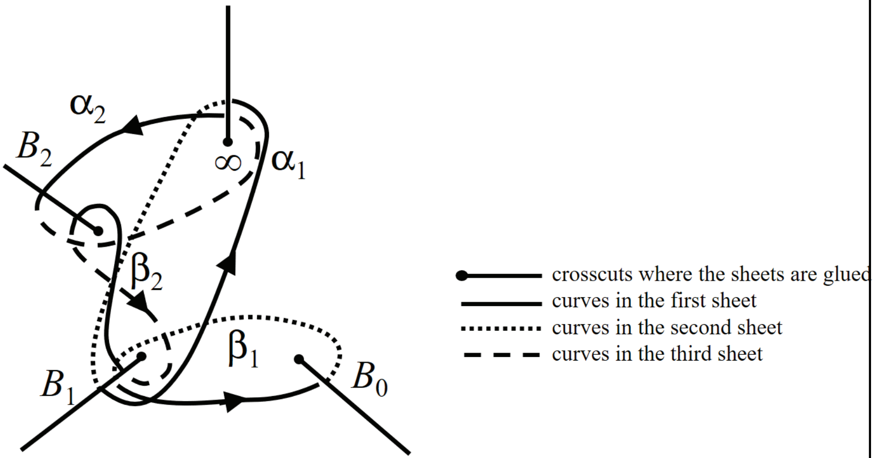
$\widetilde{X}_{\hat{0}}$: Abelianization of Path$X_{\hat{0}}$ $\kappa_X : \widetilde{X}_{\hat{0}} \to X_{\hat{0}}$
Let us assume that **they are unnormalized Abelian map.**

$$\widetilde{w}_{X_{\hat{0}}}(P) := \int_{\infty}^{P} \kappa_X^* \nu_{X_{\hat{0}}}^I, \quad \widetilde{w}_{X_{\hat{0}}}(P) : \widetilde{X} \to \mathbb{C}^g$$

$$w_{X_{\hat{0}}}(P) := \kappa_{\mathcal{J}} \left(\int_{\infty}^{P} \kappa_X^* \nu_{X_{\hat{0}}}^I \right), \quad w_{X_{\hat{0}}} : X_{\hat{0}} \to \mathcal{J}_{\hat{0}}$$

half Periods

$H_1(X_{\hat{0}}, \mathbb{Z})$



Shifted Abelian Integral

For $P_1, P_2, \cdots, P_k \in S^k \widetilde{X}_{\hat{0}}$

$$\widetilde{w}_{X_{\hat{0}}\mathfrak{s}}(P_1, P_2, \ldots, P_k) = \sum_{i}^{k} \widetilde{w}_{X_{\hat{0}}}(P_i) + \widetilde{w}_{X_{\hat{0}}}(B_0).$$

σ-function

For $\begin{bmatrix}\delta'' \\ \delta'\end{bmatrix} \in (\mathbb{Z}/2)^{2g}$ which corresponds to ξ_s, we define the σ-function as an entire function over $\mathbb{C}^2$:

$$\sigma(u) = c\mathrm{e}^{-\frac{1}{2}\ {}^t u \eta' \omega'^{-1}\ u} \sum_{n \in \mathbb{Z}^{2g}} \mathrm{e}^{\left[\pi\sqrt{-1}\left\{\ {}^t(n+\delta'')\omega'^{-1}\omega''(n+\delta'') +\ {}^t(n+\delta'')(\omega'^{-1}u+\delta')\right\}\right.}$$

$c(\neq 0)$ is a constant complex number.

σ-function

Properties of σ

1. Entire function over $\mathbb{C}^3$
2. Zeros: $\{\mathrm{div}\sigma\} \equiv \Theta = w(X_{\hat{0}})$
3. Translational formula: $\sigma_{X_{\hat{0}}}(u + \ell) = \sigma_{X_{\hat{0}}}(u) \exp(L(u + \frac{1}{2}\ell, \ell))\chi(\ell)$
4. expansion $\sigma(u) = s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + \cdots$, $s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$: Schur polynomial of $\begin{smallmatrix}\square\\\square\end{smallmatrix}$.
5. Modular invariant for $\mathrm{Sp}(2, \mathbb{Z})$

Jacobi inversion formula

Theorem (JIF)

1. For $u = w(P_1, P_2)$

$$\mu_2(P; P_1, P_2) = xy_4 - \wp_{22}(u)y_4 + \wp_{21}(u)y_5.$$

or

$$\wp_{22}(u) = \frac{y_1x_2y_2 - y_2x_1y_1}{y_1z_2 - z_1y_2}, \wp_{21}(u) = \frac{z_1x_2y_2 - z_2x_1y_1}{y_1z_2 - z_1y_2}.$$

2. For $u = w(P_1)$

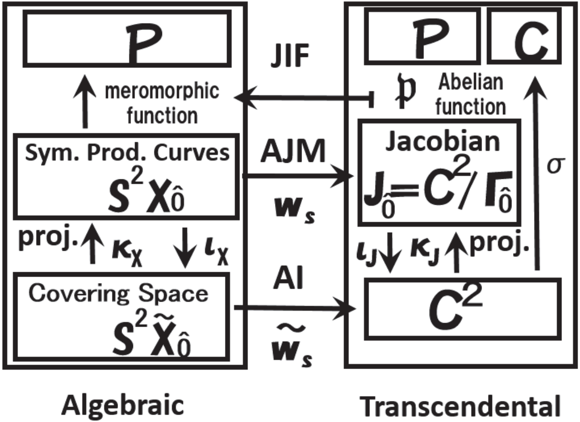
$$\mu_2(P; P_1) = y_5 - \frac{\sigma_1(u)}{\sigma_2(u)}y_4, \quad \frac{\sigma_1(u)}{\sigma_2(u)} = \frac{z_1}{y_1}$$

(Komeda-M 2013, Komeda-M-Previato 2014)

Shifted Riemann constant and shifted Abelian map

JIF

Even for singular curve, we have JIF, which was not written explicitly.



$$\mathcal{L}_{S^2 X_{\hat{0}}} \cong \mathbb{C}(\sigma_{X_{\hat{0}}} \circ \widetilde{w}_{X_{\hat{0}}\mathfrak{s}} \circ \iota_X), \ \mathcal{L}_{\mathcal{J}_{\hat{0}}} = \mathbb{C}\sigma_{X_{\hat{0}}} \iota_{\mathcal{J}}$$

Sigma for the degenerating family of curves

$$\begin{array}{ccc} X_s & \xrightarrow{s\to 0} & X_{s\to 0} \\ \downarrow & & \downarrow \\ D_\varepsilon^* & \xrightarrow{s\to 0} & \{0\} \end{array} \qquad \text{v.s.} \qquad \begin{array}{c} X_{\widehat{0}} \\ \downarrow \\ \{0\} \end{array}$$

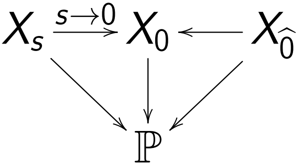
Sigma for the degenerating family of curves

$$
\begin{array}{ccc}
\mathcal{J}_s & \xrightarrow{s\to 0} & \mathcal{J}_{s\to 0} \\
\downarrow & & \downarrow \\
D_\varepsilon^* & \xrightarrow{s\to 0} & \{0\}
\end{array}
\qquad \text{v.s.} \qquad
\begin{array}{c}
\mathcal{J}_{\hat{0}} \\
\downarrow \\
\{0\}
\end{array}
$$

Sigma for the degenerating family of curves

Investigate $\mathcal{L}_{\mathcal{J}_{s\to 0}}$ and $\mathcal{L}_{\mathcal{J}_{\hat{0}}}$.

Sigma for the degenerating family of curves



The x-constant section P on D_ε

$$\pi_1(P_s) = \pi_1(P_{s'}) = x \in \mathbb{P}.$$

One forms $s \to 0$

The holomorphic one-forms (diff. of the 1st kind)

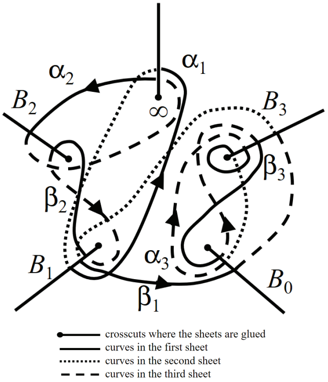
X_s	$X_{s\to 0}$ v.s. $X_{\hat{0}}$
${\nu^{\mathrm{I}}}_1 = \dfrac{dx}{3y^2}$	$\dfrac{dx}{\sqrt[3]{x^4(x-b_1)^2(x-b_2)^2}}$: not holomorphic
${\nu^{\mathrm{I}}}_2 = \dfrac{xdx}{3y^2}$	$\dfrac{dx}{3z} = {\nu^{\mathrm{I}}}_1$ of $X_{\hat{0}}$
${\nu^{\mathrm{I}}}_3 = \dfrac{dx}{3y}$	$\dfrac{dx}{3y} = {\nu^{\mathrm{I}}}_2$ of $X_{\hat{0}}$

Diff. of the 2nd kind

X_s	$X_{s\to 0}$ v.s. $X_{\hat{0}}$
$\nu^{\mathrm{II}}{}_1 = -\dfrac{(5x^2 + 2\lambda_3 x + \lambda_2)dx}{3y}$	$\dfrac{(5x^3 + 2\lambda_3 x + \lambda_2)dx}{\sqrt[3]{x^2(x-b_1)(x-b_2)}}$
$\nu^{\mathrm{II}}{}_2 = \dfrac{-2xdx}{3y}$	$\dfrac{-2xdx}{3y} = \nu^{\mathrm{II}}{}_1$ of $X_{\hat{0}}$
$\nu^{\mathrm{II}}{}_3 = -\dfrac{x^2dx}{3y^2}$	$-\dfrac{xdx}{3z} = \nu^{\mathrm{II}}{}_2$ of $X_{\hat{0}}$

Contours of Period Integral

For $\omega_{X_s a} := \int_{\infty}^{B_a} \nu^I_{X_s}, (a = 0, 1, 2, 3)$



Legendre relation of non-hyperelliptic curve

For $\omega_{X_s a} := \int_{\infty}^{B_a} \nu^I_{X_s}, (a = 0, 1, 2, 3)$

$$\omega'_{X_s 1} = \frac{1}{2}\hat{\zeta}_3(1 - \hat{\zeta}_3^2)\omega_{X_s 1}, \quad \omega'_{X_s 2} = \frac{1}{2}(1 - \hat{\zeta}_3^2)\omega_{X_s 2},$$

$$\omega'_{X_s 3} = \frac{1}{2}\Big((1 - \hat{\zeta}_3^2)\omega_{X_s 0} + \hat{\zeta}_3^2(1 - \hat{\zeta}_3)\omega_{X_s 3}\Big)$$

$$\omega''_{X_s 1} = \frac{1}{2}(\hat{\zeta}_3(1 - \hat{\zeta}_3^2)\omega_{X_s 1} + (1 - \hat{\zeta}_3^2)\omega_{X_s 0} + \hat{\zeta}_3^2(1 - \hat{\zeta}_3^2)\omega_{X_s 3})$$

$$\omega''_{X_s 2} = \frac{1}{2}(\hat{\zeta}_3 - 1)(\omega_{X_s 1} - \omega_{X_s 2}), \quad \omega''_{X_s 3} = \frac{1}{2}(\hat{\zeta}_3 - 1)(\omega_{X_s 0} - \omega_{X_s 3})$$

Integrals $s \to 0$

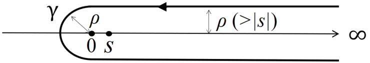
Integrals (Kazuhiko Aomoto, Feb., 2018)

When $\min\{|b_1|, |b_2|\} > s > 0$, $Im(b_1)$ and $Im(b_2) > 0$,

$$\int_\infty^0 \frac{dx}{\sqrt[3]{|x^2(x-s)^2|(x-b_1)^2)(x-b_2)^2}} = s^{-1/3} f(s^{1/3}),$$

$$\int_\gamma \frac{dx}{\sqrt[3]{x^2(x-s)^2(x-b_1)^2(x-b_2)^2}} = g(s),$$

where $f(t)$ and $g(t)$ are regular functions with respect to t around $t = 0$, and the contour γ is given by



Integrals $s \to 0$

$$\omega_{a,b} := \int_{\infty}^{B_a} \nu^{\mathrm{I}}{}_{b},(a = 0, 1, 2, 3, b = 1, 2, 3)$$

$$B_0 = (0, 0), B_1 = (b_1, 0), B_2 = (b_2, 0), B_3 := B_s = (s, 0)$$

X_s	$X_{s\to 0}$ v.s. $X_{\hat{0}}$
$\begin{pmatrix} \omega_{0,1} & \omega_{1,1} & \omega_{2,1} & \omega_{3,1} \\ \omega_{0,2} & \omega_{1,2} & \omega_{2,2} & \omega_{3,2} \\ \omega_{0,3} & \omega_{1,3} & \omega_{2,3} & \omega_{3,3} \end{pmatrix}$	$\begin{pmatrix} * & * & * & s^{-1/3}A_0(s^{-1/3}) \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$

*: finite value, $A_0(t)$: const + holo. of t.

Integrals: $\omega_{0,1} = s^{-1/3} A_1(s)$, $\ell = 1/3$

X_s	$X_{s\to 0}$ v.s. $X_{\widehat{0}}$ $(\ell = 1/3)$
ω'	$\begin{pmatrix} * & * & A_1 s^{-\ell} \\ * & * & * \\ * & * & * \end{pmatrix} \to \begin{pmatrix} * & * & \infty \\ \omega'_{X_{\widehat{0}},11} & \omega'_{X_{\widehat{0}},12} & * \\ \omega'_{X_{\widehat{0}},21} & \omega'_{X_{\widehat{0}},22} & * \end{pmatrix}$
ω'^{-1}	$\begin{pmatrix} A_{11} s^{\ell} & A_{12} s^{\ell} & A_{13} s^{\ell} \\ * & * & A_{21} s^{\ell} \\ * & * & A_{31} s^{\ell} \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \omega'^{-1}_{X_{\widehat{0}}} & 0 \end{pmatrix}$
ω''	$\begin{pmatrix} * & * & A'_1 s^{-1/3} \\ * & * & * \\ * & * & * \end{pmatrix} \to \begin{pmatrix} * & * & \infty \\ \omega''_{X_{\widehat{0}},11} & \omega''_{X_{\widehat{0}},12} & * \\ \omega''_{X_{\widehat{0}},21} & \omega''_{X_{\widehat{0}},22} & * \end{pmatrix}$

Integrals: $\omega_{0,1} = s^{-1/3} A_1(s)$, $\ell = 1/3$

X_s	$X_{s\to 0}$ v.s. $X_{\hat{0}}$ $\quad (\ell = 1/3)$
$\omega'^{-1}\omega''$	$\begin{pmatrix} A'_{12}s^\ell & A'_{13}s^\ell & * \\ * & * & * \\ * & * & * \end{pmatrix} \to \begin{pmatrix} 0 & 0 & * \\ * & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{pmatrix}$

*: finite value and $A_i(t)$: const + holo. of t.

Space

$$\mathbb{C}^3 \ni \omega'^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ * \\ * \end{pmatrix} = \omega'^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \in \mathbb{C}^2$$

Integrals: $\eta_{0,1} = B_1(s)$,

X_s	$X_{s\to 0}$ v.s. $X_{\hat{0}}$ $(\ell = 1/3)$
η'	$\begin{pmatrix} * & * & B_1 \\ * & * & * \\ * & * & * \end{pmatrix} \to \begin{pmatrix} * & * & * \\ \eta'_{X_{\hat{0}},11} & \eta'_{X_{\hat{0}},12} & * \\ \eta'_{X_{\hat{0}},21} & \eta'_{X_{\hat{0}},22} & * \end{pmatrix}$
η''	$\begin{pmatrix} * & * & B'_1 \\ * & * & * \\ * & * & * \end{pmatrix} \to \begin{pmatrix} * & * & * \\ \eta''_{X_{\hat{0}},11} & \eta''_{X_{\hat{0}},12} & * \\ \eta''_{X_{\hat{0}},21} & \eta''_{X_{\hat{0}},22} & * \end{pmatrix}$

*: finite value and $B_i(t)$: const + holo. of t.

Sigma for the degenerating family of curves

Proposition: Let
$\widehat{\mathcal{J}_{s\to 0}} := \{(u_1, u_2, u_3) \in \mathcal{J}_{s\to 0} \mid u_1 = \omega_{s,1}\} \subset \mathcal{J}_{s\to 0}$.
The limit is defined for every $s \in D_\epsilon^*$.
These integrals mean that translational formulae $\mathcal{L}_{\mathcal{J}_{s\to 0}}\big|_{\widehat{\mathcal{J}_{s\to 0}}}$ **and** $\mathcal{L}_{\mathcal{J}_{\hat{0}}}$ **agree.**

Observations

σ function (Eilbeck-Gibbons-Onishi-Yasuda 2017)

$$\sigma_{X_s}(u) := c_0 \mathrm{e}^{-\frac{1}{2}u^t\omega'^{-1}{}^t\eta' u}\theta\begin{bmatrix}\delta' \\ \delta''\end{bmatrix}\left(\frac{1}{2}\omega'^{-1}u; \omega'^{-1}\omega''\right).$$

σ function (Eilbeck-Gibbons-Onishi-Yasuda 2017)

$$c_0 = \left(\frac{(2\pi)^g}{\det \omega'}\right)^{1/2}\Delta^{-1/8}\ (\to c_0' s^{-1/3} \text{ for } s \to 0)$$

$$\Delta^{1/8} = (-729 s^4 b_2^4 b_1^4((s+b_2)^3 b_1^3 + 3sb_2(s+\tfrac{1}{4}b_2)(s+4b_2)b_1^2 + 3s^2 b_2^2(s+b_2)b_1 + s^3 b_2^3)^2)^{1/8} \sim s^{1/2}\Delta_0^{1/8}$$

σ function

For $s \to 0$, $\theta(u)$ may vanish like $s^{1/3}(\theta_0(u) + d_{>0}(s^{1/3}))$ but the pref-actor $c_0 = s^{-1/3}(c_{0,0} + d_{>0}(s^{1/3'}))$.

Observations

Fact

For every s and $u = w(P_1, P_2, P_3)$,(Eilbeck-Gibbons-Onishi-Yasuda 2017)

$$\sigma_{X_s}(u) = u_1 - u_2^2 u_3 + \frac{6}{5!} u_3^5 + \cdots$$

whereas for $X_{\hat{0}}$, ($u_3 \to v_2$ and $u_2 \to v_1$) (KMP 2013)

$$\sigma_{X_{\hat{0}}}(v) = v_1 - v_2^2 + \cdots$$

Observations

Observations around ∞^3

$$
\begin{aligned}
u_1 &= \frac{1}{5}(t_1^5 + t_2^5 + t_3^5)(1 + d_{>0}(t_a)),\\
u_2 &= \frac{1}{2}(t_1^2 + t_2^2 + t_3^2)(1 + d_{>0}(t_a)),\\
u_3 &= \frac{1}{1}(t_1 + t_2 + t_3)(1 + d_{>0}(t_a)),
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{X_s}(u) &= u_1 - u_2^2 u_3 + \tfrac{6}{5!} u_3^5 + \cdots\\
&= t_1 t_2 t_3 (t_1^2 + t_2^2 + t_3^2 + t_1 t_2 + t_1 t_2 + t_2 t_3)
\end{aligned}
$$

Observations

Observation ∞^2

$$v_1 = \frac{1}{2}(t_1^2 + t_2^2)(1 + d_{>0}(t_a)),$$
$$v_2 = \frac{1}{1}(t_1 + t_2)(1 + d_{>0}(t_a)),$$

$$\boxed{\sigma_{X_{\hat{0}}}(u) = \sigma_{X_{\hat{0}}}(v) = v_1 - v_2^2 + \cdots = t_1 t_2 + \cdots}$$

$$s_{\square\atop\square}(v_1, v_2) = \frac{\begin{vmatrix} t_1^2 & t_2^2 \\ t_1^1 & t_2^1 \end{vmatrix}}{\begin{vmatrix} t_1 & t_2 \\ 1 & 1 \end{vmatrix}} = t_1 t_2 = v_1 - v_2^2$$

Observations

The al function (MP 2013) $\omega_s = \widetilde{w}_{X_s}((s,0))$

$$\mathrm{al}_s^{(0)}(u) = \frac{\mathrm{e}^{-{}^t u \varphi_{s;0}} \sigma_{X_s}(u+\omega_s)}{\sigma_{X_s}(u)\sigma_{33}(\omega_s)}$$

$$\omega_s := \int_\infty^{B_s} \nu^{\mathrm{I}}$$

or

$$\sigma_{X_s}(u+\omega_s) = \sigma_{33}(\omega_s)\mathrm{e}^{{}^t u \varphi_{s;0}} \mathrm{al}_s^{(0)}(u)\sigma_{X_s}(u)$$

Observations

The al function (MP 2013) $u = \widetilde{w}_{X_s}(P_1, P_2, P_3)$

$$\mathrm{al}_s^{(0)}(u) = \frac{\mathrm{e}^{-{}^t u \varphi_{s;0}} \sigma_{X_s}(u + \omega_s)}{\sigma_{X_s}(u)\sigma_{33}(\omega_s)} = \frac{A_s(P_1, P_2, P_3)}{\sqrt[3]{(s - x_1)(s - x_2)(s - x_3)}}$$

$$\sigma_{33}(\omega_a) = \frac{\sqrt{2}}{\sqrt[3]{s(s - b_1)(s - b_2)}}, \quad A_s(P_1, P_2, P_3) := \frac{\begin{vmatrix} 1 & x_1 & y_1 & {x_1}^2 \\ 1 & x_2 & y_2 & {x_2}^2 \\ 1 & x_3 & y_3 & {x_3}^2 \\ 1 & s & 0 & s^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}$$

$$\sigma_{X_s}(u + \omega_s) = \frac{\sqrt{2}\mathrm{e}^{{}^t u \varphi_{s;0}}}{\sqrt[3]{s(s - b_1)(s - b_2)}} \mathrm{al}_s^{(0)}(u)\sigma_{X_s}(u)$$

Observations

For $u^{(i)} = \widetilde{w}_{X_s}(P_i)$, let us consider

$$u = u^{(1)} + u^{(2)} + u^{(3)} + \omega_s$$

and the limit $u^{(3)} \to 0, \quad P_3 \to \infty$ around $s = 0$ of

$$\sigma_{X_s}(u + \omega_s) = \frac{\sqrt{2}\mathrm{e}^{{}^t u \varphi_{s;0}}}{\sqrt[3]{s(s-b_1)(s-b_2)}} \mathrm{al}_s^{(0)}(u) \sigma_{X_s}(u)$$

noting

$$x_i = \frac{1}{t_i^3}, \quad y_i = \frac{1}{t_i^4}(1 + d_{>0}(t_i)), \quad \text{and } t_3 \to 0.$$

Observations

For $u^{(i)} = \widetilde{w}_{X_s}(P_i)$, let us consider $u = u^{(1)} + u^{(2)} + u^{(3)} + \omega_s$ and the limit $u^{(3)} \to 0, \quad P_3 \to \infty$ around $s = 0$ of
$\sigma_{X_s}(u + \omega_s) = \frac{\sqrt{2}\mathrm{e}^{{}^t u \varphi_{s;0}}}{\sqrt[3]{s(s-b_1)(s-b_2)}} \mathrm{al}_s^{(0)}(u) \sigma_{X_s}(u)$

$$\mathrm{al}_s^{(0)}(u) = \frac{1}{t_1^2 + t_1 t_2 + t_2^2} \frac{1}{t_3} (1 + d_{>0}(s, t_1, t_2, t_3))$$

$$\sigma_{X_s}(u) = t_1 t_2 (t_1^2 + t_1 t_2 + t_2^2) t_3 (1 + d_{>0}(s, t_1, t_2, t_3))$$

and thus

$$\sigma_{X_s}(u^{(1)} + u^{(2)} + \omega_s) = \frac{\sqrt{2}}{\sqrt[3]{b_1 b_2 s}} t_1 t_2 (1 + d_{>0}(s, t_1, t_2, t_3))$$

Theorem

We have the relation:

$$\sigma_{X_{\hat{0}}}(v) = \lim_{s\to 0}\left(\lim_{P_3\to\infty}\left[\frac{\sqrt[3]{b_1 b_2 s}}{\sqrt{2}}\sigma_{X_s}(u+\omega_s)\right]_{u_1=0,u_2=v_1,u_3=v_2}\right)$$

Proof: Due to the translational formulae, both are the bases of the same line bundle of $\mathcal{J}_{\hat{0}}$ and the leading terms of their expansions show the equality.

Thanks

Thank you for your attention!