The electromagnetic aspect for Yang-Mills fields

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Abstract

Let $\mathcal{A}$ be the space of irreducible connections (vector potentials) over the principal bundle $M \times SU(n)$ on a compact three-dimensional manifold M. The cotangent bundle $T^*\mathcal{A}$ has a canonical symplectic structure and there is a Hamiltonian action of the group of gauge transformations $\mathcal{G}$ on it with the moment map

$$\mathbf{J}^*(A,B) = -d_A B, \qquad A \in \mathcal{A},\ B \in T^*_A\mathcal{A}.$$

Dually, if we give the tangent space $T_A\mathcal{A}, A \in \mathcal{A}$, the riemannian metric $(a,b)_A = \int_M Tr\, a \wedge *b$, $a, b \in T_A \simeq \Omega^1(M, su(n))$, then, by virtue of the Hodge $*$-operator the tangent space also has a symplectic structure coming from that of the cotangent space, and there is a Hamiltonian action of $\mathcal{G}$ on it with the moment map

$$\mathbf{J}(A,E) = d^*_A E, \qquad A \in \mathcal{A},\ E \in T_A\mathcal{A}.$$

The Yang-Mills field $\mathcal{F}$ is defined as the subspace of the direct sum $\mathbb{T} = T\mathcal{A} \times_{\mathcal{A}} T^*\mathcal{A}$ of the tangent and cotangent bundles of $\mathcal{A}$ where we have $d^*_A E = d_A B = 0$. The symplectic structure on $\mathbb{T}$ is given by the 2-form:

$$\Omega_{(E,B)}\left(\left(\begin{array}{c} e_1 \\ \beta_1 \end{array}\right), \left(\begin{array}{c} e_2 \\ \beta_2 \end{array}\right)\right) = (\, e_2\,,\, d^*_A\beta_1\,)_1 \,-\, (\, e_1\,,\, d^*_A\beta_2\,)_1 \,,$$

for $\left(\begin{array}{c} e_i \\ \beta_i \end{array}\right) \in T_{(E,B)}\mathbb{T}\,,\ i = 1, 2.$

Let $H = H(E, B)$ be the Hamiltonian function on $\mathcal{F}$;

$$H(E, B) = \frac{1}{2}\{(d_A E, d_A E)_1 + (d_A^* B, d_A^* B)_1\}.$$

Then Hamilton's equation of motion on $(\mathcal{F}, \Omega)$ is written in the form:

$$d_A^* B + \dot{E} = 0 \quad , \qquad d_A E - \dot{B} = 0,$$

$$d_A B = 0 \quad , \qquad d_A^* E = 0 .$$

The corresponding Poisson bracket on $\mathcal{F}$ is

$$\{ \Phi , \Psi \}^{\mathbb{T}}_{(E,B)} = \left(\frac{\delta \Phi}{\delta B} , d_A^* \frac{\delta \Psi}{\delta E} \right)_1 - \left(\frac{\delta \Psi}{\delta B} , d_A^* \frac{\delta \Phi}{\delta E} \right)_1 .$$

This is a parallel formula of Marsden-Weinstein in case of the electric-magnetic field. We shall investigate the Clebsch parametrization of the Yang-Mills field $(\mathcal{F}, \Omega)$. We show that the action of $\mathcal{G}$ on $(\mathcal{F}, \Omega)$ is Hamiltonian with the moment map $\mathbb{J}(E, B) = [d_A E , *B]$. This gives a conserved quantity $\int_M [d_A E , *B]$ which is due to the non-commutativity of the gauge group.

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