Statistical Mechanics of Elastic Curves: beyond Euler's elastica 弾性曲線の統計力学：オイラーのエラスティカを超えて

第 24 回　沼津研究会–幾何，数理物理，そして量子論

Shigeki Matsutani

National Institute of Technology, Sasebo College

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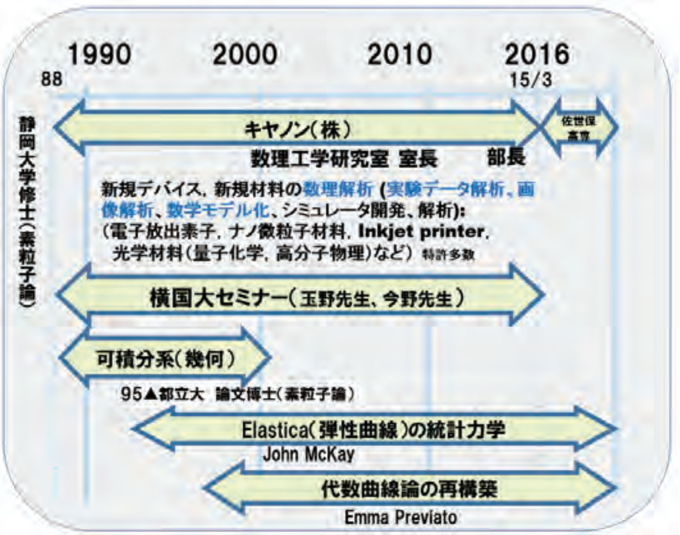
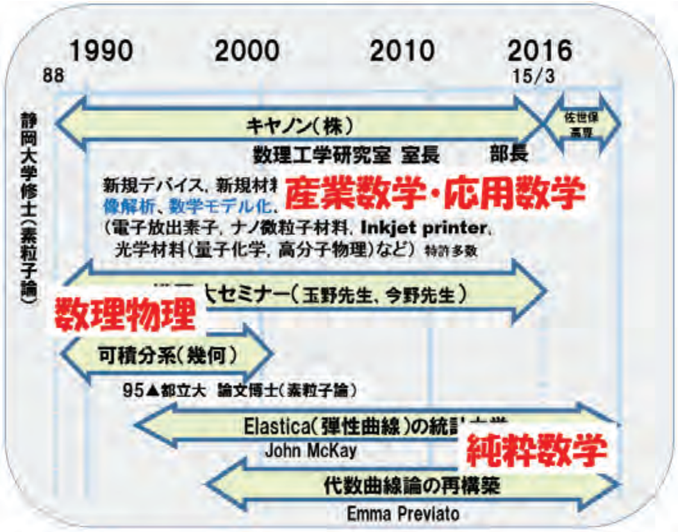
Menu

Self-Introduction



Self-Introduction



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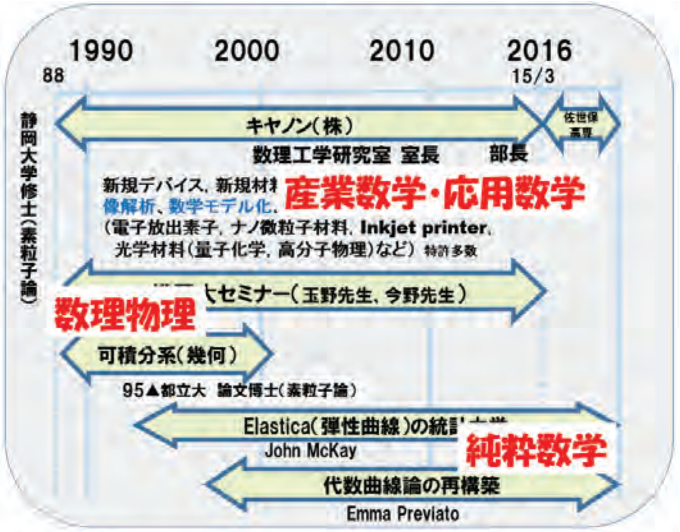
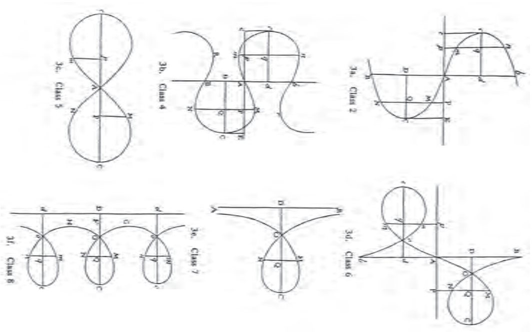
Self-Introduction



Self-Introduction

現代数学と技術との関わり合い

Elastica Problem



Immersion of Curve

$Z : S^1 \hookrightarrow \mathbb{C}$ smooth ($|\partial_s Z| = 1$).

s is arclength.

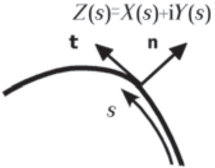
$$Z(s) = X(s) + \mathrm{i}\, Y(s),$$

$$\mathbf{t} = \partial_s Z = \mathrm{e}^{\mathrm{i}\phi},$$

$$(\phi \in C^\infty(\kappa^{-1} S^1, \mathbb{R}))$$

$$= \cos\phi + \mathrm{i} \sin\phi$$

$$\mathbf{n} = \mathrm{i}\mathbf{t} = \mathrm{i}\partial_s Z.$$



Elastica Problem

Immersion of curve

Curvature & Frenet-Serret relation

$$\mathbf{t} := \partial_s Z, \quad \partial_s \mathbf{t} = k\mathbf{n}, \quad \partial_s \mathbf{n} = -k\mathbf{t}, \quad (\partial_s^2 Z = \mathrm{i}k\partial_s Z), \qquad (1)$$

$k := \partial_s \phi$: curvature; $k = 1/$[curvature radius].

Elastica Problem (James Bernoulli (1691))

To find the shape of elastica (ideal thin elastic rod) in a plane.

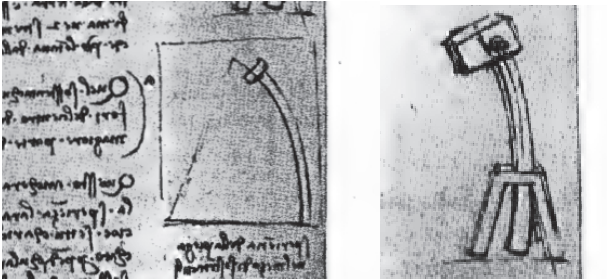
Elastica Problem

Origin of Elastica

Leonardo da Vinci (1452-1519)

Elastica Problem

Origin of Elastica



Leonardo da Vinci (1452-1519) drew the pictures of bent beams

Origin of Elastica

Galileo Galilei (1564-1654)

Elastica Problem

Origin of Elastica

Galileo Galilei (1564-1654) investigated bent beams:
It is a problem of cantilever.

Elastica Problem

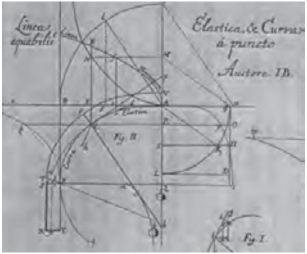
James Bernoulli (1654-1705) proposed the Elastica problem:
To find the shape of elastica (ideal thin elastic rod) in a plane.

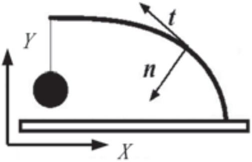
James Bernoulli (1654-1705)

Elastica Problem

Elastica Problem

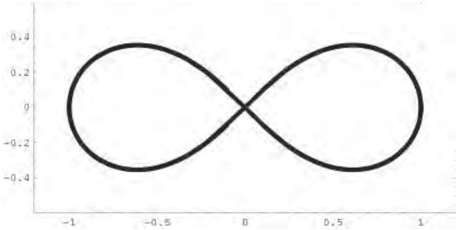
James Bernoulli (1654-1705) found the fact that the elastic force is proportional to k and the Lemniscate integral: $s = \int_X^1 \frac{dX}{\sqrt{1 - X^4}}$.

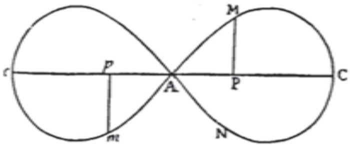




Lemniscate and Elastica

James Bernoulli defined the Lemniscate curve of eight figure.





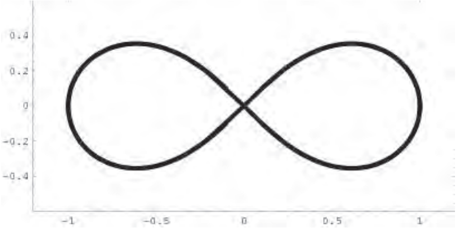
3c. Class 5

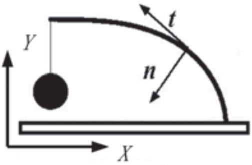
Lemniscate
$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$
$\phi_{\rm lemni}$**:tangential angle**

Elastica of Eight-Figure
$\phi_{\rm elas}$**:tangential angle**

Lemniscate and Elastica

James Bernoulli defined the Lemniscate curve of eight figure.





Lemniscate

$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$

$\phi_{\rm lemni}$**:tangential angle**

Perpendicular terminal

$$\phi_{\rm lemni} = \frac{3}{2}\phi_{\rm elas} \quad [\text{M } 1995]$$

Elastica Problem

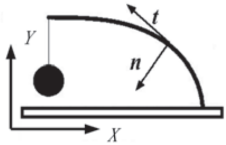
Daniel Bernoulli (1700-1782)

Elastica Problem

Elastica Problem

Daniel Bernoulli (1700-1782) discovered the least principle 1738 in a letter to Euler (1707-1783).
An elastica is realized as the least point of the energy, i.e.,
Euler-Bernoulli energy

$$\mathcal{E}[Z] := \int_{S^1} k^2(s)ds = \int_{S^1} (\partial_s \phi(s))^2 \; ds$$
$$= \int \{Z, s\}_{\mathrm{SD}} ds$$
$$= \int_{S^1} g^{-1}dg * g^{-1}dg, \quad g \in \mathrm{U}(1)$$



$\{Z, s\}_{\mathrm{SD}}$:Schwarz derivative
Elastica problem is the oldest harmonic map problem.

Elastica Problem

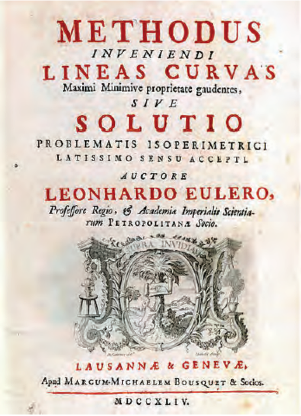
Leonhard Euler (1707-1783)

Elastica Problem

Euler's solution

By discovering the variational principle, he published the book "Method" 1744. In its Appendix, he completely solved the problems in terms of

1. **Elliptic integrals,**
2. Moduli of elliptic curves,
3. Numerical computations.


METHODUS
INVENIENDI
LINEAS CURVAS
Maximi Minimive proprietate gaudentes,
SIVE
SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI.
AUCTORE
LEONHARDO EULERO,
Professore Regio, & Academiæ Imperialis Scientiarum Petropolitanæ Socio.

LAUSANNÆ & GENEVÆ,
Apud Marcum-Michaelem Bousquet & Socios.
MDCCXLIV.

Elastica Problem

Euler's solution

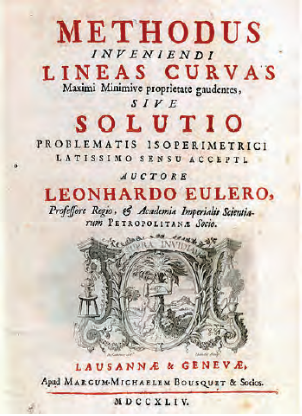
$$s = \int^X \frac{\lambda^2 dX}{\sqrt{\lambda^4 - (\alpha + \beta X + \gamma X^2)^2}},$$

$$Y = \int^X \frac{(\alpha + \beta X + \gamma X^2) dX}{\sqrt{\lambda^4 - (\alpha + \beta X + \gamma X^2)^2}}.$$

$$ak + \frac{1}{2}k^3 + \partial_s^2 k = 0.$$

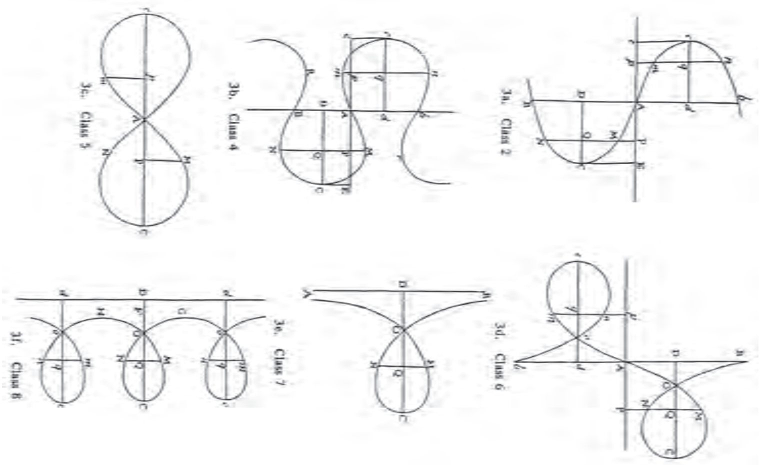
Euler relation (M-Previato 2014)

$$\begin{array}{ccc} X(s) - X_0 & = & \frac{1}{4}k(s) \\ \text{affine coordinate} & \propto & \text{affine connection} \end{array}$$

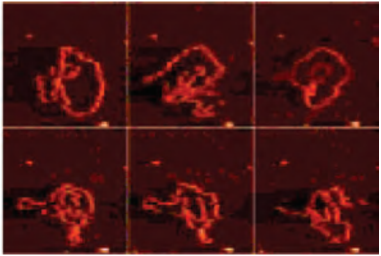


Elastica Problem

Euler's list of Elastica

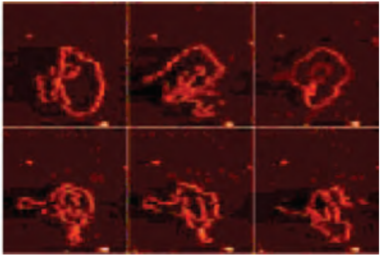


Statistical Mechanics of Elastica



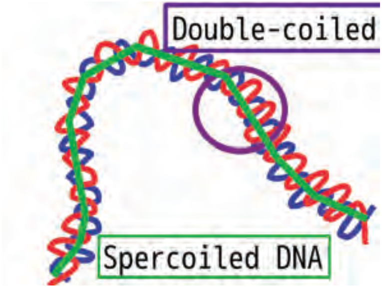
Statistical Mechanics of Elastica

Pictures of DNAs by atomic force microscopes shows the super-coils.



Pictures of DNAs by atomic force microscopes

Statistical Mechanics of Elastica



These shapes are **super-coils** rather than **double coils**.
Super-coil is weakly governed by the elastic force!
But it is **not** realized as **the least point of the Euler-Bernoulli energy**, **It is out of Euler's list**
maybe due to the **thermal effect**!

Statistical Mechanics of Elastica

Statistical Mechanics of Elastica is to evaluate the state with the Boltzmann weight $\mathrm{e}^{-\mathcal{E}[Z]\beta}(\beta > 0)$, i.e., the partition function,

$$\mathcal{Z}[\beta] = \int_{\mathbb{M}} DZ \exp(-\beta \mathcal{E}[Z])$$

Here $\mathcal{M}$ is the set of the loops in the plane,

$$\mathcal{M} := \{Z : S^1 \hookrightarrow \mathbb{C} \mid Z \in \mathcal{C}^\omega(S^1, \mathbb{C}), |dZ/ds| = 1\},$$

$$\mathrm{pr}_1 : \mathcal{M} \to \mathbb{M} := \mathcal{M}/ \sim,$$

where $\sim$ means the equivalence coming from the eulidean move.

Statistical Mechanics of Elastica

Purpose of Statistical Mechanics of Elastica

To find the natural topology and measure of $\mathbb{M}$ using the Boltzmann weight $\mathrm{e}^{-\mathcal{E}[Z]\beta}$.
As its first step, we consider the geometrical structure of $\mathbb{M}$.

Approach in Statistical Mechanics of Elastica

To find the geometrical structure of $\mathbb{M}$,
1) we consider the geometrical structure of its tangent space $T_Z\mathcal{M}$ at $Z \in \mathcal{M}$ as an infinitesimal deformation
2) using the data of $T_Z\mathcal{M}$ and its orbit, we classify $\mathcal{M}$ itself.
(M-Ônishi 2003, M-Previato 2016)

Notations

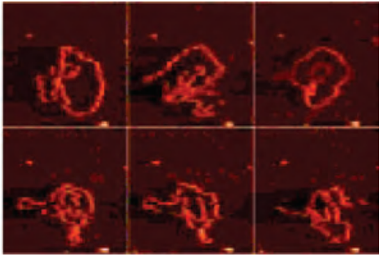
$\mathcal{A}^p(K)$: K-valued analytic p-form over S^1
(K is $\mathbb{R}$ or $\mathbb{C}$.)

Menu:Statistical Mechanics of Elastica (Quantized Elastica)

1. Infinitesimal isometric deformation
2. Infinitesimal isoenergy deformation
3. MKdV flow
4. Hyperelliptic Curves
5. Topological Properties
6. Final Remarks

Statistical Mechanics of Elastica

Infinitesimal Isometric Deformation



Infinitesimal Isometric Deformation

Tangent space $T_Z\mathcal{M}$ (= infinitesimal deformation)

To observe $T_Z\mathcal{M}$ at $Z \in \mathcal{M}$, we consider the deformation of the deformation parameter $t \in [0, \varepsilon)$ $(\varepsilon > 0)$,

$$\partial_t Z(s) = v(s)\partial_s Z(s), \quad s \in S^1, v \in \mathcal{A}^0(\mathbb{C}),$$

$$\left(v = v^{(r)} + \mathrm{i}v^{(i)}, v^{(r)}, v^{(i)} \in \mathcal{A}^0(\mathbb{R})\right)$$

Infinitesimal Isometric Deformation

Tangent space $T_Z\mathcal{M}$

Proposition

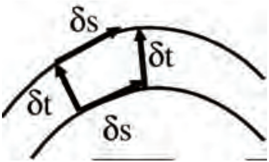
At $Z \in \mathbb{M}$, the isometric deformation ($[\partial_s, \partial_t]Z = 0$) is reduced to two equations (Goldstein-Petrichi)

$$\partial_t k = \Omega^{(II)} v^{(i)}, \qquad (2)$$

$$k v^{(i)} = \partial_s v^{(r)}. \qquad (3)$$

where

$$\Omega^{(II)} := \partial_s^2 + \partial_s(k\partial_s^{-1}k),$$



Infinitesimal Isometric Deformation

proof of Proposition

$[\partial_s, \partial_t]Z = 0$ means

proof of Proposition

$[\partial_s, \partial_t]Z = 0$ means

$$\begin{aligned}
\partial_s\partial_t Z &= \partial_s(v\partial_s Z)\\
&= (\partial_s v + \mathrm{i}kv)\partial_s Z\\
\partial_t\partial_s Z &= \partial_t(\mathrm{e}^{\mathrm{i}\varphi(s,t)})\\
&= \mathrm{i}(\partial_t\varphi)\partial_s Z
\end{aligned}$$

Infinitesimal Isometric Deformation

proof of Proposition

$[\partial_s, \partial_t]Z = 0$ means

$$\begin{aligned} \partial_s \partial_t Z &= \partial_s (v \partial_s Z) \\ &= (\partial_s v + \mathrm{i} k v) \partial_s Z \\ \partial_t \partial_s Z &= \partial_t (\mathrm{e}^{\mathrm{i}\varphi(s,t)}) \\ &= \mathrm{i}(\partial_t \varphi) \partial_s Z \end{aligned}$$

Thus $\mathrm{i}\partial_t \varphi = (\partial_s v + \mathrm{i} k v) = (\partial_s v^{(r)} - k v^{(i)}) + \mathrm{i}(\partial_s v^{(i)} + k v^{(r)})$

Infinitesimal Isometric Deformation

proof of Proposition

$[\partial_s, \partial_t]Z = 0$ means

$$\begin{aligned}\partial_s\partial_t Z &= \partial_s(v\partial_s Z)\\ &= (\partial_s v + \mathrm{i}kv)\partial_s Z\\ \partial_t\partial_s Z &= \partial_t(\mathrm{e}^{\mathrm{i}\varphi(s,t)})\\ &= \mathrm{i}(\partial_t\varphi)\partial_s Z\end{aligned}$$

Thus $\mathrm{i}\partial_t\varphi = (\partial_s v + \mathrm{i}kv) = (\partial_s v^{(r)} - kv^{(i)}) + \mathrm{i}(\partial_s v^{(i)} + kv^{(r)})$
Real part: $\partial_s v^{(r)} - kv^{(i)} = 0 \rightarrow \quad v^{(r)} = \partial_s^{-1} kv^{(i)}$

Infinitesimal Isometric Deformation

proof of Proposition

$[\partial_s, \partial_t]Z = 0$ means

$$\begin{aligned} \partial_s\partial_t Z &= \partial_s(v\partial_s Z) \\ &= (\partial_s v + \mathrm{i}kv)\partial_s Z \\ \partial_t\partial_s Z &= \partial_t(\mathrm{e}^{\mathrm{i}\varphi(s,t)}) \\ &= \mathrm{i}(\partial_t\varphi)\partial_s Z \end{aligned}$$

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imaginary part: $\partial_t k = \partial_s\partial_t\varphi = \partial_s(\partial_s v^{(i)} + kv^{(r)})$

Infinitesimal Isometric Deformation

proof of Proposition

$[\partial_s, \partial_t]Z = 0$ means

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imaginary part: $\partial_t k = \partial_s\partial_t\varphi = \partial_s(\partial_s v^{(i)} + kv^{(r)})$

$$\partial_t k = \partial_s(\partial_s + k\partial_s^{-1}k)v^{(i)}$$

Infinitesimal Isometric Deformation

proof of Proposition

$[\partial_s, \partial_t]Z = 0$ means

$$\begin{aligned}
\partial_s \partial_t Z &= \partial_s(v \partial_s Z) \\
&= (\partial_s v + \mathrm{i} k v)\partial_s Z \\
\partial_t \partial_s Z &= \partial_t(\mathrm{e}^{\mathrm{i}\varphi(s,t)}) \\
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Thus $\mathrm{i}\partial_t \varphi = (\partial_s v + \mathrm{i} k v) = (\partial_s v^{(r)} - k v^{(i)}) + \mathrm{i}(\partial_s v^{(i)} + k v^{(r)})$
Real part: $\partial_s v^{(r)} - k v^{(i)} = 0 \rightarrow \quad v^{(r)} = \partial_s^{-1} k v^{(i)}$
imaginary part: $\partial_t k = \partial_s \partial_t \varphi = \partial_s(\partial_s v^{(i)} + k v^{(r)})$

$$\partial_t k = \partial_s(\partial_s + k\partial_s^{-1} k) v^{(i)}$$

Infinitesimal Isometric Deformation

Tangent space $T_Z\mathcal{M}$

Proposition

At $Z \in \mathbb{M}$, the isometric deformation ($[\partial_s, \partial_t]Z = 0$) is reduced to two equations (Goldstein-Petrichi)

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where

$$\Omega^{(II)} := \partial_s^2 + \partial_s(k\partial_s^{-1}k),$$

Infinitesimal Isometric Deformation

Eq. (3) $\partial_s v^{(r)} = k v^{(i)}$

1. In order to find the space satisfying Eq.(3), we consider the map ℓ_d,
$$\ell_d : \mathcal{A}^0(\mathbb{R}) \to \mathcal{A}^1(\mathbb{R}), \quad \ell_d(v^{(i)}) = k v^{(i)} ds,$$
2. Let the inverse image of $d\mathcal{A}^0(\mathbb{R}) \subset \mathcal{A}^1(\mathbb{R})$ by ℓ_d be $\hat{\mathcal{A}}^0(\mathbb{R}) := \ell_d^{-1} d\mathcal{A}^0(\mathbb{R})$, which is the space statisfying Eq. (3).

Eq. (3) $\partial_s v^{(r)} = k v^{(i)}$, $v = v^{(r)} + \mathrm{i} v^{(i)}$

$\ell_r^0 : \hat{\mathcal{A}}^0(\mathbb{R}) \to \mathcal{A}^0(\mathbb{R})$: $(v^{(i)} \mapsto v^{(r)})$ because of $\partial_s v^{(r)} = k v^{(i)}$,

$$v^{(r)} = \ell_r^0(v^{(i)}) = \int_0^s k v^{(i)} ds = \int_0^s \partial_s v^{(r)} ds$$

$\ell : \hat{\mathcal{A}}^0(\mathbb{R}) \to \mathcal{A}^0(\mathbb{C})$; $(v^{(i)} \mapsto v = v^{(r)} + \mathrm{i} v^{(i)} = \ell_r^0(v^{(i)}) + \mathrm{i} v^{(i)})$.

Infinitesimal Isometric Deformation

Tangent space $T_Z\mathcal{M}$ (Space of infinitesimal isometric deformation)

For a point $Z \in \mathcal{M}$, we have

$$\ell : \hat{\mathcal{A}}^0(\mathbb{R}) \to \mathcal{A}^0(\mathbb{C}) \; ; \; (\ell(v^{(i)}) = v = v^{(r)} + \mathrm{i} v^{(i)})$$

$$\mathrm{pr}_1 : \mathcal{M} \to \mathbb{M} := \mathcal{M}/\sim, \quad :$$

$\sim$ means the equivalence coming from the eulidean move.

Infinitesimal Isometric Deformation

Tangent space $T_Z\mathcal{M}$ (Space of infinitesimal isometric deformation)

For a point $Z \in \mathcal{M}$, we have

$$\ell : \hat{\mathcal{A}}^0(\mathbb{R}) \to \mathcal{A}^0(\mathbb{C}) \ ; \ (\ell(v^{(i)}) = v = v^{(r)} + \mathrm{i}v^{(i)})$$

Proposition

(Brylinski) For a point $Z \in \mathcal{M}$, we have the map ℓ induces the bijection $\ell^\sharp$ and the surjection $\ell^\flat$:

$$\begin{array}{ccc}
\mathcal{A}^0(\mathbb{R})/\mathbb{R} \cong \hat{\mathcal{A}}^0(\mathbb{R}) & \xrightarrow{\ \ell^\sharp\ } & T_Z(\mathcal{M}) \\
& \searrow^{\ell^\flat} & \downarrow \mathrm{pr}_{1*} \\
& & T_{\mathrm{pr}_1(Z)}(\mathbb{M})
\end{array}$$

Infinitesimal Isometric Deformation

Tangent space $T_Z\mathcal{M}$ (Space of infinitesimal isometric deformation)

1. **Translation of SE(2)**
   Since $\partial_t Z = c = c_1 + \mathrm{i}c_2 \in \mathbb{C}$ means the translation,
   if $v = \dfrac{c}{\partial_s Z} = c_1 \cos\phi + c_2 \sin\phi - \mathrm{i}c_1 \sin\phi + \mathrm{i}c_2 \cos\phi$,
   it vanishes at $\mathbb{M}$.
   In fact $v^{(i)} = -c_1 \sin\phi + c_2 \cos\phi$ corresponds to
   $$v^{(r)} = \int_0^s k v^{(i)} ds = c_1 \cos\phi + c_2 \sin\phi,$$
   $$\partial_t Z = \left(v + \frac{c}{\partial_s Z} \right) \partial_s Z$$ **means translation**

2. **Rotation of SE(2)**
   $\partial_t Z = c' \partial_s Z$, $c \in \mathbb{R}$ means the rotation.
   It implies $Z = Z(s + c't)$ or $\partial_s Z = \mathrm{e}^{\mathrm{i}\phi(s+c't)} = \mathrm{e}^{\mathrm{i}\phi(s)+\mathrm{i}\phi_0}$
   It corresponds to $v^{(r)} \to v^{(r)\prime} = v^{(r)} + c'$ of ℓ_r^0.

Infinitesimal Isometric Deformation

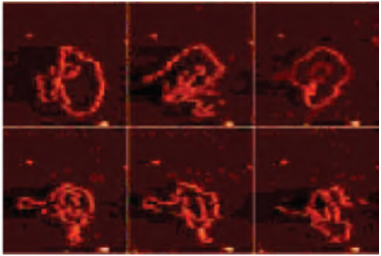
Tangent space $T_Z\mathcal{M}$ (Space of infinitesimal isometric deformation)

Proposition

For a point $Z \in \mathcal{M}$, the map $\ell^\flat$ induces the bijection $\ell^\flat$:

$$\mathcal{A}^0(\mathbb{R})/(\mathbb{R} \oplus (\mathbb{R}\cos\phi + \mathbb{R}\sin\phi)) \cong T_{\mathrm{pr}_1(Z)}(\mathbb{M})$$

Infinitesimal Isoenergy Deformation



Infinitesimal Isoenergy Deformation

isoenergy deformation

1. To consider the effect of energy $E(>0)$, we introduce
$$\mathcal{M}_E := \{Z \in \mathcal{M} \mid \mathcal{E}[Z] = E\}.$$
2. $\mathrm{pr}_{1,E} : \mathcal{M}_E \to \mathbb{M}_E := \mathcal{M}_E / \sim$

To investigate this geometric structure, we consider the subset of $T_Z\mathcal{M}$ which preseves the energy, i.e., infinitesimal isoenergy deformation :

Infinitesimal Isoenergy Deformation

isoenergy deformation

Proposition

At $Z \in \mathcal{M}$, the deformation is isoenergy, i.e., $\partial_t \mathcal{E}(Z) = 0$, if and only if $\partial_t k \in \hat{\mathcal{A}}^0(\mathbb{R})$.

proof

$$\partial_t \mathcal{E}(Z) = \partial_t \int k^2 ds = 2 \int k \partial_t k ds = \int \partial_s{}^{\exists} f ds = 0$$

because from the condition $\partial_t k \in \hat{\mathcal{A}}^0(\mathbb{R})$, $k\partial_t k ds \in d\mathcal{A}^0(\mathbb{R})$, i.e., $k\partial_t k = \partial_s{}^{\exists} f/2$, $(f \in \mathcal{A}^0(\mathbb{R}))$

Infinitesimal Isoenergy Deformation

1. $\partial_t Z$ isometric deformation in t

$$\Leftrightarrow \quad \begin{array}{l} \text{i) } v^{(i)} \in \hat{\mathcal{A}}^0(\mathbb{R}) \\ \text{ii) } \partial_t k = \Omega^{(II)} v^{(i)} \end{array}$$

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2. $\partial_t Z$ isometric and isoenergy deformation in t

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3. $\Rightarrow$ there might be another isometric deformation in another time t'

$$\Leftrightarrow \quad \text{i) } \partial_t k \in \hat{\mathcal{A}}^0(\mathbb{R}) \quad \text{ii) } \partial_{t'} k = \Omega^{(II)} \partial_t k$$

Infinitesimal Isoenergy Deformation

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$$\Leftrightarrow \quad \begin{array}{l} \text{i) } v^{(i)} \in \hat{\mathcal{A}}^0(\mathbb{R}) \\ \text{ii) } \partial_t k = \Omega^{(II)} v^{(i)} \end{array}$$
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$$\Leftrightarrow \quad \begin{array}{l} \text{i) } v^{(i)} \in \hat{\mathcal{A}}^0(\mathbb{R}) \\ \text{ii) } \partial_t k = \Omega^{(II)} v^{(i)} \\ \text{iii) } \partial_t k \in \hat{\mathcal{A}}^0(\mathbb{R}) \end{array}$$
3. $\Rightarrow$ there might be another isometric deformation in another time t'
$$\Leftrightarrow \quad \begin{array}{l} \text{i) } \partial_t k \in \hat{\mathcal{A}}^0(\mathbb{R}) \\ \text{ii) } \partial_{t'} k = \Omega^{(II)} \partial_t k = {\Omega^{(II)}}^2 v^{(i)} \end{array}$$

Infinitesimal Isoenergy Deformation

1. $\partial_t Z$ isometric deformation in t
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$$\Leftrightarrow \quad \begin{array}{l} \text{i) } \partial_t k \in \hat{\mathcal{A}}^0(\mathbb{R}) \\ \text{ii) } \partial_{t'} k = \Omega^{(II)} \partial_t k = {\Omega^{(II)}}^2 v^{(i)} \end{array}$$

$\Rightarrow$ These induce a certain hirarchy.

Infinitesimal Isoenergy Deformation

Tangent space $T_Z\mathcal{M}$

Proposition

If for $v^{(i)} \in \mathcal{A}^0(\mathbb{R})$, $\{{\Omega^{(II)}}^n v^{(i)}\}_{n=0,1,2,\dots}$ *beloing to* $\hat{\mathcal{A}}^0(\mathbb{R})$ *the parameters* $(\tilde{t}_1, \tilde{t}_2, \dots) \in [0, \varepsilon)$, *preserves the induced metric and the energy, and we have a sequence*

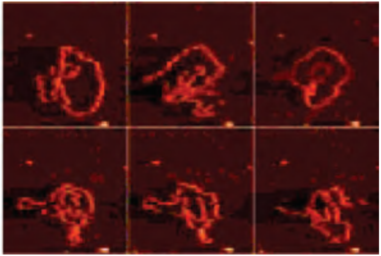
$$\partial_{\tilde{t}_1} k = \Omega^{(II)} v^{(i)},$$

$$\partial_{\tilde{t}_2} k = \Omega^{(II)} \partial_{\tilde{t}_1} k = {\Omega^{(II)}}^2 v^{(i)},$$

$$\partial_{\tilde{t}_3} k = \Omega^{(II)} \partial_{\tilde{t}_2} k = {\Omega^{(II)}}^2 \partial_{\tilde{t}_1} k = {\Omega^{(II)}}^3 v^{(i)},$$

$$\vdots$$

MKdV flow



Infinitesimal Isoenergy Deformation

Tangent space $T_Z\mathcal{M}$

Lemma

For $c \in \mathbb{R}$ and $Z \in \mathbb{M}$, the static (trivial) deformation, $Z(s+ct)$, is generatated by

$$\partial_t Z = c\partial_s Z, \Leftrightarrow \partial_t k = c\partial_s k.$$

Proposition

For the static deformation, $\mathbb{M}/\mathrm{U}(1)$ is stable, and the static deformation in $\mathcal{M}$ is isometric and isoenergy .

MKdV flow

Proposition

For $Z \in \mathbb{M}$ and $k := k[Z]$, we cosndier static deformation,

$$\partial_{t_1} k = \partial_s k,$$

and then we have the following isometric and isoenergy relations:

$$\partial_{t_2} k = \Omega^{(II)} \partial_{t_1} k = \Omega^{(II)} \partial_s k,$$

$$\partial_{t_3} k = \Omega^{(II)} \partial_{t_2} k = \Omega^{(II)^2} \partial_{t_1} k = \Omega^{(II)^2} \partial_s k,$$

$$\partial_{t_4} k = \Omega^{(II)} \partial_{t_2} k = \Omega^{(II)^2} \partial_{t_2} k = \Omega^{(II)^3} \partial_s k,$$

$$\vdots$$

These agree with the MKdV hierarchy.

MKdV flow

Orbital decompositon of $\mathcal{M}$

Since the MKdV hierarchy is integrable, we can consider the orbits in $\mathcal{M}$, $\mathbb{M}$, $\mathcal{M}_E$ and $\mathbb{M}_E$:

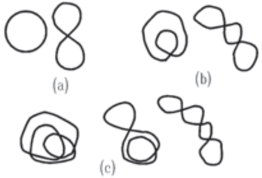
1. These orbits induce their **orbital decomposition**.
2. These orbits are described by **hyperelliptic functions** and **moduli space of hyperelliptic curves**.

$\Rightarrow$ We partially find their geometrical structure.

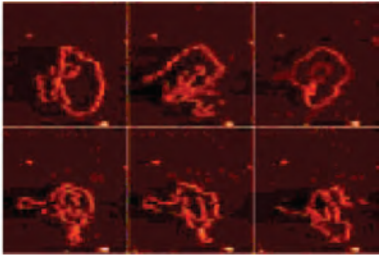
Solutions space

Open problems

1) The solution space contains **Euler's results** as genus one.
2) The solution of MKdV hierarchy is given by the **hyperelliptic curves including ∞ genus**.



Abelian Function Thoery



Abelian Function Thoery

Ellipti Function Thoery :

"The elliptic function theory is to study the algebraic properties of elliptic curves, the analytic properties of their abelian functions (=elliptic functions), and these relations."

$y^2 = (x - b_0)(x - b_1)(x - b_2)$ **standard form** **Algebraic Properties**	$\Leftrightarrow$	**Torus** σ**-func ／ entire func over** $\mathbb{C}$ **Analytic Properties**

Aim of the study of Ableian Functions

As the elliptic function theory has a power to various fields of mathematics, physics, engineer as concrete thoery of functions, we want to construct the Abelian function theory which has concrete and abstract expressions in order that it has a power to various fields.

Weierstrass Normal Form

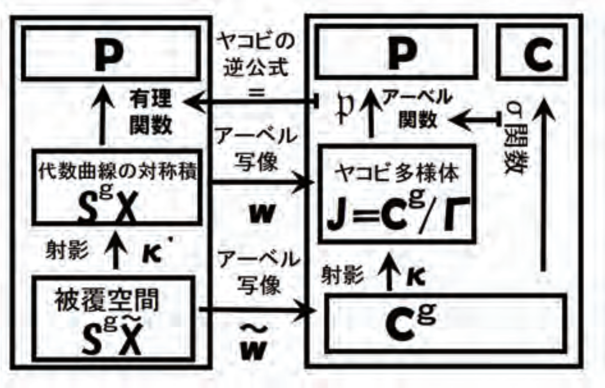
Weierstrass Normal Form

(X, P):Pointed Riemann surface $\quad P = \infty$
(X, P) is characterized by the Wierstrass gap sequence, whih is given by the numericl semi-group.
(X, ∞): $(r, s) = 1$,

$$y^r + A_1(x)y^{r-1} + \cdots + A_{r-1}(x)y + A_r(x) = 0$$

where $A_j(x)$ $(j = 1, \ldots, r-1)$ whose order is $j < js/r$) $A_r(x)$ is a s-order polynomial.

Jacobi inversion formulae



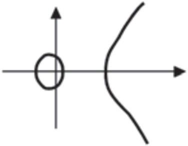
Abelian function theory for Hyperelliptic curves

As the Euler's elastica is related to elliptic function, the quantized elastica is related to the hyperelliptic function, (2003 MO, 2001, 2002 M), and naturally contains the Euler's elastica.

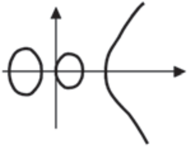
A hyperelliptic curve C_g of genus g ($g > 0$) is given by,

$$y^2 = (x - b_1)(x - b_2) \cdots \cdots (x - b_{2g+1}),$$

where b_j's are complex numbers.



$g = 1$ case



$g = 2$ case

Abelian function theory for Hyperelliptic curves

Hyperelliptic Integrals

Hyperelliptic complete integrals :

$$\omega'_{ij} := \int_{\alpha_i} \nu^I_j, \quad \omega''_{ij} := \int_{\beta_i} \nu^I_j, \quad i, j = 1, \ldots, g,$$

$$\eta'_{ij} := \int_{\alpha_i} \nu^{II}_j, \quad \eta''_{ij} := \int_{\beta_i} \nu^{II}_j, \quad i, j = 1, \ldots, g,$$

where hyperelliptic differentials, 1st and 2nd kinds:

$$\nu^I_i = \frac{x^{i-1}dx}{2y}, \quad \nu^{II}_i = \frac{(x^{g+i-1} + \sum_{j=1}^{g+i-2} a_{ij} x^j)dx}{2y}.$$

for certain a_{ij} of b_i's, $(i = 1, \ldots, g)$.

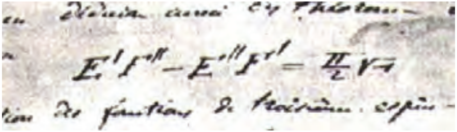
Abelian function theory for Hyperelliptic curves

Symplectic structure as Legendre relations

Legendre relations as the symplectic structure:

$$\omega'\eta'' - \omega''\eta' = \frac{\pi}{2}\sqrt{-1}I_g$$

This is the same as a part of Galois's letter to A. Chevalier:



Abelian function theory for Hyperelliptic curves

Hyperelliptic Jacobian

For a symmetric product space of C_g, $\mathrm{S}^g(C_g)$, the Abelian map is defined by

$$u := (u_1, \cdots, u_g) : \mathrm{S}^g(C_g) \longrightarrow \mathbb{C}^g,$$

$$\left(u_k((x_1, y_1), \cdots, (x_g, y_g)) := \sum_{i=1}^{g} \int_{\infty}^{(x_i, y_i)} \frac{x^{k-1} dx}{2y} \right).$$

The hyperelliptic Jacobian:

$$\mathcal{J}_g = \mathbb{C}^g / \Lambda, \quad \Lambda = < \omega', \omega'' >_{\mathbb{Z}} .$$

Abelian function theory for Hyperelliptic curves

theta function and sigma function

$\mathbb{T} = {\omega'}^{-1}\omega''$. The θ function on $\mathbb{C}^g$ with modulus $\mathbb{T}$ and characteristics $\mathbb{T}a + b$ is given by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z) = \theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\mathbb{T})$$
$$= \sum_{n\in\mathbb{Z}^g} \exp\left[2\pi\mathrm{i}\left\{\frac{1}{2}\ {}^t(n+a)\mathbb{T}(n+a) + {}^t(n+a)(z+b)\right\}\right]$$

for g-dimensional complex vectors a and b.
The σ-function is given by

$$\sigma(u) = \gamma_0 \exp\left\{-\frac{1}{2}\ {}^t u\eta'{\omega'}^{-1}u\right\}\vartheta\begin{bmatrix} \delta'' \\ \delta' \end{bmatrix}(\frac{1}{2}{\omega'}^{-1}u;\mathbb{T})$$

where δ and δ' are half-integer characteristics.

$\wp$ function

$$\wp_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u),$$

$$\zeta_i = \frac{\partial}{\partial u_i} \log \sigma(u)$$

$$\mathrm{al}_r := \sqrt{(b_r - x_1)(b_r - x_2)\cdots(b_r - x_g)} = \gamma_0' \frac{\mathrm{e}^{-\eta_r u}\sigma(u + \omega_r)}{\sigma(\omega_r)\sigma(u)},$$

Euler's results from a modern point of view

Euler's elastica and symplectic structure

$$\boxed{Z(s) = (-\zeta(s) + (a/6)s)/\mathrm{i},}$$

The symplectic structure in Jacobian is given by

$$\langle ds, \zeta(s)ds\rangle = 1$$

and

$$\omega'\eta'' - \omega''\eta' = \frac{\pi}{2}\mathrm{i}.$$

It means that for the space

$$G := \{(s, Z(s)) | s \in S^1\} \subset S^1 \times Z(S^1)$$

$T_* G$ has the "symplectic structure" $ds \wedge dZ$.

Hyperelliptic Solutions and Quantized Elastica

Theorem (2002, 2010 M) 1) For the hyperelliptic curve C_g, by lettings $s := u_g$, $Z_r \in \mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E}$ $(r = 1, 2, \cdots, 2g + 1)$ is given by

$$\partial_s Z_r(s) = \mathrm{al}_r(s)^2, \quad Z_r(s) = b_r^g s - \sum_{i=1}^{g} \zeta_i(s) b_r^{i-1}.$$

2) $Z_r(u \in \mathcal{J}_g)$ is isoenergy flows!!!
3) The energy is given by the hyperelliptic integrals:

$$\oint_{\alpha_a} k_r^2 ds = -4\eta'_{ag} + 2(\lambda_{2g} + b_r)\omega'_{ag}$$

4) $\mathrm{Vol}(\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E})$ is the volume of the real subspace in the Jacobi variety $\mathcal{J}_g$.

Hyperelliptic Solutions and Quantized Elastica

Remark 1) The shape of quantized elastica is

$$Z_r(s) = b_r^g s - \sum_{i=1}^g \zeta_i(s) b_r^{i-1},$$

whereas that of Euler's elastica is

$$Z'(s) = (a/6)s - \zeta(s) \text{ for } (Z'(s) = Z(s)/\sqrt{-1}).$$

2) The energy of quantized elastica is

$$\oint k^2 ds = -4\eta'_{ag} + 2(\lambda_{2g} + b_r)\omega'_{ag},$$

whereas that of Euler's elastica is

$$\oint k^2 ds = -4\eta' + 2(e_1)\omega'.$$

3) The generalization of Euler's relation is

$$Z(u) - Z(u - \omega) = \sum_i^g b^{i-1} \partial_i \log \partial_{t_1} Z.$$

Hyperelliptic Solutions and Quantized Elastica

Remark

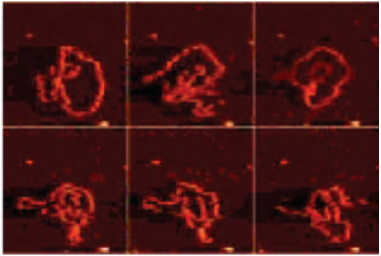
4) The shape of quantized elastica is

$$\begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_{g+1} \end{pmatrix} = \begin{pmatrix} b_1^g & b_1^{g-1} & b_1^{g-2} & \cdots & b_1 & 1 \\ b_2^g & b_2^{g-1} & b_2^{g-2} & \cdots & b_2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{g+1}^g & b_{g+1}^{g-1} & b_{g+1}^{g-2} & \cdots & b_{g+1} & 1 \end{pmatrix} \begin{pmatrix} s \\ \zeta_g \\ \vdots \\ \zeta_1 \end{pmatrix}.$$

$$\langle \zeta_r dt_{g-r}, dt_v \rangle = \delta_{r,v} \textbf{ means } \langle \sum_i \pi_{r,i} Z_i dt_{g-r}, dt_v \rangle = \delta_{r,v},$$

which is a "symplectic structure" in $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$.

Topological Properties



Topological Properties of Moduli of Quantized Elastica

Lemma (Maclachlan)

The modulus space of conformal equivalence classes of compact Riemann surfaces of genus g is simply connected.

MKdV hierarchy

For $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},g} \to \mathfrak{M}_{\mathrm{elas},g}, \quad (Z(\mathbb{T}^g) \mapsto pt)$, we have

$$\mathfrak{M}_{\mathrm{elas},g} \subset \mathfrak{M}_{\mathrm{hyp},g}, \quad \mathfrak{M}_{\mathrm{hyp},g} \sim pt.$$

Lemma (MO 2003)

Due to the relations $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},g} \setminus \mathcal{M}^{\mathbb{C}}_{\mathrm{elas},g-1} \sim \mathbb{T}^{g-1}$ and

$$pt \hookrightarrow S^1 \hookrightarrow \mathbb{T}^2 \hookrightarrow \mathbb{T}^3 \hookrightarrow \mathbb{T}^4 \hookrightarrow \mathbb{T}^5 \hookrightarrow \cdots,$$

we have

$$\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},1} \hookrightarrow \mathcal{M}^{\mathbb{C}}_{\mathrm{elas},2} \hookrightarrow \mathcal{M}^{\mathbb{C}}_{\mathrm{elas},3} \hookrightarrow \cdots.$$

Topological Properties of Moduli of Quantized Elastica

Theorem (Bott-Tu)

The cohomology of the loop space ΩS^n over S^n is given by

$$\mathrm{H}^p(\Omega S^n, \mathbb{R}) = \mathbb{R}\delta_{p \bmod (n-1), 0}.$$

For $n = 2$ case, the ring structure is given by

$$\mathrm{H}^*(\Omega S^2, \mathbb{R}) = \mathbb{R}[x]/(x^2) \cdot \mathbb{R}[e],$$

where $\mathrm{degree}(e) = 2$ and $\mathrm{degree}(x) = 1$.

$$\mathrm{H}^*(\Omega S^2, \mathbb{R}) = \mathbb{R} + \mathbb{R}x + \mathbb{R}e + \mathbb{R}xe + \mathbb{R}e^2 + \mathbb{R}xe^2 + \cdots .$$

Topological Properties of Moduli of Quantized Elastica

A loop space

Since $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$ is topologically decomposed by genus, we have:

Theorem (MO 2003)

For the forgetful functor $\mathrm{for}: \mathit{Diff} \to \mathit{Top}$, we have

$$\mathrm{H}^*(\Omega S^2, \mathbb{R}) = \mathrm{H}^*(\mathrm{for}(\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}), \mathbb{R})$$

i.e., for $\mathrm{H}^*(\Omega S^2, \mathbb{R}) = \mathbb{R}[x]/(x^2) \cdot \mathbb{R}[e]$, $\mathrm{H}^*(\mathrm{for}(\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}), \mathbb{R}) = \Lambda_{\mathbb{R}}[dt_1, \epsilon]$, where $\Lambda_{\mathbb{R}}[dt_1, \epsilon]$ is a ring generated by dt_1 and

$$\epsilon = dt_1 + dt_2 \wedge (dt_1 i_{\partial_1}) + dt_3 \wedge (dt_1 i_{\partial_1}) + \cdots$$

with the wedge product and the degree: $\mathrm{degree}(dt_i) = 1$:

$$\mathrm{H}^*(\mathrm{for}(\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}), \mathbb{R}) = \mathbb{R} + \mathbb{R}dt_1 + \mathbb{R}\epsilon + \mathbb{R}\epsilon dt_1 + \mathbb{R}\epsilon^2 + \mathbb{R}\epsilon^2 dt_1 + \cdots .$$

Proof:

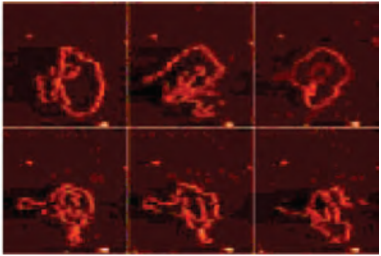
Since $\epsilon \cdot 1 = dt_1$, and $\epsilon^{n-1} \cdot dt_1 = \epsilon^n \cdot 1 = dt_n \wedge dt_{n-1} \wedge \cdots \wedge dt_2 \wedge dt_1$, we have

$$\begin{aligned}\Lambda_{\mathbb{R}}[dt_1, \epsilon] &= \mathbb{R} + \mathbb{R}dt_1 + \mathbb{R}\epsilon + \mathbb{R}\epsilon dt_1 + \mathbb{R}\epsilon^2 + \mathbb{R}\epsilon^2 dt_1 + \cdots \\ &= \mathbb{R} + \mathbb{R}dt_1 + \mathbb{R}dt_1 \wedge dt_2 + \mathbb{R}dt_1 \wedge dt_2 \wedge dt_3 + \cdots .\end{aligned}$$

Due to the **Bäcklund transformation**, $\mathcal{M}_{\mathrm{elas}}^{\mathbb{C}}$ is topologically given as **a telescopic type space** related to these genera. Hence we have

$$\mathrm{H}^*(\mathrm{for}(\mathcal{M}_{\mathrm{elas}}^{\mathbb{C}}), \mathbb{R}) = \Lambda_{\mathbb{R}}[dt_1, \epsilon].$$

Final Remark



Final Remark

Open problems

1. These relations are closely related to $\log \dfrac{Z(p) - Z(q)}{p - q}$, which is also related to replicable functions in Monster group by John McKay.
   Investigate this fact!!!!
2. Show the explicit expression of quantized elastica or quantized elastica of genus $g > 1$ in terms of computer graphic and so on.
3. Show the degenerate limit from the quantized elasticas of g to $g - 1$.

Final Remark

Open problems

1. A quantized elastica in (p, q)-dimensional Minkswski space with so(p, q) and generalized MKdV equation.
2. **Willmore surface** (Polynakov extrinsic string) and MNV hierarchy (M 1999),
3. A geometrical object expressed by **generalized Weierstrass representation** of submanifold Dirac operator (M 2008, 2009),
4. **Diff/SDiff** for a manifold which B. Khesin (**Arnold-Khesin**) considers, or fluid dynamics.

Reference

open problems

Since we partially have the hyperelliptic solutions of loop solitons (M 2002), we will consider the moduli space.

Reference

1. S. Matsutani, , *Hyperelliptic loop solitons with genus* g*: investigation of a quantized elastica*, , J. Geom. Phys., , **43** (2002) 146-162
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4. S.Matsutani, Emma Previato, *From Euler's elastica to the mKdV hierarchy, through the Faber polynomials*, , J. Math. Phys., **57** (2016) 081519; arXiv:1511.08658

Thank you!!