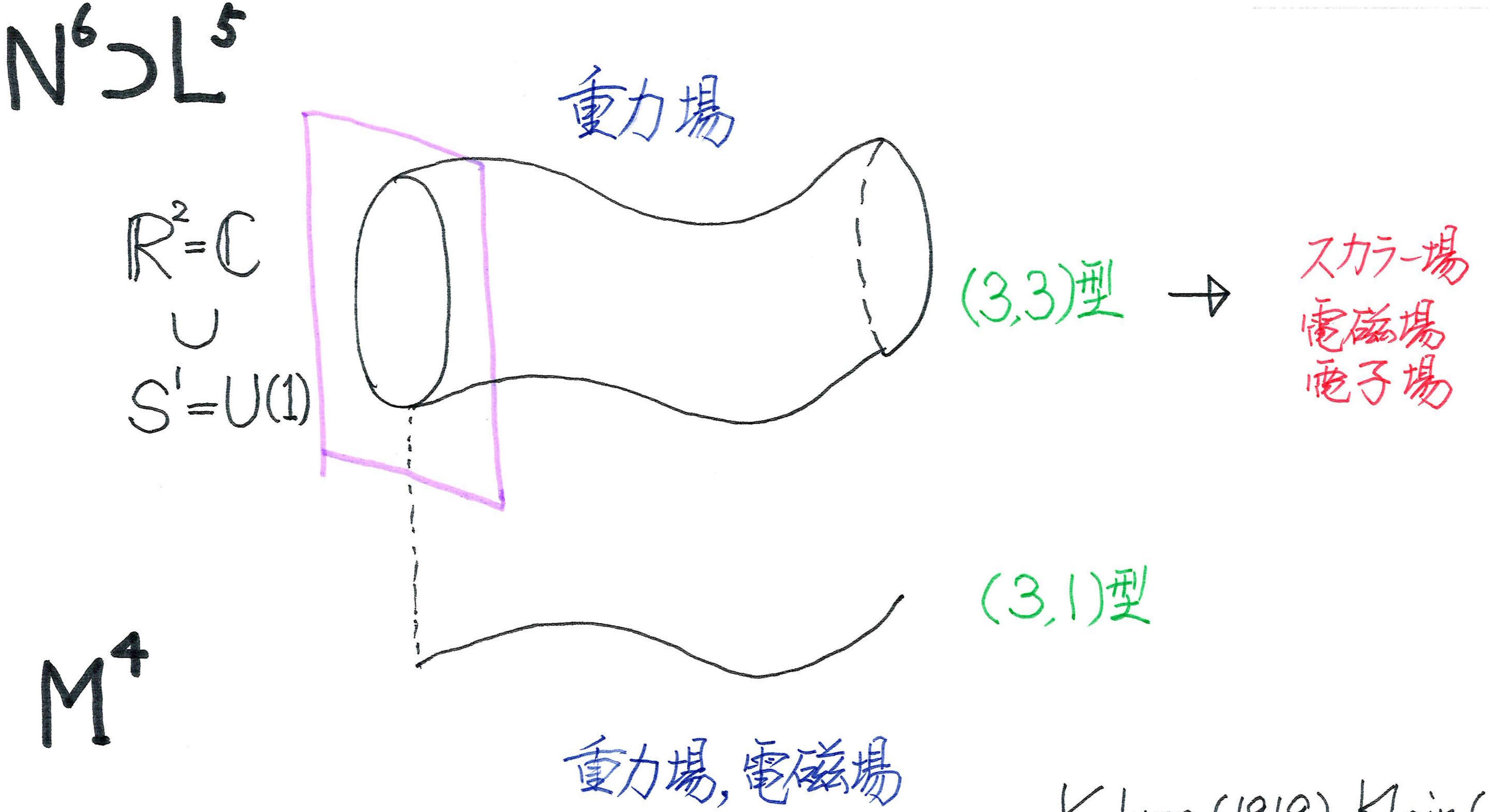
Kaluza-Klein理論

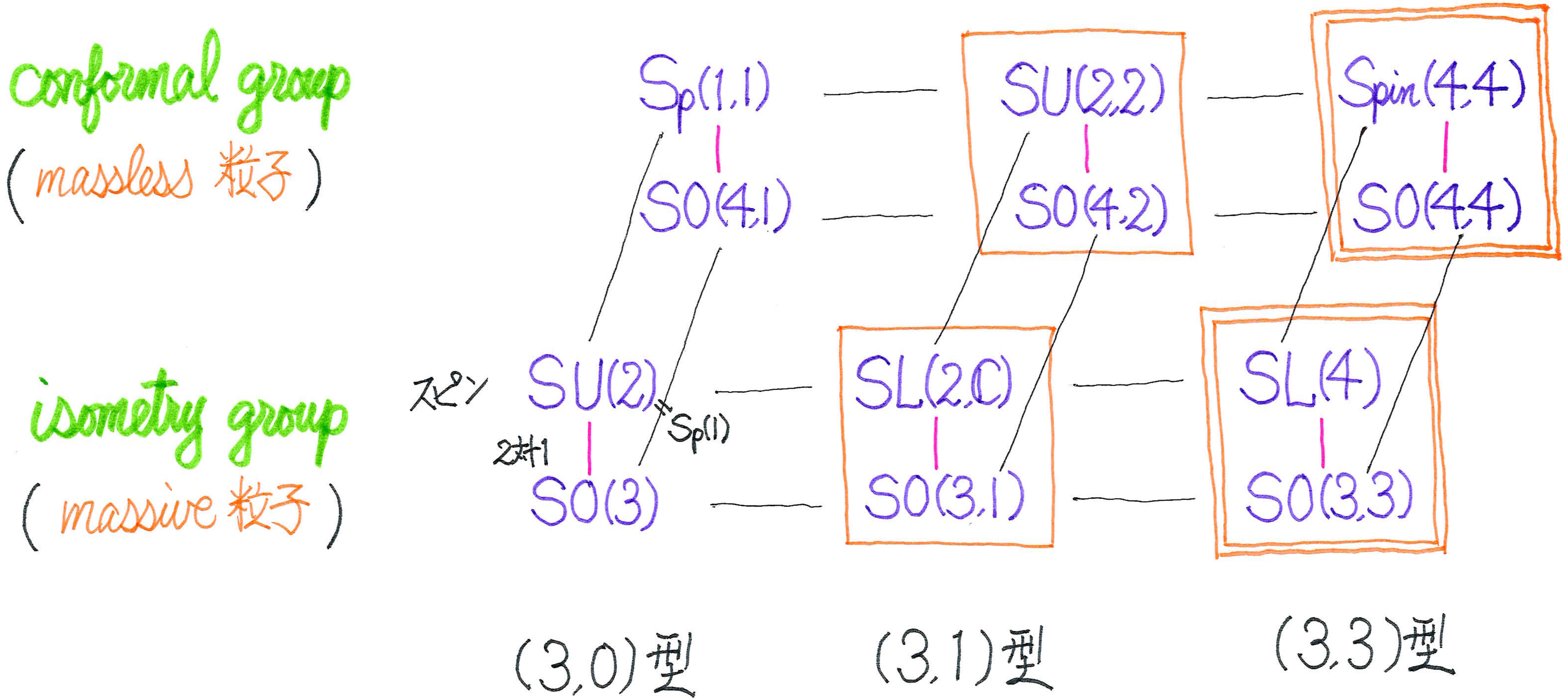
と

QED

待田芳德



- × Kaluza (1919), Klein (1926)
- × Kostant (1988), MIT


conformal group
(massless 粒子)
isometry group
(massive 粒子)
Sp(1,1)
SO(4,1)
SU(2,2)
SO(4,2)
Spin(4,4)
SO(4,4)
スピン
SU(2)
≅Sp(1)
2対1
SO(3)
SL(2,C)
SO(3,1)
SL(4)
SO(3,3)
(3,0)型
(3,1)型
(3,3)型

ここでは

- 光, 電子, 陽電子の3対性

6次元において！

- 伝播関数, S行列, Feynman積分における発散積分の正則化

ツイスター積分表示

6次元において！

cf. 双対性

- 超対称性　ボゾン↔フェルミオン
  10次元, 4次元, 2次元
- 超弦理論
  S双対, T双対
  AdS-CFT双対

cf. 4次元

4次元(3,1)型時空

$\mathbb{R}^4$: $x=(x^0,x^1,x^2,x^3)$ 4次元Minkowski空間

$ds^2=(dx^0)^2-(dx^1)^2-(dx^2)^2-(dx^3)^2$ (3,1)型 $g=(g_{mn})_{0\le m,n\le 3}=\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$

$\square=\left(\frac{\partial}{\partial x^0}\right)^2-\left(\frac{\partial}{\partial x^1}\right)^2-\left(\frac{\partial}{\partial x^2}\right)^2-\left(\frac{\partial}{\partial x^3}\right)^2$

1) スカラー場 $\varphi(x)$ $(\square+m^2)\varphi=0$ Klein-Gordon eq.

2) $\underset{\text{光}}{\text{電磁場}}$ $A(x)=\sum_{n=0}^{3}A_n(x)dx^n$
(ベクトル場)

$$\begin{cases}\square A_n=0 \quad (n=0,1,2,3) & \text{波動方程式}\\ \sum_{n=0}^{3}\frac{\partial A_n}{\partial x^n}=0 & \text{Lorentz条件}\end{cases}$$

3) 電子場 $\psi(x)=\begin{pmatrix}\psi_1(x)\\\psi_2(x)\\\hdashline\psi_3(x)\\\psi_4(x)\end{pmatrix}$ $\left(\underbrace{\sum_{\nu=0}^{3}\gamma^\nu\frac{\partial}{\partial x^\nu}}_{\gamma}+m\right)\psi=0$ Dirac eq.

(スピノル場)
$\frac{1}{2}$

SL(2,C)の表現空間

γ^ν : 4×4のガンマ行列

$\underset{6\text{次元}}{\text{SO(3,1)-不変}}$ $\underset{\text{回転+ブースト}}{\underset{6}{\text{Lorentz群}}}+\underset{4}{\text{平行移動}}=\underset{10}{\text{Poincaré群}}$

$\xrightarrow{m=0\ \text{massless}}$

$\underset{15\text{次元}}{\text{SO(4,2)-不変}}$ 共形群 = Poincaré群 + $\underset{\rho}{\text{相似変換}}$ + $\underset{\sigma}{\text{反転}}$

$SO(4,2)=\{SO(3,1)\cdot\mathbb{R}^{\cup}\sigma\}$

○ SO(p,2)のBruhat分解

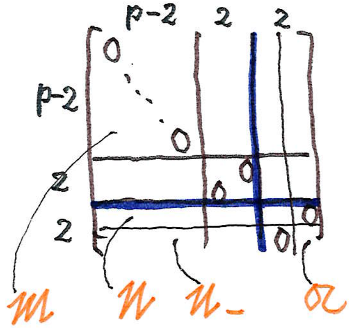
$$G = SO(p,2) = \{ \underset{\det g = 1}{g \in GL(p+2)} \mid {}^t g J g = J \}$$

$$\mathfrak{g} = \mathfrak{o}(p,2) = \{ X \in \mathfrak{gl}(p+2) \mid {}^t X J + J X = 0 \}$$

$$\begin{tikzcd}[column sep=small]
= & \mathfrak{m} \arrow[d, phantom, "\shortparallel"] & \oplus & \mathfrak{a} \arrow[d, phantom, "1"] & \oplus & \mathfrak{n} \arrow[d, phantom, "p"] & \oplus & \mathfrak{n}_- \arrow[d, phantom, "p"] \\
& \mathfrak{o}(p-1,1) \arrow[d] & & {} \arrow[d] & & {} \arrow[d] & & {} \arrow[d] \\
& M \arrow[d, phantom, "\cong"] & & A & & N & & N_- \\
& SO(p-1,1) & & & & & &
\end{tikzcd}$$

$p=4$, $SO(4,2)$

$$J = \begin{array}{c|c} & \begin{array}{ccc} p-2 & 2 & 2 \end{array} \\ \begin{array}{c} p-2 \\ 2 \\ 2 \end{array} & \left[\begin{array}{c|c|c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & & \\ \hline & \begin{matrix} & 1 \\ 1 & \end{matrix} & \\ \hline & & \begin{matrix} & 1 \\ 1 & \end{matrix} \end{array}\right] \end{array} \sim I_{p,2} = \begin{array}{c|c} & \begin{array}{cc} p & 2 \end{array} \\ \begin{array}{c} p \\ 2 \end{array} & \left[\begin{array}{c|c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \\ \hline & \begin{matrix}-1 & \\ & -1\end{matrix} \end{array}\right] \end{array}$$



$$G \doteqdot MANN_- \cup MAN\sigma N_-$$

$$\doteqdot \underbrace{MAN}_{P}N_- \cong SO(p-1,1) \cdot \mathbb{R}_+ \cdot \mathbb{R}^p \cdot \mathbb{R}^p$$

$$\underset{\hat{M}}{\underset{\cap}{m}} = \begin{bmatrix} \hat{m} & \\ & 1_2 \end{bmatrix}, \quad \underset{\hat{A}}{\underset{\cap}{a_t}} = \begin{bmatrix} 1_p & \\ & \begin{matrix} e^t & \\ & e^{-t} \end{matrix} \end{bmatrix}, \quad \underset{\hat{N}}{\underset{\cap}{n_\xi}} = \begin{bmatrix} 1_{p-2} & & 0 & \hat{\xi} \\ & 1_2 & & \begin{matrix}\xi_p \\ \xi_{p-1}\end{matrix} \\ -{}^t\hat{\xi} & -\xi_{p-1} \ \xi_p & 1 & \boxed{} \\ 0 & & & 1 \end{bmatrix}, \quad \underset{\hat{N}_-}{\underset{\cap}{x_\eta}} = \begin{bmatrix} 1_{p-2} & & \hat{\eta} & 0 \\ & 1_2 & \begin{matrix}\eta_{p-1} \\ \eta_p\end{matrix} & \\ 0 & & 1 & \\ -{}^t\hat{\eta} & \eta_p \ \eta_{p-1} & \boxed{} & 1 \end{bmatrix}$$

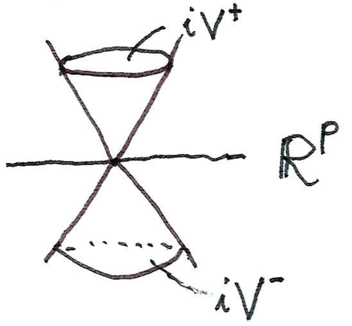
$$-\frac{1}{2}(|\hat{\xi}|^2 + 2\xi_{p-1}\xi_p) \qquad -\frac{1}{2}(|\hat{\eta}|^2 + 2\eta_{p-1}\eta_p)$$

$$N_- \cong \mathbb{R}^p \subset Q^p = P \backslash G \cong S^{p-1} \underset{\mathbb{Z}_2}{\times} S^1$$

$$T^{\pm} = \cancel{\frac{SO(p,2)}{SO(p) \times SO(2)}} \cong \mathbb{R}^p \times iV^{\pm}$$

$[y,y]>0$, $y^0 \gtrless 0$

Ⅳ型 対称有界領域
Silov 境界がQ^p



○ $SO(p,2)$の$\mathbb{R}^p$への作用

$x \in N_- = \mathbb{R}^p \subset Q^p$

$g \in SO(p,2)$

$$x \in \left[\begin{array}{c|cc|c|c} 1_{p-2} & & & -\hat{x} & 0 \\ \hline & 1 & & -x_p & \\ & & 1 & -x_{p-1} & \\ \hline 0 & & & 1 & \\ \hline {}^t\hat{x} & x_{p-1} & x_p & \square & 1 \end{array}\right] \qquad \frac{1}{2}[x,x] = -\frac{1}{2}(|\hat{x}|^2 + 2x_{p-1}x_p)$$

$$(x\cdot g)_k = \frac{\sum_{j=1}^{p} x^j g_{jk} + \frac{1}{2}[x,x]g_{p+1,k} + g_{p+2,k}}{\underbrace{\sum_{j=1}^{p} x^j g_{j,p+2} + \frac{1}{2}[x,x]g_{p+1,p+2} + g_{p+2,p+2}}_{分母}} \qquad 1 \leq k \leq p$$

$$\sum_{j=1}^{p} x^j g_{j,p+2} + \frac{1}{2}[x,x]g_{p+1,p+2} + g_{p+2,p+2} = \delta(x,g)$$

2次分数変換

$$\left(\begin{array}{l} \det\left(\frac{\partial(x\cdot g)}{\partial x}\right) = \delta(x,g)^{-p} \\ \frac{\partial(x\cdot g)}{\partial x} = \underbrace{\hat{m}(x,g)}_{\widehat{SO}(p-1,1)}\delta(x,g)^{-1} \end{array} \right.$$

- $M \cong SO(p-1,1) \ni g = \left[\begin{array}{c|c} \hat{g} & \\ \hline & 1_2 \end{array}\right]$ $\qquad (x\cdot g)_k = \sum_{j=1}^{p} x^j g_{jk}$
- $A \ni g = \left[\begin{array}{c|cc} 1_p & & \\ \hline & e^t & \\ & & e^{-t} \end{array}\right]$ $\qquad (x\cdot g)_k = \frac{1}{e^t} x^k$ 相似変換
- $N_- \ni g$ $\qquad (x\cdot g)_k = x^k + g^k$ 平行移動

⊙ $\sigma = \left[\begin{array}{c|cc|cc} 1_{p-2} & & & & \\ \hline & & 1 & & \\ & 1 & & & \\ \hline & & & & 1 \\ & & & 1 & \end{array}\right]$ $\qquad \sigma^2 = 1$ Weyl群の元 $\qquad$ 反転

$$\left\{ \begin{array}{l} x'_k = \frac{x_k}{[x,x]/2} = \frac{2x_k}{[x,x]} \qquad 1 \leq k \leq p-2 \\ x'_{p-1} = \frac{2x_p}{[x,x]} \\ x'_p = \frac{2x_{p-1}}{[x,x]} \end{array} \right.$$

※ SL(2,C)とSO(3,1)の関係

- $H(2)=\{Z\in M_2(\mathbb{C})\mid Z^*=Z\}$ Hermite行列

  $h\in SL(2,\mathbb{C})$に対して、 $Z\longmapsto \underline{D'(h)}:=hZh^*$

- Pauliのσ行列 $\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \sigma_4=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $H(2)$の基底

  $H(2)\ni Z=\begin{pmatrix}x_3+x_4 & x_1-ix_2\\ x_1+ix_2 & -x_3+x_4\end{pmatrix}=\sum_{j=1}^{4}x_j\sigma_j$

- $\phi: H(2)\ni Z=\sum_{j=1}^{4}x_j\sigma_j\longmapsto x=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\in\mathbb{R}^4$ $\phi(D'(h)Z)=D(h)\phi(Z)=\underline{D(h)x}$

$$\begin{tikzcd}
H(2)\ni Z \arrow[r, mapsto, "h"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & D'(h)Z\in H(2) \arrow[d, "\phi"'] \\
\mathbb{R}^4\ni x=\phi(Z) \arrow[r, mapsto, "D(h)"'] & D(h)x=\phi(D'(h)Z)\in\mathbb{R}^4
\end{tikzcd}$$

× $\det Z=-x_1^2-x_2^2-x_3^2+x_4^2$で、$\det(hZh^*)=\det Z$

$\mathbb{R}^4$上の$\langle x,x\rangle=-x_1^2-x_2^2-x_3^2+x_4^2$で、$D(h)$によって不変

× $D: SL(2,\mathbb{C})\ni h\longmapsto D(h)\in SO(3,1)$

$\mathrm{Ker}\,D=\pm I_2$

- $Z=\sum_{j=1}^{4}x_j\sigma_j$, $g\in D(h)$, に対して、

  $D'(h)Z=hZh^*=\sum_{j=1}^{4}x_j'\sigma_j=(\sigma_1\sigma_2\sigma_3\sigma_4)\begin{pmatrix}x_1'\\x_2'\\x_3'\\x_4'\end{pmatrix}=(\sigma_1\sigma_2\sigma_3\sigma_4)g\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$

  よって、$(h\sigma_1h^*, h\sigma_2h^*, h\sigma_3h^*, h\sigma_4h^*)=(\sigma_1\sigma_2\sigma_3\sigma_4)g$

○ 誘導表現

$\rho : \underset{P=\ }{} M\underset{\substack{(\\ \text{スカラー}}}{A}\underset{\substack{\backslash \\ \text{trivial}}}{N} \longrightarrow GL(V)$ 表現

$\vartheta = \rho|_M : \underset{\substack{\| \\ SO(p-1,1)}}{M} \longrightarrow GL(V)$ Lorentz表現

1) trivial
2) ベクトル表現 $V = \mathbb{R}^p$
3) スピン表現 $\frac{1}{2}$

誘導表現

$\Gamma(\underbrace{MAN\backslash G}_{\substack{P= \quad \| \\ N_-}}, G\times_P V) \doteqdot \Gamma(\mathbb{R}^p, V)$

1) ϑ : trivial スカラー場

2) ϑ : ベクトル ベクトル場, 電磁場

3) ϑ : スピン$\frac{1}{2}$ スピノル場, 電子場

※ $G \supset H$

Gの表現 → Hの表現 表現の制限
Hの表現 → Gの表現 誘導表現

HのVへの表現
↳ Gの等質ベクトル束 $G\times_H V \to G/H$ への作用
↳ Gの $\Gamma(G\times_H V)$ への表現

$g \in G$

$$(T_g\varphi)(x) = \vartheta(m(x,g))\varphi(x\cdot g)\delta(x,g)^{-\frac{p}{2}+\lambda}$$

* $\det\left(\frac{\partial(x\cdot g)}{\partial x}\right) = \delta(x,g)^{-p}$

* $(T^{\vartheta,\lambda}, \mathcal{H}_{\vartheta,\lambda})$

※ Poisson変換P

cf. 複素関数論，調和関数
Poisson積分表示

- Poisson核 $P_{\mu,\nu}(\omega,\eta)=\dfrac{1}{[\omega-\eta,\omega-\eta]^{\frac{\mu}{2}}}\left(\dfrac{[\omega-\bar{\omega},\omega-\bar{\omega}]^{\frac{1}{2}}}{[\omega-\eta,\omega-\eta]}\right)^{\nu}$

  $\omega\in T=\mathbb{R}^p+iV^p$
  $\eta\in\mathbb{R}^p$

  * $P_{\mu,\nu}(\omega\cdot g,\eta\cdot g)=P_{\mu,\nu}(\omega,\eta)\,\delta(\omega,g)^{\frac{\mu}{2}}\delta(\eta,g)^{\frac{\mu}{2}+\nu}$
  $\mathcal{D}_{\mu,\nu}$

- Poisson変換 $(P\varphi)(\omega)=\displaystyle\int_{\mathbb{R}^p}P_{\mu,\nu}(\omega,\eta)\varphi(\eta)d\eta$

- $\nu=0$とする． cf. $\mu=0$

  $\phi\in\mathcal{D}_{-\frac{p}{2}-\lambda}\Rightarrow(T_g\phi)(\omega)=\phi(\omega\cdot g)\delta(\omega,g)^{-\frac{p}{2}-\lambda}$

  * Poisson変換Pに対して，
  $P:\mathcal{B}_{-\frac{p}{2}+\lambda}\longrightarrow\mathcal{D}_{-\frac{p}{2}-\lambda}$

  $\lambda=-1$とする

  $P:\mathcal{B}_{-\frac{p}{2}-1}\longrightarrow\mathcal{D}^{\pm}_{-\frac{p}{2}+1}$

  $$(P\varphi)(\omega)=\int_{\mathbb{R}^p}\frac{1}{[\omega^{\pm}\eta,\omega^{\pm}\eta]^{\frac{p-2}{2}}}\varphi(\eta)d\eta$$

○図式

自由場

※ 纏絡作用素（からみ合い、もつれ合い）

$$S : \underset{T_g}{\underset{\circlearrowleft}{V}} \longrightarrow \underset{T'_g}{\underset{\circlearrowleft}{V'}} \qquad ST_g = T'_g S$$

・$\mathrm{Ker}S, \mathrm{Im}S$ は表現部分空間

- スカラー場

（ベクトル場も同様）

$$\begin{tikzcd}
\underset{\lambda=1}{\mathfrak{H}_{-\frac{p}{2}+1}} \arrow[r, "\Box_p"] \arrow[ddr, "\Box_p"] & \underset{\lambda=-1}{\mathfrak{H}_{-\frac{p}{2}-1}} \arrow[ddl, "P_p"] \\
\cup & \cup \\
\mathrm{Ker}\Box_p \cong \mathcal{D}^+_{-\frac{p}{2}+1} \oplus \mathcal{D}^-_{-\frac{p}{2}+1} & \mathrm{Im}\Box_p \cong \mathcal{D}^+_{-\frac{p}{2}-1} \oplus \mathcal{D}^-_{-\frac{p}{2}-1}
\end{tikzcd}$$

・$p=4$ $\quad \mathfrak{H}_{-1} \longrightarrow \mathfrak{H}_{-3}$

- スピノル場

$$\begin{tikzcd}
\underset{\lambda=\frac{1}{2}}{\mathfrak{H}_{\vartheta, -\frac{p}{2}+\frac{1}{2}}} \arrow[r, "D_p"] \arrow[ddr, "D_p"] & \underset{\lambda=-\frac{1}{2}}{\mathfrak{H}_{{}^t\vartheta^{-1}, -\frac{p}{2}-\frac{1}{2}}} \arrow[ddl, "P_p"] \\
\cup & \cup \\
\mathrm{Ker}D_p \cong \mathcal{D}^+_{\vartheta, -\frac{p}{3}+1} \oplus \mathcal{D}^-_{\vartheta, -\frac{p}{2}+1} & \mathrm{Im}D_p \cong \mathcal{D}^+_{{}^t\vartheta^{-1}, -\frac{p}{2}+\frac{1}{2}} \oplus \mathcal{D}^-_{{}^t\vartheta^{-1}, -\frac{p}{2}-\frac{1}{2}}
\end{tikzcd}$$

・$p=4$ $\quad \mathfrak{H}_{\vartheta, -\frac{3}{2}} \longrightarrow \mathfrak{H}_{{}^t\theta^{-1}, -\frac{5}{2}}$

○ 相互作用

- QED (量子電磁力学)　U(1)-ゲージ理論

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \underbrace{D_\mu}_{\partial_\mu + ieA_\mu} - m)\psi \boxed{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}} \longrightarrow -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu$$

Lorentzゲージ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$= \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \underbrace{-\underset{\text{結合定数}}{e}\bar{\psi}\gamma^\mu A_\mu \psi}_{\mathcal{L}_I} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$m=0$

$$\begin{cases} (i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu)\psi = 0 \\ \Box A^\mu = e\bar{\psi}\gamma^\mu\psi \end{cases}$$

解は？

- $(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu)\psi = \Psi$

$$\begin{tikzcd}
\mathcal{H}_{\substack{\vartheta_{\frac{1}{2}},\frac{1}{2}\\ \lambda}} \oplus \mathcal{H}_{\substack{\vartheta_{1},1\\ \lambda}} \arrow[r, "\hat{\partial}+e\hat{A}"] & \mathcal{H}_{\substack{{}^t\vartheta^{-1}_{\frac{1}{2}},-1\\ \lambda}} \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
(\psi, A) \arrow[r, mapsto] \arrow[loop below, "T_g"] & \Psi \arrow[loop below, "T_g"]
\end{tikzcd}$$

Kerは？

$$*\ \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = \partial_\mu(i\bar{\psi}\gamma^\mu),\ \frac{\partial\mathcal{L}}{\partial\psi} = -e\bar{\psi}\gamma^\mu A_\mu - m\bar{\psi}$$

$$\therefore i\gamma^\mu\partial_\mu\psi - m\psi = e\gamma^\mu A_\mu\psi$$

$m=0$のとき

$$(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu)\psi = 0$$

$e=0$のとき

$$i\gamma^\mu\partial_\mu\psi = 0$$ (自由場)

$$*\ \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu),\ \frac{\partial\mathcal{L}}{\partial A_\mu} = -e\bar{\psi}\gamma^\mu\psi$$

$$\therefore \boxed{\partial_\nu F^{\mu\nu}} = e\bar{\psi}\gamma^\mu\psi$$

$\longrightarrow \Box A^\mu$

$e=0$のとき

$$\begin{cases} \Box A^\mu = 0 \\ \partial_\mu A^\mu = 0 \end{cases}$$ (自由場)

6次元(3,3)型時空

○ SO(4,4)のBruhat分解

$G = SO(4,4) = \{ g \in GL(8) \mid {}^t g J g = J \}$ 　$\det g = 1$

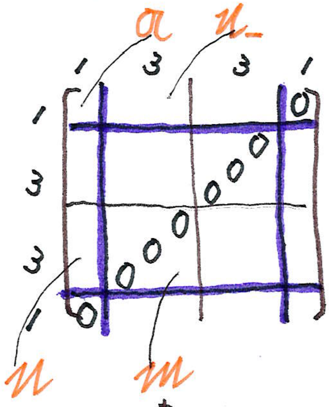
$J = \left[\begin{array}{cc|cc} & & & 1 \\ & & ⋰ & \\ \hline & ⋰ & & \\ 1 & & & \end{array}\right] \sim I_{3,3} = \left[\begin{array}{c|c} 1_3 & \\ \hline & -1_3 \end{array}\right]$

$\mathfrak{g} = \mathfrak{o}(4,4) = \{ X \in \mathfrak{gl}(8) \mid {}^t X J + J X = 0 \}$

28次元

$$= \underset{\overset{\|}{\mathfrak{o}(3,3)}}{\mathfrak{m}} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$$

$$\begin{array}{cccc} 15\downarrow & 1\downarrow & 6\downarrow & 6\downarrow \\ \underset{\underset{SO(3,3)}{\wr\|}}{M} & A & N & N_- \end{array}$$



$$m = \left[\begin{array}{c|c|c} 1 & & \\ \hline & \hat{m} & \\ \hline & & 1 \end{array}\right], \quad a_t = \left[\begin{array}{c|c|c} e^t & & \\ \hline & I_6 & \\ \hline & & e^t \end{array}\right], \quad x = \left[\begin{array}{c|c|c} 1 & x_1 x_2 x_3 x_4 x_5 x_6 & \boxed{\cdot} \\ \hline & I_6 & \begin{matrix} -x_6 \\ -x_5 \\ -x_4 \\ -x_3 \\ -x_2 \\ -x_1 \end{matrix} \\ \hline & & 1 \end{array}\right]$$

$m \in \hat{M}$, $a_t \in \hat{A}$, $x \in N_- \cong \mathbb{R}^6$

$-Q(x) = -(x_1 x_6 + x_2 x_5 + x_3 x_4)$

$$G = MANN_- \cup MAN\sigma N_-$$

$$\doteqdot \underbrace{MAN}_{P}N_- \cong SO(3,3) \cdot \mathbb{R}_+ \cdot \mathbb{R}^6 \cdot \mathbb{R}^6$$

$$\sigma = \left[\begin{array}{c|c|c} & & -1 \\ \hline & \begin{matrix} & & 1 \\ & ⋰ & \\ 1 & & \end{matrix} & \\ \hline -1 & & \end{array}\right] \in SO(4,4)$$

反転

$G = \{ SO(3,3) \cdot \mathbb{R} \cup \sigma \}$

$$N_- \cong \mathbb{R}^6 \subset Q^6 = {}_{P}\backslash G \cong S^3 \underset{\mathbb{Z}_2}{\times} S^3$$

○ SO(4,4)のR^6への作用

$x \in N_- = \mathbb{R}^6 \subset Q^6$

$g \in SO(4,4)$

$g : x \longmapsto x' = xg$ $\quad x'_k = \dfrac{g_{0k} + \sum_{j=1}^{6} x_j g_{jk} - Q(x) g_{7k}}{g_{00} + \sum_{j=} x_j g_{j0} - Q(x) g_{70}}$ $\quad 1 \leq k \leq 6$ $\quad$ 2次分数変換

- $M \cong SO(3,3) \ni g = \begin{bmatrix} 1 & & \\ & \tilde{g} & \\ & & 1 \end{bmatrix}$ $\quad x' = x\tilde{g}$, $\; x'_j = \sum_{k=1}^{6} x_k \tilde{g}_{kj}$
- $A \ni g = \begin{bmatrix} \tau^{-1} & & \\ & I_6 & \\ & & \tau \end{bmatrix}$ $\quad x' = \tau x$ $\quad$ 相似変換

⊙ $\sigma = \begin{bmatrix} & & 1 \\ & \begin{smallmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{smallmatrix} & \\ -1 & & \end{bmatrix}$ $\quad \sigma^2 = 1$ $\quad$ 反転

$$\begin{cases} x' = x\sigma \\ x'_k = \dfrac{x_{7-k}}{Q(x)} \quad 1 \leq k \leq 6 \end{cases}$$

$$[dx', dx'] = \frac{1}{Q(x)^2}[dx, dx]$$

※ SL(4)とSO(3.3)の関係

- $A(4)=\{X\in M_4(\mathbb{R}) \mid {}^tX=-X\}$ 交代行列

  $u\in SL(4)$に対して，$X\longmapsto {}^tuXu$

- $A(4)\ni X=\begin{pmatrix} 0 & -x_1 & x_2 & -x_3 \\ x_1 & 0 & -x_4 & -x_5 \\ -x_2 & x_4 & 0 & -x_6 \\ x_3 & x_5 & x_6 & 0 \end{pmatrix}=\sum_{k=1}^{6}x_k\Gamma^k=(x_1x_2x_3x_4x_5x_6)\begin{pmatrix}\Gamma^1\\ \Gamma^2\\ \Gamma^3\\ \Gamma^4\\ \Gamma^5\\ \Gamma^6\end{pmatrix}$

- $$\begin{tikzcd} A(4)\ni X \arrow[r, "u\,\in SL(4)"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & X'={}^tuXu \arrow[d] \\ \mathbb{R}^6\ni x \arrow[r, "\tilde{u}\,\in SO(3,3)"'] & x'=x\tilde{u} \end{tikzcd}$$

- $${}^tuXu={}^tu\left(\sum_{k=1}^{6}x_k\Gamma^k\right)u=\sum_{k=1}^{6}x_k\,{}^tu\Gamma^ku=(x_1x_2x_3x_4x_5x_6)\begin{pmatrix}{}^tu\Gamma^1u\\ {}^tu\Gamma^2u\\ {}^tu\Gamma^3u\\ {}^tu\Gamma^4u\\ {}^tu\Gamma^5u\\ {}^tu\Gamma^6u\end{pmatrix}$$

$$=\sum_{k=1}^{6}x'_k\Gamma^k=(x'_1x'_2x'_3x'_4x'_5x'_6)\begin{pmatrix}\Gamma^1\\ \Gamma^2\\ \Gamma^3\\ \Gamma^4\\ \Gamma^5\\ \Gamma^6\end{pmatrix}$$

$$=(x_1x_2x_3x_4x_5x_6)\tilde{u}\begin{pmatrix}\Gamma^1\\ \Gamma^2\\ \Gamma^3\\ \Gamma^4\\ \Gamma^5\\ \Gamma^6\end{pmatrix}$$

よって，${}^tu\Gamma u=\tilde{u}\Gamma$.

ガンマ行列

$\Gamma^1=\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$, $\Gamma^2=\begin{bmatrix} 0 & & 1 & \\ & 0 & & \\ -1 & & 0 & \\ & & & 0 \end{bmatrix}$, $\Gamma^3=\begin{bmatrix} 0 & & & -1 \\ & 0 & & \\ & & 0 & \\ 1 & & & 0 \end{bmatrix}$

$\Gamma^4=\begin{bmatrix} 0 & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & 0 \end{bmatrix}$, $\Gamma^5=\begin{bmatrix} 0 & & & \\ & 0 & & -1 \\ & & 0 & \\ & 1 & & 0 \end{bmatrix}$, $\Gamma^6=\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{bmatrix}$

$\det X=(\mathrm{pf}X)^2=(x_1x_6+\underset{-(-x_2x_5)}{\underline{x_2x_5}}+x_3x_4)^2$で，

$\det({}^tuXu)=\det X$

$\mathbb{R}^6$上の$\langle x,x\rangle=x_1x_6+x_2x_5+x_3x_4$で，

$\tilde{u}$によって不変

○・スカラー場 $\varphi(x) \longrightarrow (T_u\varphi)(x) = \varphi(x\cdot\tilde{u})$

・ベクトル場 $A(x) = \sum_{n=1}^{6} A_n(x)dx_n = \begin{bmatrix} A_1(x) \\ A_2(x) \\ A_3(x) \\ A_4(x) \\ A_5(x) \\ A_6(x) \end{bmatrix} \longrightarrow (T_uA)(x) = \tilde{u}A(x\cdot\tilde{u})$

・スピノル場 $\psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix} \longrightarrow (T_u\psi)(x) = u\cdot\psi(x\cdot\tilde{u})$

○ $\Gamma'^k := \Gamma^{7-k}$

$$\begin{aligned} \Gamma'^1 &= \Gamma^6 \\ \Gamma'^2 &= \Gamma^5 \\ \Gamma'^3 &= \Gamma^4 \\ \Gamma'^4 &= \Gamma^3 \\ \Gamma'^5 &= \Gamma^2 \\ \Gamma'^6 &= \Gamma^1 \end{aligned}$$

$$X = \begin{bmatrix} 0 & -x_1 & x_2 & -x_3 \\ x_1 & 0 & -x_4 & -x_5 \\ -x_2 & x_4 & 0 & -x_6 \\ x_3 & x_5 & x_6 & 0 \end{bmatrix}, \quad X' = \begin{bmatrix} 0 & -x_6 & x_5 & -x_4 \\ x_6 & 0 & -x_3 & -x_2 \\ -x_5 & x_3 & 0 & -x_1 \\ x_4 & x_2 & x_1 & 0 \end{bmatrix}$$

$D\psi := \Gamma'\partial\psi$

$A'\psi := \Gamma'A\psi$

$$D\psi_\alpha(x) = \sum_{n,\beta} \Gamma'^n_{\alpha\beta}\partial_{x_n}\psi_\beta(x)$$

$$A'_\alpha\psi_\alpha(x) = \sum_{n,\beta} \Gamma'^n_{\alpha\beta}A_n(x)\psi_\beta(x)$$

$$D = \Gamma'\partial = \begin{bmatrix} 0 & -\partial_{x_6} & \partial_{x_5} & -\partial_{x_4} \\ \partial_{x_6} & 0 & -\partial_{x_3} & -\partial_{x_2} \\ -\partial_{x_5} & \partial_{x_3} & 0 & -\partial_{x_1} \\ \partial_{x_4} & \partial_{x_2} & \partial_{x_1} & 0 \end{bmatrix}$$

$$(\Gamma\partial)(\Gamma'\partial) = -\Box I_4$$

$$\Box = \frac{\partial^2}{\partial x_1\partial x_6} + \frac{\partial^2}{\partial x_2\partial x_5} + \frac{\partial^2}{\partial x_3\partial x_4}$$

・$D\psi(x) \longrightarrow (T_uD\psi)(x) = {}^t u^{-1}\cdot D\psi(x\cdot\tilde{u})$

・$A'\psi(x) \longrightarrow (T_uA'\psi)(x) = {}^t u^{-1}\cdot A'\psi(x\cdot\tilde{u})$

絡み合い作用素である:

・$\underset{T_u}{\underset{\circlearrowleft}{\psi(x)}} \longmapsto \underset{T_u}{\underset{\circlearrowleft}{D\psi(x)}}$

・$\underset{T_u}{\underset{\circlearrowleft}{\{\psi(x), A(x)\}}} \longmapsto \underset{T_u}{\underset{\circlearrowleft}{A'\psi(x)}},\ \underset{T_u}{\underset{\circlearrowleft}{(D+A')\psi(x)}}$

○誘導表現

$\rho : \underset{\underset{P}{\shortparallel}}{M}\underset{\text{スカラー}}{A}\underset{\text{trivial}}{N} \longrightarrow GL(V)$ 表現

$\vartheta = \rho|_M : \underset{\underset{SO(3,3)}{\shortparallel}}{M} \longrightarrow GL(V)$ Lorentz表現

$\Gamma(MAN\backslash G, G\times_P V) \doteqdot \Gamma(\mathbb{R}^6, V)$

※ $$\left(\begin{aligned} & dx' = \frac{\partial(x\cdot g)}{\partial x} dx \\ & \frac{\partial(x\cdot g)}{\partial x} = \underbrace{u(x,g)}_{\substack{\shortparallel \\ \tilde{u} \\ \cap \\ SO(3,3)}} \cdot \underbrace{\delta(x,g)^{-1}}_{\substack{\shortparallel \\ \det\left(\frac{\partial x_j'}{\partial x_k}\right)^{\frac{1}{6}}}} \quad \left(\begin{aligned} & \delta(x,g) = \det\left(\frac{\partial x_j'}{\partial x_k}\right)^{-\frac{1}{6}} \\ & \det\left(\frac{\partial x_j'}{\partial x_k}\right) = \delta(x,g)^{-6} \end{aligned} \right) \end{aligned} \right.$$

$g \in G$

1) ϑ : trivial　スカラー場　$(T_g\varphi)(x) = \varphi(x\cdot g)\,\delta(x,g)^{-3+\lambda}$

2) ϑ : ベクトル表現　ベクトル場　$(T_gA)(x) = \tilde{u}A(x\cdot g)\delta(x,g)^{-3+\lambda}$

3) ϑ : スピン表現　スピノル場　$(T_g\psi)(x) = u\,\psi(x\cdot g)\delta(x,g)^{-3+\lambda}$

$-\frac{p}{2}+\lambda \overset{p=6}{\doteqdot} -3+\lambda$

- $(T_gD\psi)(x) = {}^t\{\underbrace{\vartheta_{\frac{1}{2}}(u(x,g))}_{u}\}^{-1} D\psi(x\cdot g)\,\delta(x,g)^{\underline{-4}+\lambda}$
- $(T_gA'\psi)(x) = {}^t\{\underbrace{\vartheta_{\frac{1}{2}}(u(x,g))}_{u}\}^{-1} A'\psi(x\cdot g)\,\delta(x,g)^{\underline{-4}+\lambda}$

- $g = g_{\tilde{u}} \in SO(3,3)$　　$\delta(x,g) = 1$
- $g = g_\tau$: 相似変換　　$\delta(x,g) = \tau^{-1}$
- $g = g_\sigma$: 反転
  $x' = \sigma(x) \in \mathbb{R}^6,\ x_k' = \frac{x_{7-k}}{Q(x)}$
  $X' = \sigma(X) \in A(4),\ \sigma(X) = X^{-1}$
  $\delta(x,g) = (\det X)^{\frac{1}{2}} = Q(x)$

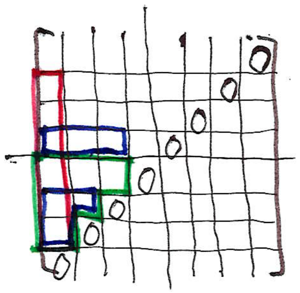
絡み合い作用素である：

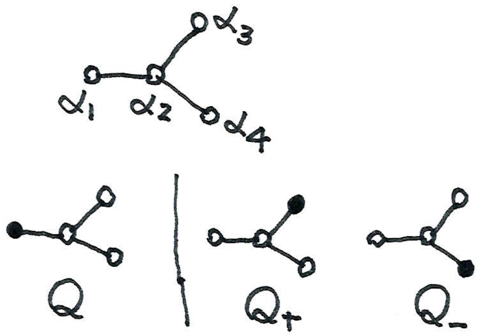
- $$\begin{tikzcd} \psi(x) \arrow[r] \arrow[loop below, "T_g"] & D\psi(x) \arrow[loop below, "T_g"] \end{tikzcd}$$
- $$\begin{tikzcd} \{\psi(x), A(x)\} \arrow[r] \arrow[loop below, "T_g"] & A'\psi(x), \arrow[loop below, "T_g"] & (D+A')\psi(x) \arrow[loop below, "T_g"] \end{tikzcd}$$

※ Radon変換 R

$\underset{\substack{\| \\ P\backslash G}}{Q^6}$; $\underset{\substack{\| \\ P_+\backslash G}}{Q_+^6}$, $\underset{\substack{\| \\ P_-\backslash G}}{Q_-^6}$

いずれも(3,3)型計量





$\tau_{13} : Q \longrightarrow Q_+$
$\tau_{14} : Q \longrightarrow Q_-$
$\tau_{34} : Q_+ \longrightarrow Q_-$
いずれも diffeo.

$\sigma\alpha_i = \alpha_{\underset{\|}{\sigma(i)}}$ $_j$

$\sigma_{ij} P_i = P_j$

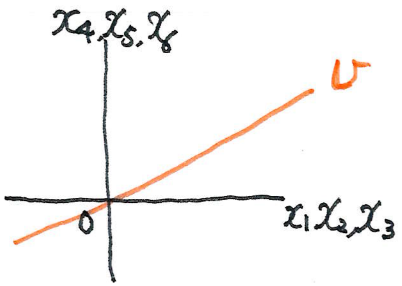
$\tau_{ij}\ P_i\backslash G = P_j\backslash G$

$x = (x_1, x_2, x_3, x_4, x_5, x_6) \in Q$
$v = (v_1, v_2, v_3, v_4, v_5, v_6) \in Q_\pm$

generic ヌル3-平面

$$\begin{cases} x_4 = v_2x_1 + v_1x_2 + v_4 \\ x_5 = v_3x_1 - v_1x_3 - v_5 \\ x_6 = -v_3x_2 - v_2x_3 + v_6 \end{cases}$$

$dx_1dx_6 + dx_2dx_5 + dx_3dx_4 = 0$



$\varphi(x)$に対して, $(R\varphi)(v) = \int_v \varphi(x_1, x_2, x_3 ; x_4, x_5, x_6)\, dx_1 \wedge dx_2 \wedge dx_3$

$R : \mathcal{B}_{-3+\lambda}(Q) \longrightarrow \mathcal{B}_{-1+\lambda}(Q_\pm) \cong \mathcal{B}_{-1+\lambda}(Q)$

○図式　自由場

- スカラー場
（ベクトル場も同様）

$$\begin{tikzcd}
\mathfrak{h}_{\underset{\lambda=1}{\underline{-2}}} \arrow[r, "\square"] \arrow[ddr, "\square"] & \mathfrak{h}_{\underset{\lambda=-1}{\underline{-4}}} \arrow[ddl, "R"'] \\
\cup & \cup \\
\mathrm{Ker}\,\square \cong \mathfrak{h}_{-2}(Q_+) \oplus \mathfrak{h}_{-2}(Q_-) & \mathrm{Im}\,\square \cong \mathfrak{h}_{-4}(Q_+) \oplus \mathfrak{h}_{-4}(Q_-)
\end{tikzcd}$$

- スピノル場

$$\begin{tikzcd}
\mathfrak{h}_{\vartheta,\underset{\lambda=\frac{1}{2}}{\underline{-\frac{5}{2}}}} \arrow[r, "D"] \arrow[ddr, "D"] & \mathfrak{h}_{{}^t\vartheta^{-1},\underset{\lambda=-\frac{1}{2}}{\underline{-\frac{7}{2}}}} \arrow[ddl, "R"'] \\
\cup & \cup \\
\mathrm{Ker}\,D \cong \mathfrak{h}_{\vartheta,-\frac{5}{2}}(Q_+) \oplus \mathfrak{h}_{\vartheta,-\frac{5}{2}}(Q_-) & \mathrm{Im}\,D \cong \mathfrak{h}_{{}^t\vartheta^{-1},-\frac{7}{2}}(Q_+) \oplus \mathfrak{h}_{{}^t\vartheta^{-1},-\frac{7}{2}}(Q_-)
\end{tikzcd}$$

○ 相互作用

- $I = \int_{\mathbb{R}^6} \left({}^t\psi(x) \underset{\Gamma'\partial}{\underline{D}} \psi(x) + k\, {}^t\psi(x) \underset{\Gamma' A}{\underline{A'}} \psi(x) - \frac{1}{2}(\partial A)^2 \right) \underset{dx_1 \wedge \cdots \wedge dx_6}{\underline{dx}}$

  SO(4,4)で不変 $\Longleftrightarrow \lambda = \frac{1}{2}$ $\quad (\delta(x,g)^{-6})$

  運動方程式

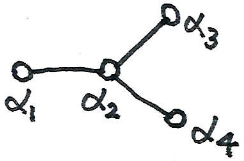
  $$\begin{cases} (D + kA')\psi = 0 \\ \Box A = k\, {}^t\psi \Gamma' \psi \end{cases}$$

- $(D + kA')\psi = \Psi$

$$\begin{tikzcd}
\mathfrak{h}_{\vartheta_{\frac{1}{2}}, \frac{1}{2}} \oplus \mathfrak{h}_{\vartheta_1, 1} \arrow[r, "D+kA'"] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \mathfrak{h}_{{}^t\vartheta^{-1}_{\frac{1}{2}}, -1} \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] \\
(\psi, A) \arrow[r, mapsto] & \Psi \\
\underset{T_g}{\circlearrowright} & \underset{T_g}{\circlearrowright}
\end{tikzcd}$$

◎ 光, 電子, 陽電子の3対性

○ Q ; Q_+, Q_-

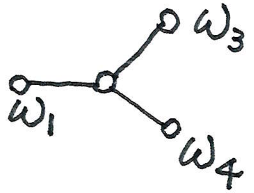


$$\begin{tikzcd}
 & \text{Q での点 x} \arrow[dl, "{\{\Pi_+ \in Q_+ \mid x \text{を通るヌル3平面} \Pi_+\}}"'] \arrow[dr, "{\{\Pi_- \in Q_- \mid x \text{を通るヌル3平面} \Pi_-\}}"] & \\
\text{Q_+ でのヌル3-平面} & & \text{Q_- でのヌル3-平面} \\
\text{Q_- での点 Π_-} \arrow[u, "{\{\Pi_+ \in Q_+ \mid \Pi_+ \cap \Pi_- \text{は} Q \text{でのヌル2-平面}\}}"] \arrow[dr] & & \text{Q_+ での点 Π_+} \arrow[u, "{\{\Pi_- \in Q_- \mid \Pi_- \cap \Pi_+ \text{は} Q \text{でのヌル2-平面}\}}"'] \arrow[dl] \\
 & \text{Q でのヌル3-平面} &
\end{tikzcd}$$

幾何的

$Cl(4,4)$
$\cup$

○ $Q^6 \subset P^7 = P(V)$, $V^8 = \underbrace{W \oplus W'}_{\text{ヌル4平面}}$ ……… Spin(4,4)の標準表現 ω_1



$S^+ = \Lambda^{\text{even}} W = \underset{1}{\Lambda^0 W} \oplus \underset{6}{\Lambda^2 W} \oplus \underset{1}{\Lambda^4 W}$
$S^- = \Lambda^{\text{odd}} W = \underset{4}{\Lambda^1 W} \oplus \underset{4}{\Lambda^3 W}$
8次元 …… Spin(4,4)の$+\frac{1}{2}, -\frac{1}{2}$のスピン表現 ω_3 ω_4

($\Lambda^4 W \cong \mathbb{R}$から2次形式$\langle , \rangle_{V, S^+, S^-}$

$V \times S^+ \longrightarrow S^-$, $V \times S^- \longrightarrow S^+$ bilinear

$Q^+ \subset P(S^+)$　Q^+での点 − $s \in S^+$ → $\{v \in V \mid v \cdot s = 0\}$: Vでのヌル4平面 − Qでのヌル3平面
$Q^- \subset P(S^-)$　Q^-での点 − $t \in S^-$ → $\{v \in V \mid v \cdot t = 0\}$: Vでのヌル4平面 − Qでのヌル3平面

$S^+ \times S^- \longrightarrow V$: $\langle v, s \cdot t \rangle_V = \langle v \cdot s, t \rangle_{S^-}$
(s,t)　　$s \cdot t$

Qでの点 − $v \in V$ → $v \cdot s = 0$なるs
　　　　　　　　→ $v \cdot t = 0$なるt

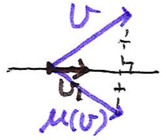
代数的

○ $A := V \oplus S^+ \oplus S^-$　$(v,s,t) \longmapsto \langle v \cdot s, t \rangle_{S^-}$ 3次形式

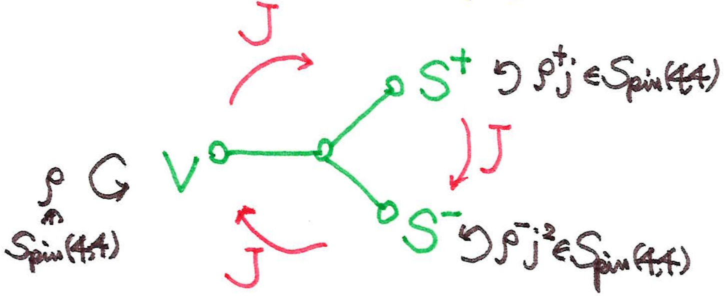
J : order 3 $\begin{cases} V \to S^+ \\ S^+ \to S^- \\ S^- \to V \end{cases}$

$\left.\begin{array}{l} v_1 \in V \text{ s.t. } \langle v_1, v_1 \rangle_V = 1 \\ s_1 \in S^+ \text{ s.t. } \langle s_1, s_1 \rangle_{S^+} = 1 \end{array}\right)$ fix $\Rightarrow t_1 = v_1 \cdot s_1$ s.t. $\langle t_1, t_1 \rangle_{S^-} = 1$

× $\mu : S^+ \longleftrightarrow S^-, V \circlearrowleft$　$\mu(s) = v_1 \cdot s$ $(s \in S^+)$
　$\mu^2 = id$　$\mu(v) = 2\langle v, v_1 \rangle_V v_1 - v$ $(v \in V)$



× $\nu : V \longleftrightarrow S^-, S^+ \circlearrowleft$　$\nu(v) = v \cdot s_1$ $(v \in V)$
　$\nu^2 = id$　$\nu(s) = 2\langle s, s_1 \rangle_{S^+} s_1 - s$ $(s \in S^+)$



$J := \mu \circ \nu : V \xrightarrow{\nu} S^- \xrightarrow{\mu} S^+$

○ $$\begin{array}{l} G=\mathrm{Spin}(4,4)\\ \cup \\ H=\mathrm{Spin}(3,3)=SL(4)\end{array}$$

$$V^8=\langle \underbrace{e_0,e_1,e_2,e_3}_{W} \mid \underbrace{e_4,e_5,e_6,e_7}_{W'} \rangle$$

$\cup$ e_0,e_7 : fix

$$U^6=\langle \underbrace{e_1,e_2,e_3}_{W_U} \mid \underbrace{e_4,e_5,e_6}_{W'_U} \rangle \quad \circlearrowright \mathrm{Spin}(3,3)$$

$$\left.\begin{array}{l} S_U^+=\Lambda^{even}W_U=\underset{1}{\Lambda^0}\oplus\underset{3}{\Lambda^2}\\ S_U^-=\Lambda^{odd}W_U=\underset{3}{\Lambda^1}\oplus\underset{1}{\Lambda^3}\end{array}\right\} 4次元 \circlearrowright SL(4,4)$$

$$\begin{tikzcd} G \circlearrowright & V_8 \arrow[r, "J"] & S^+_8 \arrow[d, "J"] \\ & & S^-_8 \arrow[ul, "J"] \end{tikzcd}$$

制限

$$\begin{tikzcd} H \circlearrowright & U_6 \arrow[r, "J"] & S_{U\,4}^+ \arrow[d, "J"] \\ & & S_{U\,4}^- \arrow[ul, "J"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} U_4/\mathrm{Ker}J \arrow[r, "J"] & S_U^+ \arrow[d, "J"] \\ & S_U^- \arrow[ul, "J"] \end{tikzcd}$$

誘導表現

× $\Gamma(\underset{固定}{Q},\underset{上の3つ}{\square})$

× $\Gamma(\bigcirc,\square)$

Q,Q_+,Q_- 固定

以上. 9つの同値

$$\begin{tikzcd} 光 \arrow[r] & 電子 \arrow[d] \\ & 陽電子 \arrow[ul] \end{tikzcd}$$

偏極, 偏光?

◎ 伝播関数, Feynman積分の超関数, 発散積分の正則化

ツイスター積分表示, 留数

○ **S行列**　相互作用での粒子の散乱

初めの状態と終りの状態の散乱行列 〜 遷移確率

Dyson

$$S = U(+\infty, -\infty)$$ 断熱因子 $e^{-\varepsilon|t|}$

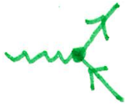
$$= T\exp\left(i\int d^4x \mathcal{L}_I(x)\right)$$ 時間順序積

$$= 1 + \sum_{n\geq 1} \frac{1}{n!}\int_{\mathbb{R}^4} d^4x_1 \cdots \int_{\mathbb{R}^4} d^4x_n S_n(x_1, \cdots, x_n)$$ 摂動展開

$$S_n(x_1, \cdots, x_n) = i^n T(\mathcal{L}_I(x_1) \cdots \mathcal{L}_I(x_n))$$

$$S_1(x) = i\mathcal{L}_I(x) = ie : \bar{\Psi}(x)\hat{A}(x)\psi(x) :$$ 正規順序積

QED



$$S_2(x,y) = i^2 T(\mathcal{L}_I(x)\mathcal{L}_I(y))$$

$$= -e^2 \sum_{m,n} T(: J^m(x)A_m(x) : : J^n(x)A_n(x) :)$$ $J^m(x) = \bar{\Psi}(x)\gamma^m\psi(x)$ 電流場

Compton散乱　自己エネルギー　真空偏極

- $\langle\beta|S|\alpha\rangle$

  $\langle 0|S|0\rangle$

- Feynman図 ― 内線, 頂点, 外線

  Feynman積分 ― 時空 → 運動量 → パラメータ積分　構造関係？

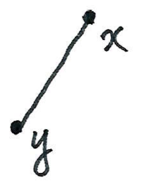
（Feynman，因果的）

○ 伝播関数

Green関数，プロパゲータ

以下，場の演算子（生成，消滅）

1) スカラー場 $\varphi(x)$



$$\varphi(x)\varphi(y)=\langle 0|T\varphi(x)\varphi(y)|0\rangle=\underline{\Delta^F(x-y)}$$

2点関数　　真空期待値

$$=\frac{-i}{(2\pi)^4}\int_{\mathbb{R}^4}d^4p\,\frac{e^{-ip(x-y)}}{\boxed{m^2-p^2-i0}}$$

運動量積分表示（Fourier変換）

massless のとき，Δ_0^F

$$(\Box+m^2)\Delta^F(x-y)=-i\delta(x-y)$$

2) 電磁場 $A(x)$

$$A_m(x)A_n(y)=\langle 0|TA_m(x)A_n(y)|0\rangle=ig^{mn}\underline{D_0^F(x-y)}$$

$$D_0^F(x-y)=-\frac{1}{(2\pi)^4}\int_{\mathbb{R}^4}d^4p\,\frac{e^{-ip(x-y)}}{\boxed{p^2+i0}}$$

$$\Box D_0^F(x-y)=-\delta(x-y)$$

3) 電子場 $\psi(x)$

$$\psi_\alpha(x)\bar{\psi}_\beta(y)=\langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle=-i\underline{S^F_{\alpha\beta}(x-y)}$$

$$S^F_{\alpha\beta}(x-y)=\frac{1}{(2\pi)^4}\int_{\mathbb{R}^4}d^4p\,\boxed{\frac{(m-\hat{p})_{\alpha\beta}}{m^2-p^2-i0}}e^{-ip(x-y)}$$

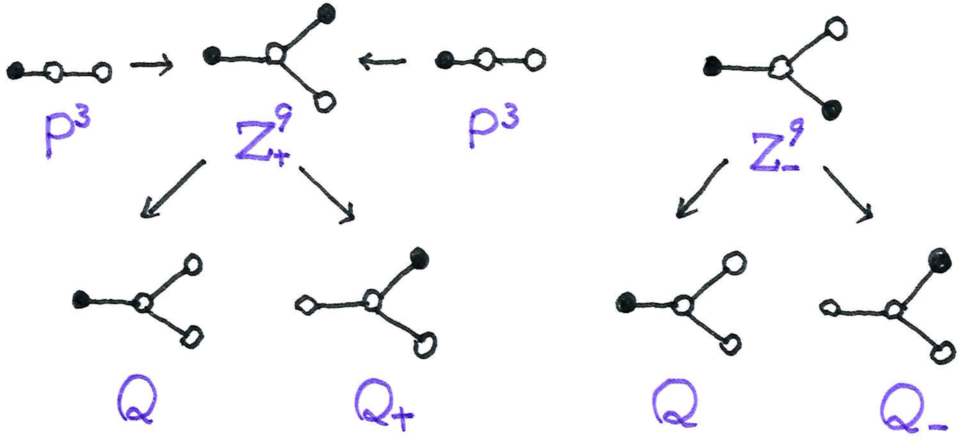
$$(i\hat{\partial}+m)_{\alpha\beta}\Delta^F(x-y)=S^F_{\alpha\beta}(x-y)$$

$$(i\hat{\partial}-m)S^F_{\alpha\beta}(x-y)=-\delta(x-y)\delta_{\alpha\beta}$$

○基本解(伝播関数)のツイスター積分表示

A·M (1999) (n,n)型, (n,n-1)型

				積分	ツイスター	
	4	(2,2)	—	1次元	⊂ 3	
Q	6	(3,3)	—	3	⊂ 6	Q_+, Q_-
	8	(4,4)	—	6	⊂ 10	
	10	(5,5)	—	10	⊂ 15	
	12	(6,6)	—	15	⊂ 21	
	⋮					



(p,2)型のとき, Poisson 積分表示

(p,p)型のとき, Radon 〃

- $$\Delta_0^F = \frac{1}{\underbrace{(x_1x_6+x_2x_5+x_3x_4)^2}_{=q}} \rightsquigarrow \Box\Delta_0^F=0$$

$\mathrm{res}\,\Delta_0^F \cdots\cdots \delta_1(q)$

$\mathrm{res}\int_{q=0} q(x-y)^2\varphi(y)dy=\varphi(x)$

$\Box\,\mathrm{res}\,\Delta_0^F(x-y)=\delta(y)$

- $$S_0^F=\underbrace{D}_{\Gamma'\partial}\Delta_0^F=\frac{-2}{(x_1x_6+x_2x_5+x_3x_4)^3}\underbrace{\begin{pmatrix}0&-x_1&x_2&-x_3\\x_1&0&-x_4&-x_5\\-x_2&x_4&0&-x_6\\x_3&x_5&x_6&0\end{pmatrix}}_{\Gamma} \rightsquigarrow \underbrace{D'}_{\Gamma\partial}S_0^F=0$$

- 電磁場は, D_0^F

$$\Gamma=\begin{pmatrix}0&-x_1&x_2&-x_3\\x_1&0&-x_4&-x_5\\-x_2&x_4&0&-x_6\\x_3&x_5&x_6&0\end{pmatrix},\quad D'=\Gamma\partial=\begin{pmatrix}0&-\partial_1&\partial_2&-\partial_3\\\partial_1&0&-\partial_4&-\partial_5\\-\partial_2&\partial_4&0&-\partial_6\\\partial_3&\partial_5&\partial_6&0\end{pmatrix}$$

$$\Gamma'=\begin{pmatrix}0&-x_6&x_5&-x_4\\x_6&0&-x_3&-x_2\\-x_5&x_3&0&-x_1\\x_4&x_2&x_1&0\end{pmatrix},\quad D=\Gamma'\partial=\begin{pmatrix}0&-\partial_6&\partial_5&-\partial_4\\\partial_6&0&-\partial_3&-\partial_2\\-\partial_5&\partial_3&0&-\partial_1\\\partial_4&\partial_2&\partial_1&0\end{pmatrix}$$

$$D'D=\Box I_4$$

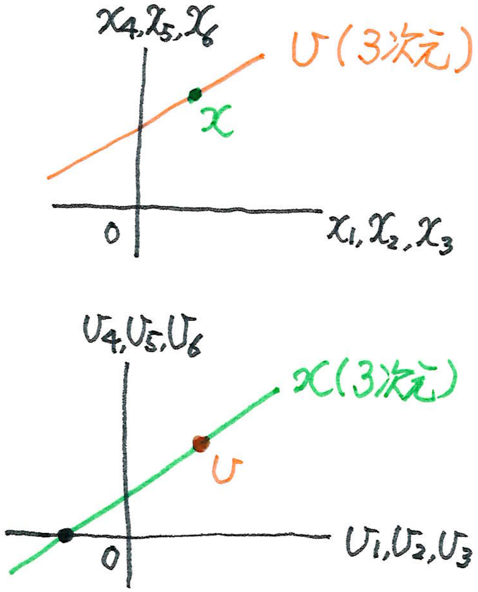
generic

$$\begin{cases} x_4 = U_2 x_1 + U_1 x_2 + U_4 \\ x_5 = U_3 x_1 - U_1 x_3 - U_5 \\ x_6 = -U_3 x_2 - U_2 x_3 + U_6 \end{cases}$$

$$dx_1 dx_6 + dx_2 dx_5 + dx_3 dx_4 = 0$$

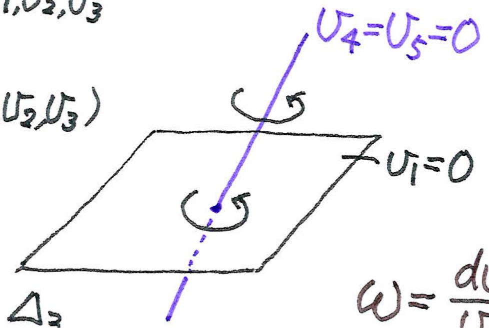
$$\begin{cases} U_4 = -x_2 U_1 - x_1 U_2 + x_4 \\ U_5 = -x_3 U_1 + x_1 U_3 - x_5 \\ U_6 = x_3 U_2 + x_2 U_3 + x_6 \end{cases}$$

$$dU_1 dU_6 + dU_2 dU_5 + dU_3 dU_4 = 0$$



- $\omega := \dfrac{dU_1 \wedge dU_2}{U_4^2 U_5 U_6} + \dfrac{dU_1 \wedge dU_3}{U_4 U_5^2 U_6} + \dfrac{dU_2 \wedge dU_3}{U_4 U_5 U_6^2}$

$x : (U_1, U_2, U_3)$



$\omega = \dfrac{dU_2 \wedge dU_3}{U_4 U_5 U_6^2}$

× $d\omega = 0$ on $x \cong \mathbb{R}^3 . \mathbb{C}^3$

× $[\omega] \in H^2(x - \bigcup_{i=4}^{6} \{U_i = 0\}, \mathbb{C})$

× $\omega \sim \alpha\,(d\log U_2 \wedge d\log U_3 - d\log U_1 \wedge d\log U_3 + d\log U_1 \wedge d\log U_2)$

α : $(x_1 x + x_2 x_5 + x_3 x_4)^{-2}$

⊗ $\dfrac{1}{(2\pi i)^2} \displaystyle\int_\Delta \omega = (x_1 x_6 + x_2 x_5 + x_3 x_4)^{-2}$ residue

Δ : $C_1 \Delta_1 + C_2 \Delta_2 + C_3 \Delta_3 \ (C_1 + C_2 + C_3 = 1)$

- $\partial_i \displaystyle\int_\Delta \omega = \int_\Delta \partial_i \omega$

$\partial_1 \omega, \partial_2 \omega, \partial_3 \omega$
$\partial_4 \omega, \partial_5 \omega, \partial_6 \omega$

- $\Delta_0^F(x-y)\,\Delta_0^F(y-z)$

$= \displaystyle\int_{\mathbb{R}^3 \times \mathbb{R}^3} \omega_1(x-y) \wedge \omega_2(y-z)$

$= \displaystyle\int_{\mathbb{R}^3} \omega_1(x-y) \int_{\mathbb{R}^3} \omega_2(y-z)$