Lagrange対をもつ Monge-Ampère系

待田芳德

——石川剛郎さんとの共同研究

(I) 2変数Monge-Ampère方程式

$$F = A(rt-s^2) + Br + 2Cs + Dt + E = 0$$

（2独立変数 x, y；1未知関数 z
$p = z_x, q = z_y$；$r = z_{xx}, s = z_{xy}, t = z_{yy}$
（A, B, C, D, E は，x, y, z, p, q の関数

- × 線形，準線形を含む
- × $\mathrm{Hess} = \begin{vmatrix} r & s \\ s & t \end{vmatrix} = rt - s^2 = f(x, y, z, p, q)$

① $r + t = 0$ （Laplace eq.） $\wr$ $\mathrm{Hess} = 1$

② $r - t = 0,\ s = 0$ （wave eq.） $\wr$ $\mathrm{Hess} = -1$

③ $r = 0$ （定常 heat eq.） $\wr$ $\mathrm{Hess} = 0$ — 接触変換 （cf. 古典ゲージ変換）

$J^1(\mathbb{R}^2, \mathbb{R}) = \mathbb{R}^5 : (x, y, z, p, q)$ J^2 ではない

- $\theta = dz - p\,dx - q\,dy$ 接触形式
  $d\theta = dx \wedge dp + dy \wedge dq$
- $\omega = A\,dp \wedge dq + B\,dp \wedge dy + C(dx \wedge dp + dq \wedge dy) + D\,dx \wedge dq + E\,dx \wedge dy$ 2-form

- 解：$L^2 \overset{\iota}{\hookrightarrow} J^1$ s.t. $\iota^*\theta = 0,\ \iota^*\omega = 0$ （$\iota^*(dx \wedge dy) \neq 0$）古典解

∵ $z = \varphi(x, y)$ とすると，$p = \frac{\partial \varphi}{\partial x}, q = \frac{\partial \varphi}{\partial y}$；$dp = \frac{\partial^2 \varphi}{\partial x^2}dx + \frac{\partial^2 \varphi}{\partial x \partial y}dy,\ dq = \frac{\partial^2 \varphi}{\partial x \partial y}dx + \frac{\partial^2 \varphi}{\partial y^2}dy$

M-A方程式を幾何的にどうとらえるか？
解を幾何的にどうとらえるか？

Monge-Ampère系

- (M^{2n+1}, D)：接触多様体，$D = \mathrm{Ker}\,\theta = \{\theta = 0\} \subset TM$
- $\mathcal{M} = \langle \theta, \omega \rangle_{diff}$　ω : n-form
- 解：$L^n \overset{\iota}{\hookrightarrow} M^{2n+1}$ integral mfd. s.t. $\iota^*\theta = 0$, $\iota^*\omega = 0$
  (幾何的解)

cf. 代数方程式 ⟷ 多項式環のイデアル
　　多様体
　　微分方程式 ⟷ 微分形式 $\Omega(M)$ の微分イデアル

$(M, D, \mathcal{M}) \cong (M', D', \mathcal{M}')$ 同型
$\Leftrightarrow \exists \Phi : M \to M'$ diffeo. s.t. $\Phi_* D = D'$, $\Phi^* \mathcal{M}' = \mathcal{M}$

$n = 2$

- $J^2 = \mathbb{R}^8 : (x, y, z, p, q, r, s, t) \supset \Sigma^7 = \{F = 0\}$
  $\downarrow$
  $J^1 = \mathbb{R}^5 : (x, y, z, p, q)$　with $\mathcal{M} = \langle \theta, \omega \rangle$
  $\downarrow$
  $J^0 = \mathbb{R}^3 : (x, y, z)$

$\begin{cases} \theta_0 = dz - p\,dx - q\,dy \\ \theta_1 = dp - r\,dx - s\,dy \\ \theta_2 = dq - s\,dx - t\,dy \end{cases}$

$\theta = dz - p\,dx - q\,dy$

- $L^2 \longrightarrow J^0$　古典解
  $L^2 \longrightarrow J^1$　幾何的解　・多価性　・特異性 波面，コースティック
  $\qquad\qquad\downarrow$
  $\qquad\qquad J^0$
  $L^2 \longrightarrow J^1$　双対性でみる
  $J^0 \swarrow \quad \searrow (J^0)^*$

Legendre変換　$(x, y, z, p, q) \longleftrightarrow (X, Y, Z, P, Q)$

$X = p$, $Y = q$, $Z = xp + yq - z$, $P = x$, $Q = y$

(x, y, z)　　(X, Y, Z)

部分Legendre変換
$(x \to p,\ p \to -x,\ z \to z - xp)$

① $(r+t=0)$ $\omega = dp\wedge dy + dx\wedge dz \longleftrightarrow (\mathrm{Hess}=1)$ $\omega = dx\wedge dy - dp\wedge dz$

② $(r-t=0)$ $\omega = dp\wedge dy - dx\wedge dz \longleftrightarrow (\mathrm{Hess}=-1)$ $\omega = dx\wedge dy + dp\wedge dz$

$(s=0)$ $\omega = dx\wedge dp,\ dz\wedge dy$

③ $(r=0)$ $\omega = dp\wedge dy \longleftrightarrow (\mathrm{Hess}=0)$ $\omega = dp\wedge dz$

拡張

- $\mathrm{Hess}=0,\ r=0\ ;\ s=0 \Longrightarrow$ 分解可能M-A系 (decomposable) $\omega = \omega_1\wedge\cdots\wedge\omega_n$ $\begin{cases} \text{Lagrange型} \\ \text{非Lagrange型} \end{cases}$
- $\mathrm{Hess}=1,\ \mathrm{Hess}=-1 \Longrightarrow$ 双分解可能 M-A系 (bi-decomposable) $\omega = \omega_1 - \omega_2$ — Lagrange対型
- $r+t=0,\ r-t=0 \Longrightarrow$ 複素分解可能M-A系 (complex decomposable) $\omega = \mathrm{Im}\,\Omega$ — CR型

(II) Lagrange対型Monge-Ampère系 (M^{2n+1}, D, ω)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ rk$=2n$ $\quad$ n-form

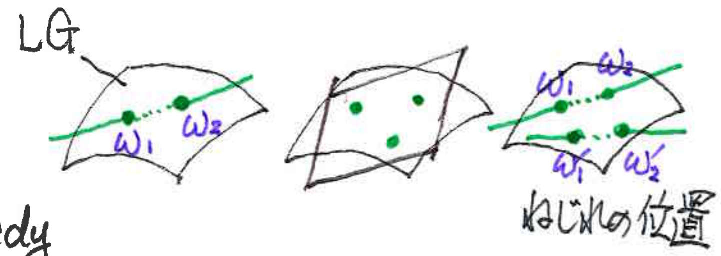
bi-decomposable M-A系 $\mathcal{M} = \langle \theta, \omega \rangle$

$$\begin{cases} D = E_1 \oplus E_2 : & \text{Lagrange対} \\ \omega = \omega_1 - \omega_2 \text{ s.t.} & i_u\omega_1 = 0\ (u \in E_2),\ i_v\omega_2 = 0\ (v \in E_1) \\ & \omega_1|_{E_1} : \text{volume form on } E_1,\ \omega_2|_{E_2} : \text{volume form on } E_2 \end{cases}$$

- $GL(n,\mathbb{R}) \times GL(n,\mathbb{R})$ almost product str. on D
  $\cup$
  $GL(n,\mathbb{R})$ Lagrange対
  $\cup$
  $SL(n,\mathbb{R})$ M-A系 (volume form)

- × $\mathcal{M}$は一意にω (mod θ)を決める. i.e. $\mathcal{M} = \langle \theta, \omega \rangle = \langle \theta, \omega' \rangle \Longrightarrow \omega' = \lambda\omega + \eta \wedge \theta$
  × $\omega \in \mathcal{M}$は一意に対(E_1, E_2)を決める. $n \geq 3$ i.e. $\omega = \omega_1 - \omega_2, (E_1, E_2)$ $= \omega_1' - \omega_2', (E_1', E_2')$ $\Rightarrow E_1 = E_1', E_2 = E_2'$
  × $\mathcal{M}$の同型写像は対(E_1, E_2)を保存する. $n \geq 3$

$LG = \{\text{Lag.v.s.}\} \subset Gr(n, V^{2n}) \xrightarrow{\text{Plücker}} P(\wedge^n V)$
$\qquad\qquad\qquad\qquad$ symplectic



(注) $n=2$では成り立たない. $M = \mathbb{R}^5 : (x,y,z,p,q)$, $\theta = dz - p\,dx - q\,dy$
Hess$=1$. $\omega = dx \wedge dy - dp \wedge dq$
$\qquad = d(x+p) \wedge dy - dp \wedge d(y+q)$

○例

1) $\mathbb{R}^{n+1}$上 Hess$=C$ → $\mathbb{R}^{2n+1}$上 M-A系

$M=\mathbb{R}^{2n+1}:(x,z,p)=(x_1,\cdots,x_n,z,p_1,\cdots,p_n)$

$\theta = dz-\sum_{i=1}^{n} p_i dx_i$ contact form

$D=\{\theta=0\}$

$\quad = E_1 \oplus E_2$

$E_1=\left\langle \frac{\partial}{\partial x_1}+p_1\frac{\partial}{\partial z}, \cdots, \frac{\partial}{\partial x_n}+p_n\frac{\partial}{\partial z}\right\rangle = \mathrm{Ker}\,\pi_{2*}$

$E_2=\left\langle \frac{\partial}{\partial p_1}, \cdots, \frac{\partial}{\partial p_n}\right\rangle = \mathrm{Ker}\,\pi_{1*}$

$$\begin{tikzcd}[column sep=small] & M=\mathbb{R}^{2n+1}\ (x,z,p) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ W_1=\mathbb{R}^{n+1} & & W_2=\mathbb{R}^{n+1} \end{tikzcd}$$

$\pi_1(x,z,p)=(x,z)$ $\qquad \pi_2(x,z,p)=(p,\tilde{z})$, $\tilde{z}=\sum_{i=1}^{n}p_i x_i - z$

$\Omega_1 = c(dz\wedge dx_1\wedge\cdots\wedge dx_n)$ on W_1 $\qquad (c\in\mathbb{R})$

$\Omega_2 = -d\tilde{z}\wedge dp_1\wedge\cdots\wedge dp_n$ on W_2

$\omega_1 = i_R(\pi_1^*\Omega_1) = c\,dx_1\wedge\cdots\wedge dx_n$ on M $\qquad (R=\frac{\partial}{\partial z}$: Reeb v.f.$)$

$\omega_2 = i_R(\pi_2^*\Omega_2) = dp_1\wedge\cdots\wedge dp_n$ on M

$\omega=\omega_1-\omega_2$

$D=E_1\oplus D_2$ → $M=\langle\theta,\omega\rangle$ Lagrange対型M-A系 $\qquad W_1$上 Hess$=c$

W_2上 Hess$=\frac{1}{c}$

2) E^{n+1} 上 $K=c$ → T_1E^{n+1} 上 M-A系

ガウス曲率

$$M = T_1E^{n+1} = E^{n+1}\times S^n \subset E^{n+1}\times E^{n+1} : (x,y)=(x_1,\cdots,x_{n+1},y_1,\cdots,y_{n+1})$$

$$\theta = \sum_{i=1}^{n+1} y_i dx_i \Big|_{E^{n+1}\times S^n} \quad \text{contact form}$$

$$D = \{\theta = 0\} = E_1 \oplus E_2$$

$$E_1 = \left\{ u = \xi_1\frac{\partial}{\partial x_1}+\cdots+\xi_{n+1}\frac{\partial}{\partial x_{n+1}} \,\middle|\, \xi_1 y_1+\cdots+\xi_{n+1}y_{n+1}=0 \right\} = \mathrm{Ker}\,\pi_{2*}$$

$E_2 = \left\{ v = \eta_1\frac{\partial}{\partial y_1}+\cdots+\eta_{n+1}\frac{\partial}{\partial y_{n+1}} \,\middle|\, v \text{は} S^n \text{に接する} \right\} = \mathrm{Ker}\,\pi_{1*}$

$$\begin{array}{ccc} & M = E^{n+1}\times S^n \ (x,y) & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ W_1 = E^{n+1} & & W_2 = \mathbb{R}\times S^n \end{array}$$

$\pi_1(x,y)=x \qquad \pi_2(x,y) = (z = x\cdot y,\ y)$

$$\Omega_1 = c\,dx_1\wedge\cdots\wedge dx_{n+1} \quad \text{on } W_1$$

$\left(R = y_1\frac{\partial}{\partial x_1}+\cdots+y_{n+1}\frac{\partial}{\partial x_{n+1}} : \text{Reeb v.f.} \right)$

$$\Omega_2 = dz\wedge\sum_{i=1}^{n+1}\left((-1)^{i+1} y_i\, dy_1\wedge\cdots\wedge\check{dy_i}\wedge\cdots\wedge dy_{n+1}\right)\Big|_{\mathbb{R}\times S^n}$$

$$\omega_1 = i_R(\pi_1^*\Omega_1) = c\,(y_1 dx_2\wedge\cdots\wedge dx_{n+1} - y_2 dx_1\wedge dx_3\wedge\cdots\wedge dx_{n+1}+\cdots+(-1)^n y_{n+1} dx_1\wedge\cdots\wedge dx_n)$$

$$\omega_2 = i_R(\pi_2^*\Omega_2) = y_1 dy_2\wedge\cdots\wedge dy_{n+1} - y_2 dy_1\wedge dy_3\wedge\cdots\wedge dy_{n+1}+\cdots+(-1)^n y_{n+1} dy_1\wedge\cdots\wedge dy_n$$

$\omega = \omega_1 - \omega_2$

$E = E_1 \oplus E_2$ → $M = \langle \theta, \omega \rangle$ Lagrange対型M-A系

Mの解のW_1への射影は $K=c$ のn次元 subvariety

○ クラス

各クラスは接触変換で不変 $(n \geqq 3)$

{ Lagrange対型 M-A系 } ex. 一般の Riemann mfd. W^{n+1} での $K=C$

$\cup$

{ bi-integrable Lag. 対型 M-A系 } E_1, E_2 : 完全積分可能

$\cup$

{ Hesse M-A系 } $\mathrm{Hess}(z) = f(x,z,p)\ (\neq 0)$

$\cup$

{ Euler-Lagrange M-A系 } $\mathrm{Hess}(z) = f_1(x,z)\cdot f_2(p,\tilde{z})\ (\neq 0)$

ex. $E^{n+1}, S^{n+1}, H^{n+1}$ での $K=C$

$\cup$

{ flat M-A系 } $\mathrm{Hess}(z) = c\ (\neq 0)$

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ W_1 = M/\mathcal{E}_2 & & W_2 = M/\mathcal{E}_1 \end{array}$$

E_1 : 水平的
E_2 : 垂直的

$$\begin{array}{ccc} & M = P(T^*P^{n+1}) & \\ \swarrow & & \searrow \\ W_1 = P^{n+1} & & W_2 = (P^{n+1})^* \end{array}$$

○ 対称性

Thm { Mの自己同型群は、有限次元で、高々$(n+1)^2$次元である。$n \geqq 3$

attainする例(1)

$$G = \left\{ \bar{A} = \begin{pmatrix} \frac{1}{k} & 0 & 0 \\ b & A & 0 \\ c & {}^t a & k \end{pmatrix} \middle| k \in \mathbb{R}^\times, A \in SL(n,\mathbb{R}), a, b \in \mathbb{R}^n, c \in \mathbb{R} \right\} \subset PGL(n+2, \mathbb{R})$$

Gは $M = \mathbb{R}^{2n+1} : (x,z,p)$ 上 transitive に act する：

$$\bar{A}(x,z,p) = \left(k(Ax+b),\ k(kz - {}^t a x - c),\ k\left(p - \tfrac{1}{k}{}^t a\right)A^{-1}\right)$$

そのとき、$\theta \longmapsto k^2\theta,\ \omega \longmapsto k^n\omega$ よって $\mathrm{Aut}(M) = G,\ \dim G = (n+1)^2$

㊟ $n=2$では成り立たない。$\mathrm{Hess} = -1 \sim s = z_{xy} = 0$ において、$(x,y) \longmapsto (X(x), Y(y))$ は無限次元

(Ⅲ) CR型 Monge-Ampère系 (M^{2n+1}, D, ω)

← complex decomposable M-A系 $\mathcal{M} = \langle \theta, \omega \rangle$ (D: rk=2n, ω: n-form)

$$\begin{cases} D : \text{complex str. } J \\ \quad\ \text{complex volume form } \Omega \\ \omega \text{ st. } \omega|_D = \operatorname{Im}\Omega = \frac{1}{2i}(\Omega - \bar{\Omega}) \end{cases}$$

- $GL(n,\mathbb{C})$ — almost complex str. on D

$$\begin{array}{cc|cl} & GL(n,\mathbb{C}) & & \\ & \cup & & \\ & U(n) & U(1,1) & \text{symplectic str.} \\ & \cup & \cup & \\ & SU(n) & SU(1,1) & \text{M-A系 (complex volume form)} \end{array}$$

- × 方程式 — Sasaki-Einstein, AdS/CFT
  cf. Calabi-Yau

  × 解 — special Legendrian
  cf. calibration, special Lagrangian

$$\left(\begin{array}{ccccc} Gr(n,V) & \supset & LG & \supset & SLG \\ & & \| & & \| \\ & & U(n)/SO(n) & & SU(n)/SO(n) \\ \dim = n^2 & & \frac{n(n+1)}{2} & & \frac{n(n+1)}{2} - 1 \end{array} \right.$$

○例

1) • $M^5 = \mathbb{R}^5 : (x, y, z, p, q)$ $\quad \theta = dz - p\,dx - q\,dy$

J : $d\omega_1 = dx + i\,dp$, $d\omega_2 = dy + i\,dq$ $\quad$ (1,0)-form

$\left\langle \frac{1}{2}\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - i\frac{\partial}{\partial p}\right), \frac{1}{2}\left(\frac{\partial}{\partial y} + q\frac{\partial}{\partial z} - i\frac{\partial}{\partial q}\right) \right\rangle$ (1,0)-vector $\quad D \cong \mathbb{C}^2$

$\Omega = d\omega_1 \wedge d\omega_2 = dx \wedge dy - dp \wedge dq + i(dx \wedge dq + dp \wedge dy)$ $\quad$ SU(2)-str.

$\omega = dx \wedge dq + dp \wedge dy$ s.t. $\omega|_D = \mathrm{Im}\,\Omega$ $\quad$ SU(2)型 M-A系

$z = f(x, y)$: 解 $\Longrightarrow \Delta(f) = 0$ $(\sim \mathrm{Hess}(f) = 1)$

• $M^7 = \mathbb{R}^7 : (x_1, x_2, x_3, z, p_1, p_2, p_3)$ $\quad \theta = dz - p_1 dx_1 - p_2 dx_2 - p_3 dx_3$

J : $d\omega_1 = dx_1 + i\,dp_1$, $d\omega_2 = dx_2 + i\,dp_2$, $d\omega_3 = dx_3 + i\,dp_3$ $\quad$ (1,0)-form

$\left\langle \frac{1}{2}\left(\frac{\partial}{\partial x_i} + p_i\frac{\partial}{\partial z} - i\frac{\partial}{\partial p_i}\right)_{i=1,2,3} \right\rangle$ (1,0)-vector $\quad D \cong \mathbb{C}^3$

$\Omega = d\omega_1 \wedge d\omega_2 \wedge d\omega_3$ $\quad$ SU(3)-str.

$\omega = dx_1 \wedge dx_2 \wedge dp_3 + dx_1 \wedge dp_2 \wedge dx_3 + dp_1 \wedge dx_2 \wedge dx_3 - dp_1 \wedge dp_2 \wedge dp_3$ s.t. $\omega|_D = \mathrm{Im}\,\Omega$

$z = f(x_1, x_2, x_3)$: 解 $\Longrightarrow \Delta(f) - \mathrm{Hess}(f) = 0$ $\quad$ SU(3)型 M-A系

- $M^{2n+1} = \mathbb{R}^{2n+1} : (x_1, \cdots, x_n, z, p_1, \cdots, p_n)$ $\quad \theta = dz - \sum_{i=1}^{n} p_i dx_i$

  $J : dw_1 = dx_1 + i dp_1, \cdots, dw_n = dx_n + i dp_n$ (1,0)-form

  $\left\langle \frac{1}{2}\left(\frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial z} - i \frac{\partial}{\partial p_i}\right)_{i=1,\cdots,n} \right\rangle$ (1,0)-vector $\quad D \cong \mathbb{C}^n$ (Levi form 定値 強擬凸

  $\Omega = dw_1 \wedge \cdots \wedge dw_n$ $\quad$ $SU(n)$-str.

  ω s.t. $\omega|_D = \mathrm{Im}\,\Omega$ $\quad$ $SU(n)$型 M-A系 (自己同型群は有限次元で、高々$(n+1)^2$次元 ($n \geq 3$) Lag対型と同じ

  $z = f(x_1, \cdots, x_n)$: 解 $\Rightarrow$ $\sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \sigma_{2k+1}(\mathrm{Hess}(f)) = 0$

  (cf. $\mathbb{C}^n$ での special Lag. eq.)

Riemann多様体 W^{n+1} での austere部分多様体 L^n の

$SU(n)$型M-A系 on $M^{2n+1} = T_1 W$ (きびしい、簡素な) shape operatorが固有値の正、負の対をもつ

- austere $\Rightarrow$ minimal i.e. $H=0$
- 特に n:奇数 $\Rightarrow$ $H=0, K=0$ (Hessian : Hess − Laplacian : Δ
- $n=2$ i.e. M^5; $W^3 \supset L^2$: austere $\Leftrightarrow$ $H=0$ ガウス曲率K − 平均曲率H
- $n=3$ i.e. M^7; $W^4 \supset L^3$: austere $\Leftrightarrow$ $H=0, K=0$

2) $M^5 = \mathbb{R}^5 : (x, y, z, p, q)$ $\theta = dz - p\,dx - q\,dy$

- J : $d\omega_1 = \underset{\sim\sim\sim}{dx + i\,dy}$, $d\omega_2 = \underset{\sim\sim\sim}{dp - i\,dq}$ (1,0)-form

  $\left\langle \frac{1}{2}\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - i\left(\frac{\partial}{\partial y} + q\frac{\partial}{\partial z}\right)\right), \frac{1}{2}\left(\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right) \right\rangle$ (1,0)-vector $D \cong \mathbb{C}^2$

  $\Omega = d\omega_1 \wedge d\omega_2 = dx\wedge dp + dy \wedge dq + i(-dx\wedge dq + dy\wedge dp)$ (Levi form (1,1)型

  $\omega = -dx\wedge dq + dy \wedge dp$ s.t. $\omega|_D = \mathrm{Im}\,\Omega$ SU(1,1)-str.

  $z = f(x,y)$: 解 $\Rightarrow$ $\Delta(f) = 0$ ($\sim \mathrm{Hess}(f) = 1$) SU(1,1)型 M-A系

- J : $d\omega_1 = dx + i\,dy$, $d\omega_2 = dp - i\,dq \underset{\sim\sim\sim}{- \frac{c}{2}(dx - i\,dy)}$ (1,0)-form

  $\left\langle \frac{1}{2}\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + \frac{c}{2}\frac{\partial}{\partial p} - i\left(\frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + \frac{c}{2}\frac{\partial}{\partial q}\right)\right), \frac{1}{2}\left(\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right) \right\rangle$ (1,0)-vector $D \cong \mathbb{C}^2$
  ($c \neq 0$)

  $z = f(x,y)$: 解 $\Rightarrow$ $\Delta(f) = c$ ($\neq 0$)

Riemann多様体 W^3 での $\left\{\begin{array}{ll} \text{極小曲面} & : H = 0 \\ \text{平均曲率一定曲面 (CMC)} & : H = c\ (\neq 0) \end{array}\right\}$ L^2 の
$\mathbb{E}^3, S^3, H^3$

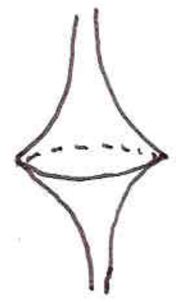
SU(1,1)型 M-A系 on $M^5 = T_1 W$

(IV) 解, 特異点

- Lagrange型M-A系 $\mathcal{M}=\langle\theta,\omega\rangle$, $D=E_1\oplus E_2$, $\omega=\omega_1-\omega_2$
  - $\iota: L^n \longrightarrow M^{2n+1}$ が解, $p\in L$

$$\iota_* T_pL \cap E_{1|p}=\{0\} \Longleftrightarrow \iota_* T_pL \cap E_{2|p}=\{0\}$$
$$\iota_* T_pL \cap E_{1|p}\neq\{0\} \Longleftrightarrow \iota_* T_pL \cap E_{2|p}\neq\{0\}$$

  - 大域的解
    - × $\mathrm{Hess}=C\ (>0)$ global sol. $\Rightarrow$ 2次多項式
    - × $K=C\ (>0)$ compact sol. $\Rightarrow$ 球面
    - × $K=C\ (<0)$ complete sol. $\Rightarrow$ ない



それ以外, 解の特異点

- CR型M-A系 $\mathcal{M}=\langle\theta,\omega\rangle$, J on D, $\omega|_D=\mathrm{Im}\,\Omega$

SU(1,1)型M-A系 on M^5 に対して,

$\iota: L^2 \longrightarrow M^5$ が解

$\Longleftrightarrow$ J-正則 Legendrian

(cf. SU(n)型に対して, 解はJ-直線を含まない

Legendre射影(wave front)が極小曲面, CMC曲面のとき, ガウス写像へは正則, 調和写像

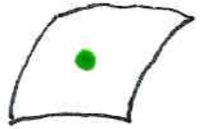
$n=2$

- $\iota : L^2 \longrightarrow M = \mathbb{R}^5 \quad (x,y,z,p,q) \qquad \theta = dz - p\,dx - q\,dy$

  $\iota^*\theta = 0$

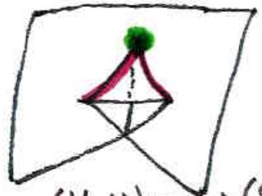
  $$\begin{array}{c} M=\mathbb{R}^5 \\ \downarrow \pi \\ W=\mathbb{R}^3 \quad (x,y,z) \end{array}$$

  Legendre射影 $\pi\circ\iota$ の generic な Legendre特異点：

  immersion A_1 

  cuspidal edge A_2 

  $(u,v) \longmapsto (x,y,z,p,q) = (u, v^2, \frac{2}{3}v^3, 0, v)$

  swallow tail A_3 

  $(u,v) \longmapsto (x,y,z,p,q) = (u, v^3+uv, \frac{3}{4}v^4+\frac{1}{2}uv^2, -\frac{1}{2}v^2, v)$

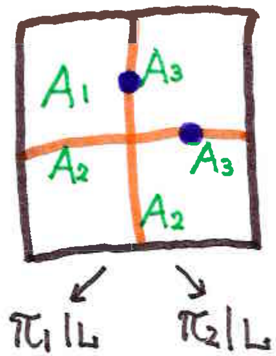
- <u>Hess$=C$, $\Delta=C$</u>

  $\iota : L^2 \xrightarrow{\text{Leg.}} M = \mathbb{R}^5 \quad (x,y,z,p,q)$

  $\pi_1 : M \to W_1 = \mathbb{R}^3 \quad (x,y,z)$, $\quad \pi_2 : M \to W_2 = \mathbb{R}^3 \quad (p,q,xp+yq-z)$

  Legendre射影 $\pi_1\circ\iota$, $\pi_2\circ\iota$ の generic な π_1-, π_2-Legendre特異点

  (方程式なし)

  6つ

$\pi_1\circ\iota$		$\pi_2\circ\iota$
A_1	—	A_1
A_1	—	A_2
A_2	—	A_1
A_2	—	A_2
A_1	—	A_3
A_3	—	A_1

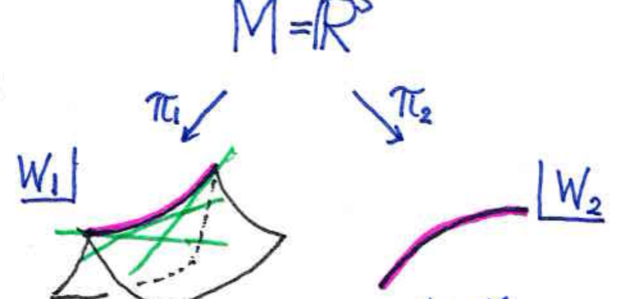
  <u>$K=C$, $H=C$</u>

  $\iota : L^2 \xrightarrow{\text{Leg.}} M = E^3 \times S^2 \subset \mathbb{R}^3 \quad (x,y)$

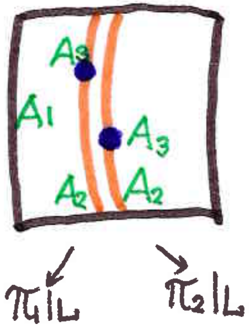
  $\pi_1 : M \to W_1 = E^3 \quad (x)$, $\quad \pi_2 : M \to W_2 = \mathbb{R}\times S^2 \quad (x\cdot y, y)$

Prop. $\mathrm{Hess}=0$ の generic な解の特異点は，
$(K=0)$　　可展面（柱面，錐面；接線曲面）

$$\left(\begin{array}{ccc} \underline{\pi_1\circ L} & & \underline{\pi_2\circ L} \\ A_2, A_3 & \text{——} & \text{generic な非特異曲線} \\ \text{generic な非特異曲線} & \text{——} & A_2, A_3 \end{array}\right.$$



Thm. Lagrange 対型　$\mathrm{Hess}=1$ の generic な解の特異点は，
$(\mathrm{Hess}=-1, K=1, K=-1)$



4つ
$$\left(\begin{array}{ccc} \underline{\pi_1\circ L} & & \underline{\pi_2\circ L} \\ A_1 & \text{——} & A_1 \\ A_2 & \text{——} & A_2 \\ A_2 & \text{——} & A_3 \\ A_3 & \text{——} & A_2 \end{array}\right.$$

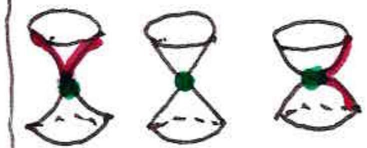
Thm. SU(1,1)型

Laplace eq.　Poisson eq.
$\Delta=0$, $\Delta=C$ の generic な解の特異点は，

$\underline{\pi_1\circ L}$　　$\underline{\pi_2\circ L}$
A_1 —— A_1
A_1 —— 
 —— A_1


(2,3)型カスプ曲線の回転面

極小曲面　CMC曲面
$H=0$, $H=C$ の generic な解の特異点は，

$\underline{\pi_1\circ L}$　　$\underline{\pi_2\circ L}$
A_1 —— A_1
A_1 —— 
 —— A_1

$H=0$		
$H=C$ $K<0$	$H=C$ $K=0$	$H=C$ $K>0$