Electromagnetic aspect of Yang-Mills fields

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$\mathcal{A}$: the space of irreducible connections (vector potentials) over the principal bundle $P = M \times SU(n)$.

$\mathcal{A}$ is an affine space modeled with the vector space $\Omega^1(M, Lie\, G)$.

$$T_A\mathcal{A} = \Omega^1(M,\ Lie\, G))$$

$\mathcal{G}$: The group of (pointed) gauge transformations:

$$\mathcal{G} = \Omega^0(M, Ad_G P) = \Omega^0(M, G).$$

$$\mathcal{A} \times \mathcal{G} \ni (A, g) \longrightarrow g^{-1}Ag + g^{-1}dg\,.$$

$$\mathcal{A} \xrightarrow{\pi} \mathcal{B} = \mathcal{A}/\mathcal{G}; \quad modulispace.$$

1 Geometric pre-Quantization (Kostant-Souriou)

(i) Find a pre-symplectic form ω on $\mathcal{B}$.

(ii) Give a line bundle with connection

$$(\mathcal{L},\, \theta) \longrightarrow \mathcal{B}$$

such that the curvature of θ is ω.

Example 1[(Atiyah-Bott, 1982)]

(i) Let Σ be a surface (2-dimensional manifold).

$$T_A\mathcal{A}(\Sigma) \simeq T_A^*\mathcal{A}(\Sigma) \simeq \Omega^1(\Sigma, LieG)$$

$$\omega_A(a,b) = \int_\Sigma tr(ba) - \int_\Sigma tr(ab) = 2\int_\Sigma tr(ba).$$

Then $(\mathcal{A}(\Sigma),\, \omega\,)$ is a symplectic manifold.

(ii) $\exists$ a line bundle with connection

$$(\mathcal{L},\, \theta) \longrightarrow \mathcal{B}$$

such that the curvature of θ is ω.

Jeffery-Weitsman, Meinrenken, $\cdots$

Example 2(Kori, 2011)

(i) Let $P = X \times SU(n)$ be the trivial $SU(n)-$principal bundle on a four-manifold X. There exists a pre-symplectic structure on the space of irreducible connections $\mathcal{A}(X)$ given by the 2-form

$$\sigma_A(a,b) = \frac{1}{8\pi^3}\int_X tr[(ab-ba)F] - \frac{1}{24\pi^3}\int_M tr[(ab-ba)A]\,. \quad (1.1)$$

Where $F = dA + \frac{1}{2}[A, A]$.

(ii) $\exists$ a line bundle with connection

$$(\mathcal{L},\, \theta) \longrightarrow \mathcal{B}$$

such that the curvature of θ is σ. (Chern-Simons quantization)

For 3-dimensional manifolds there would be no presymplectic structure on the space of connections $\mathcal{A}$.

(Remark) There is a presymplectic form on the space of flat connections $\mathcal{A}^\flat(\mathcal{M})$

In the following we shall deal with the geometric pre-quantization of the sapce of connections over 3-manifolds, where there would be no presymplectic structure.

BUT

- A quantization would not be the quantizationof the space of vector potentials $\mathcal{A}$, but should be the space of fields $\subset T^*\mathcal{A}$.
- The cotangent sapce $T^*\mathcal{A}$ is always symplectic.
- May be anticipated the theory of Bohr-Sommerfeld Quantization $\simeq$ semi-classical approximation over $T^*\mathcal{A}$

Today's talk:

Hamiltonian formalism on $T\mathcal{A} \oplus T^*\mathcal{A}(M)$ with $\dim M = 3$.

$=$ Maxwell like equation of motion

2 Revision: Maxwell equations

2.1 4-dimensional vector potentials $\Longrightarrow$ 3-dimensional Maxwell field equations

$\mathcal{A}_{Max}$: The space of $U(1)$-connections on R^4.

$$\mathcal{A}_{Max} \ni \hat{A} = A_1 dx^1 + A_2 dx^2 + A_3 dx^3 + \phi\, dt$$

$\hat{d}$: = the exterior differentiation on R^4.

$$F = \hat{d}\,\hat{A} \;=\; B + Edt \qquad \text{curvature}$$

$$B = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2\,,$$

$$E = E_1 dx^1 + E_2 dx^2 + E_3 dx^3.$$

$$B_i = \frac{\partial}{\partial x^j} A_k - \frac{\partial}{\partial x^k} A_j, \qquad E_i = \frac{\partial}{\partial x^i}\phi - \frac{\partial}{\partial t} A_i\,.$$

(1)

$$\hat{d}F = \hat{d}\,\hat{d}\hat{A} = 0 \tag{2.1}$$

$\Longleftrightarrow$

$$\sum_{i=1}^{3} \frac{\partial}{\partial x^1} B_i = 0, \qquad \frac{\partial}{\partial x^j} E_k - \frac{\partial}{\partial x^k} E_j + \dot{B}_i = 0\,.$$

$$i.e. \quad div\, B = 0\,, \qquad \nabla \times E + \dot{B} = 0$$

$\Longleftrightarrow$

3-dim. expression:

$$\begin{cases} dB = 0 & \nexists \text{ magnetic monopole} \\ dE + \dot{B} = 0 & \text{Faraday's law} \end{cases} \tag{2.2}$$

where $d = \sum_{i=1}^{3} \frac{\partial}{\partial x^i} dx^i \, : \, =$ the exterior differentiation on R^3.

(2.2) is invariant under the action of gauge group $U(1)$. That is, (2.2) is a **field equation**.

(2)

$\star$: Hodge operator on R^4.

We have

$$\hat{d} \star F = \mathbf{j} \wedge dt + \rho\,. \tag{2.3}$$

for

$\exists$ 2-form; $\mathbf{j} = j_1 dx^2 \wedge dx^3 + j_2 dx^3 \wedge dx^1 + j_3 dx^1 \wedge dx^2$,

$\exists$ 3-form; $\rho\, dx^1 dx^2 dx^3$.

$\Longleftrightarrow$

3-dim. expression:

$$\begin{aligned} d * E &= \rho dx^1 \wedge dx^2 \wedge dx^3, \\ d * B + *\dot{E} &= \mathbf{j} \end{aligned}$$

where $*$ is the Hodge operator on R^3.

That is,

$$\begin{cases} d^* E = \rho \quad \Longleftrightarrow \quad div\, E = \rho\,; \quad \text{Gauss law} \\ d^* B + \dot{E} = *\mathbf{j} \Longleftrightarrow \nabla \times B + \dot{E} = \mathbf{j}; \quad \text{Ampère's law} \end{cases} \tag{2.4}$$

Maxwell's eqation consists of ;

- The field equation (2.2) , i,e, the defining equation of the electro-magnetic field.
- The conserved quantities (2.4) , i.e. the momentum of the action of the gauge transformation.

(3)

Marsden-Weinstein introduced the Poisson bracket on the electro-magnetic field; (Vortex type Poisson bracket):

$$\{\,\Phi\,,\,\Psi\,\}_{(E,B)} \;=\; \left(\frac{\delta\Phi}{\delta B}\,,\,curl\,\frac{\delta\Psi}{\delta E}\right)_1 - \left(\frac{\delta\Psi}{\delta B}\,,\,curl\,\frac{\delta\Phi}{\delta E}\right)_1 . \tag{2.5}$$

For the Hamiltonian function

$$H = H(E,B) = \frac{1}{2}\left(||dE||^2 + ||d^*B||^2\right), \tag{2.6}$$

the equation of motion

$$\dot{\Phi} \;=\; \{\,\Phi\,,\,H\}_{(E,B)}$$

is nothing but the the Maxwell equation (Faradey and Ampère with 0-current:

$$\dot{E}\,+\,d^*B\,=0\,,\qquad \dot{B}\,+\,dE\,=0\,. \tag{2.7}$$

□

Why vortex formula?

We can look from Faraday's equation and Ampère equation the actual motion of the magnetmeter or the electricity in the coil. By vorticity potentials are realized (visible) as fields.

- We glance over this fact in VORTICITY of fluid mechanics.

$B \subset \mathrm{R}^3$ simply connected.

- $\mathcal{G} = SVect(B)$: the divergent free vector fields on S^3.

The space of vorticity vector fields is

$$\nabla \times \mathcal{G} = \{\omega = \nabla \times \mathbf{v}; \quad \mathbf{v} \in \mathcal{G}\}.$$

- $\forall \mathbf{u} \in \mathcal{G}$, there is a unique (mod. ∇f) solution $\mathbf{v} \in \mathcal{G}$ of

$$\nabla \times \mathbf{v} = \mathbf{u}\,.$$

That is given by the Biot-Savart's formula:

$$\mathbf{v}(y) = BS(\mathbf{u}) = -\frac{1}{4\pi}\int_B \frac{\mathbf{u}(x) \times (x-y)}{|x-y|^3} d^3x,$$

Hence

$$\nabla \times \mathcal{G} \xrightarrow{BS \simeq} \mathcal{G}\,.$$

On the other hand $\mathcal{G} = SVect(B)$ and $\nabla \times \mathcal{G}$ are in duality by

$$(\omega, \mathbf{v}) = \int_B \omega \cdot \mathbf{v}\ d^3x$$

Hence

$$\mathcal{G}^* \simeq \nabla \times \mathcal{G}. \tag{2.8}$$

$$
\begin{array}{ccc}
\mathcal{G} = SVect(B) & \xrightarrow{A\,\simeq} & \Omega^1(B)/d\Omega^0(B) \simeq \mathcal{G}^* \\
\nabla\times \downarrow \simeq \uparrow BS & & d \downarrow \simeq \uparrow d^* G \\
\mathcal{G}^* = \nabla \times SVect(B) & \xrightarrow{i\,.\,vol.\simeq} & Z^2(B, \partial B)
\end{array}
\tag{2.9}
$$

Here, on the RHS

$$
\Omega^1(B)/d\Omega^0(B) \xrightarrow{d\,\simeq} Z^2(B, \partial B) = \{\beta \in \Omega^2(B);\, d\beta = 0, \quad \beta|\partial B = 0\}\,,
$$

and the Green function G gives the solution $\beta = G\nu \in Z^2(B, \partial B)$ of $\Delta\beta = \nu$ for $\nu \in Z^2(B, \partial B)$

$$
\begin{aligned}
SVect(B) \ni \mathbf{v} &\longrightarrow \mathbf{v}^\flat = A\mathbf{v} \in \Omega^1(B)/d\Omega^0(B) \\
\nabla \times SVect(B) \ni \nabla \times \mathbf{v} = \omega &\longrightarrow d\mathbf{v}^\flat = \omega^\flat = i_\omega vol \in Z^2(B, \partial B), \\
SVect(B) \ni \mathbf{v} &\longrightarrow i_\mathbf{v} vol \in Z^2(B, \partial B),
\end{aligned}
$$

- Vortex representation

Euler equation for incompressible flow:

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0, \qquad \exists p, \tag{2.10}$$

$$\text{rewritten} \quad \Longrightarrow$$

$\dot{\mathbf{v}} = \mathbf{v} \times \omega + \nabla q, \quad q = -(p + \frac{1}{2}\mathbf{v} \cdot \mathbf{v}),$

Then

$$\begin{aligned} \dot{\omega} &= \nabla \times \dot{\mathbf{v}} = \nabla \times (\mathbf{v} \times \omega) + \nabla \times \nabla q \\ &\quad = (\omega \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\,\omega - (div\,\mathbf{v})\omega + (div\,\omega)\mathbf{v} \\ &= (\omega \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\omega = L_{\mathbf{v}}\omega. \end{aligned}$$

$\dot{\omega} = L_{\mathbf{v}}\omega$: Vortex type Euler equation.

The above vortex type Euler's equation shows that the vorticity vector fields are always on the flow of velocity vector fields; (Lord Kelvin's theorem)

THIS LECTURE provides the following subjects.

M : a compact 3-dimensional manifold X.

$\mathcal{A}$: the space of irreducible connections (vector potentials) over the principal bundle $M \times SU(n)$.

♣

$\mathbb{T} = T\mathcal{A} \times_{\mathcal{A}} T^*\mathcal{A}$: Whittney's direct sum of the tangent and cotangent bundles of $\mathcal{A}$.

The symplectic structure on $\mathbb{T}$ is given by the 2-form:

$$\Omega_{(E,B)}\left(\begin{pmatrix} e_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ \beta_2 \end{pmatrix}\right) = (\, e_2 \, , \, d_A^* \beta_1 \,)_1 \, - \, (\, e_1 \, , \, d_A^* \beta_2 \,)_1 \ ,$$

for $\begin{pmatrix} e_i \\ \beta_i \end{pmatrix} \in T_{(E,B)}\mathbb{T} \, , \, i = 1, 2.$

$\diamondsuit$

(1)

The Yang-Mills field $\mathcal{F}$ is defined as a subspace of $(\mathbb{T}, \Omega\,)$.

$\mathcal{F}$ is rather a symplectic reduction, or a horizontal lift of the reduced space.

We shall prove the Maxwell equations on $\mathcal{F}$:

$$\left\{ \begin{array}{cc} d_A^* B \,+\, \dot{E} \,=\, 0\,, & d_A E \,-\, \dot{B} \,=\, 0, \\ d_A B \,=\, 0\ , & d_A^* E \,=0\,. \end{array} \right. \tag{2.11}$$

The first two equations are the Hamilton equations of motion derived from the symplectic structure on $\mathbb{T}$*,*

The second equations come from the action of the group of gauge transformations $\mathcal{G}$ *on* $\mathcal{A}$*,*

The second equations are the defining equations of $\mathcal{F}$.

(2)

The corresponding Poisson bracket on $\mathcal{F}$ is

$$\{\,\Phi\,,\,\Psi\,\}^{\mathbb{T}}_{(E,B)} \;=\; \Omega_{(E,B)}\,(\,X_\Phi\,,\,X_\Psi\,) \tag{2.12}$$

$$= \left(\frac{\delta\Phi}{\delta B}\,,\,d_A^*\frac{\delta\Psi}{\delta E}\right)_1 - \left(\frac{\delta\Psi}{\delta B}\,,\,d_A^*\frac{\delta\Phi}{\delta E}\right)_1 \tag{2.13}$$

This is a parallel formula of Marsden-Weinstein in case of the electric-magnetic field.

$$\{ \Phi \, , \, \Psi \}_{(E,B)} \; = \; \left(\frac{\delta \Phi}{\delta B} \, , \, curl \, \frac{\delta \Psi}{\delta E} \right)_1 \; - \; \left(\frac{\delta \Psi}{\delta B} \, , \, curl \, \frac{\delta \Phi}{\delta E} \right)_1 \, . \tag{2.14}$$

For the Hamiltonian function

$$H = H(E, B) = \frac{1}{2} \left(||dE||^2 + ||d^* B||^2 \right) , \tag{2.15}$$

the equation of motion

$$\dot{\Phi} \; = \; \{ \Phi \, , \, H \}_{(E,B)}$$

is nothing but the the Maxwell equation (Faradey and Ampère with 0-current:

$$\dot{E} \, + \, d^* B \, = 0 \, , \qquad \dot{B} \, + \, dE \, = 0 \, . \tag{2.16}$$

□

♡

(1)

We show that the action of $\mathcal{G}$ on $(\mathcal{F}, \Omega)$ is Hamiltonian with the moment map

$$\mathbb{J}(\,E,\,B\,) = [\,d_A E\,,\,*B\,]\,.$$

This gives a conserved quantity $\int_M [\,d_A E\,,\,*B\,]$ that is due to the non-commutativity of the gauge group.

(2)

There is a symplectic parametrization (Clebsch variables) of $\mathcal{F}$ by the tangent space of the moduli space $T(\mathcal{A}/\mathcal{G})$.

3 Canonical structure on $T^*\mathcal{A}$

M : compact, connected and oriented m-dimensional riemannian manifold.
$P \xrightarrow{\pi} M$: a principal G-bundle, $G = SU(N)$, $N \geq 2$.
(In the sequel, mostly supposed to be trivial: $P = M \times G$.)

$\mathcal{A} = \mathcal{A}(M)$: the space of *irreducible* L^2_{s-1} connections over P. (s will be abbreviated); An affine space modeled by $\Omega^1_{s-1}(M, LieG)$.

Tangent space of $\mathcal{A}$ at $A \in \mathcal{A}$ is

$$T_A\mathcal{A} = \Omega^1_{s-1}(M, Lie\, G)\,. \tag{3.1}$$

Cotangent space of $\mathcal{A}$ at A is

$$T^*_A\mathcal{A} = \Omega^{m-1}_{s-1}(M, Lie\, G)\,), \tag{3.2}$$

The pairing $\langle a,\, \alpha\rangle_A \;= \int_M\, tr(\, a \wedge \alpha\,)$ of $\alpha \in T^*_A\mathcal{A}$ and $a \in T_A\mathcal{A}$ is given by

$$\langle \phi \otimes X, \psi \otimes Y\rangle = (\phi\,,\,\psi\,)_{s-1}\, tr(XY),$$

for $\psi \in \Omega^{m-1}(M)$, $\phi \in \Omega^1(M)$, and $X, Y \in Lie\, G$.

[Notation]
Tangent bundle $R = T\mathcal{A}$,
Cotangent bundle $S = T^*\mathcal{A}$.

The point of S is denoted by
$S \ni (A, \lambda)$ with $A \in \mathcal{A}$ and $\lambda \in T_A^*\mathcal{A}$.

The tangent space to the cotangent space S at the point $(A, \lambda) \in S$:

$$T_{(A,\lambda)}S \equiv T_A\mathcal{A} \oplus T_A^*\mathcal{A} = \Omega^1(M,\ Lie\, G) \oplus \Omega^{m-1}(M.\, Lie\, G). \tag{3.3}$$

Any tangent vector $\mathbf{a} \in T_{(A,\lambda)}S$ has the form $\mathbf{a} = \begin{pmatrix} a \\ \alpha \end{pmatrix}$ with $a \in T_A\mathcal{A}$ and $\alpha \in T_A^*\mathcal{A}$.

The canonical 1-form θ on the cotangent space S is defined as follows:

$$\theta_{(A,\lambda)}(\begin{pmatrix} a \\ \alpha \end{pmatrix}) = \langle\, \lambda, \pi_* \begin{pmatrix} a \\ \alpha \end{pmatrix} \rangle_A = \int_M tr\, a \wedge \lambda, \tag{3.4}$$

for any tangent vector $\begin{pmatrix} a \\ \alpha \end{pmatrix} \in T_{(A,\lambda)}S$.

Let ϕ be a 1-form on $\mathcal{A}$. We have the following characteristic property:

$$\phi^*\theta = \phi. \tag{3.5}$$

The canonical 2-form ω on S is defind by

$$\omega = \widetilde{d}\theta. \tag{3.6}$$

ω is a non-degenerate closed 2-form on the cotangent space S.

The exterior differential $\widetilde{d}$ on $\mathcal{A}$ will be explained in the following.

canonical 1-form θ :

$$\theta_{(A,\lambda)}(\begin{pmatrix} a \\ \alpha \end{pmatrix}) = \langle\, \lambda, \pi_* \begin{pmatrix} a \\ \alpha \end{pmatrix} \rangle_A = \int_M tr\, a \wedge \lambda,$$

Lemma 3.1. *The derivation of the 1-form θ ; is given by*

$$(\partial_{(A,\lambda)}\, \theta) \begin{pmatrix} a \\ \alpha \end{pmatrix} = \int_M tr\, a \wedge \alpha\,, \qquad for\ \forall \begin{pmatrix} a \\ \alpha \end{pmatrix} \in T_{(A,\lambda)}S\,.$$

In fact $\quad (\partial_{(A,\lambda)}\theta)(\begin{pmatrix} a \\ \alpha \end{pmatrix}) = \lim_{t \longrightarrow 0} \frac{1}{t} \int_M (\, tr\, a \wedge (\lambda + t\alpha) - tr\, a \wedge \lambda\,) = \int_M tr\, a \wedge \alpha.$

Proposition 3.2.

$$\omega_{(A,\lambda)}\,(\begin{pmatrix} a \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ \beta \end{pmatrix}) = \int_M tr[\, b \wedge \alpha - a \wedge \beta\,,]$$

for $\begin{pmatrix} a \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ \beta \end{pmatrix} \in T_{(A,\lambda)}S$

Proposition follows from Lemma 1.1 and (3.6).

Canonical 2-form ω is a symplectic form on S

Let $\Phi = \Phi(A, \lambda)$ be a function on the cotangent space S. The Hamitonian vector field X_Φ of Φ is defined by the formula:

$$(\widetilde{d}\, \Phi)_{(A,\lambda)} \; = \; \omega(\quad \cdot \quad , X_\Phi(A, \lambda) \,). \tag{3.7}$$

We look for $X_\Phi(A, \lambda) = \begin{pmatrix} b \\ \beta \end{pmatrix}$ by compairing

$$\text{(1)} \quad (\widetilde{d}\, \Phi)_{(A,\lambda)} \begin{pmatrix} a \\ \alpha \end{pmatrix} \; = \; \langle\, a \,,\, \frac{\delta \Phi}{\delta A} \,\rangle_A \, + \, \langle\, \frac{\delta \Phi}{\delta \lambda} \,,\, \alpha \,\rangle_A$$

$$= \int_M tr[\, a \wedge \frac{\delta \Phi}{\delta A} \,] + \int_M tr[\, \frac{\delta \Phi}{\delta \lambda} \wedge \alpha \,]$$

and

$$\text{(2)} \quad \omega_{(A,\lambda)} \,(\begin{pmatrix} a \\ \alpha \end{pmatrix} , \begin{pmatrix} b \\ \beta \end{pmatrix}) = \int_M tr[\, b \wedge \alpha - a \wedge \beta \,]$$

Then we have

$$X_\Phi \; = \; \begin{pmatrix} \frac{\delta \Phi}{\delta \lambda} \\ - \frac{\delta \Phi}{\delta A} \end{pmatrix} . \tag{3.8}$$

3.1 Tangent space of $\mathcal{A}$

We assume that P is a trivial bundle $P = M \times G$.

$\mathcal{A}$ being an affine space modelled by the vector space
$\Omega^1(M,\, ad\, P \;= \Omega^1(M,\, su(n))$,
the tangent space at the point $A \in \mathcal{A}$ is

$$T_A\mathcal{A} \;=\; \Omega^1(M, su(n))\,.$$

The inner product on $T_A\mathcal{A}$?F

$$(\,a\,,\,b\,)_1 \;=\; \int_M Tr\, a \wedge *b \qquad \forall a, b \in T_A\mathcal{A}\,.$$

Denote a point of $R = T\mathcal{A}$ by
$(A, p) \in R$ with $A \in \mathcal{A}$ and $p \in T_A\mathcal{A}$.
The tangent space of R is

$$T_{(A,p)}R = T_A\mathcal{A} \oplus T_A\mathcal{A}.$$

a tangent vector $\mathbf{a} \in T_{(A,p)}R$ is given by

$$\mathbf{a} = \begin{pmatrix} a \\ x \end{pmatrix} \qquad \text{with } a,\, x \in T_A\mathcal{A}.$$

The symplectic structure on R is defined by the formula

$$\sigma_{(A,p)}\left(\begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix} \right) \;=\; (\,b\,,\,x\,)_1 \;-\; (\,a\,,\,y\,)_1\,,$$

for all $\begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix} \in T_{(A,p)}R\,.$

3.2 The action of the group of gauge transformations $\mathcal{G}$ on $R = T\mathcal{A}$ and $S = T^*\mathcal{A}$, and the corresponding moment maps

$\mathcal{G}(M) = \Omega^0(M, Ad_G P) = \Omega^0_s(M,\, Ad\, G)$ acts on $\mathcal{A}$ by

$$g \cdot A = g^{-1}Ag + g^{-1}dg, \qquad A \in \mathcal{A},\, g \in \mathcal{G}$$

- $\mathcal{G}(M)$ acts on $R = T_A\mathcal{A}$ by

$$a \longrightarrow Ad_{g^{-1}}\, a = g^{-1}ag$$

and on $S = T^*_A\mathcal{A}$ by its dual

$$\alpha \longrightarrow g^{-1}\alpha g\,.$$

The canonical 1-form θ and 2-form ω on $S = T^*\mathcal{A}$ are $\mathcal{G}$-invariant.

- $Lie\,\mathcal{G} = \Omega^0(M, Lie\, G)$.

For $\xi \in Lie\,\mathcal{G}$, the fundamental vector field ξ_S on S is given by

$$\xi_S(A, \lambda) = \frac{d}{dt} \exp t\xi \cdot \begin{pmatrix} A \\ \lambda \end{pmatrix} = \begin{pmatrix} d_A\xi \\ [\lambda, \xi] \end{pmatrix}.$$

- The dual space of $Lie\,\mathcal{G}$ is

$$(Lie\,\mathcal{G})^* \;=\; \Omega^m(M, Lie\, G)$$

with the dual pairing:

$$\langle \mu,\, \xi \rangle \;=\; \int_M tr\,(\mu\,\xi\,)\,, \qquad \forall \xi \in \mathcal{G}_0\,,\; \mu \in \Omega^m(M, Lie\, G).$$

3.3 moment map on S

The *moment map* of the action of $\mathcal{G}$ on $(S\,,\omega)$

$$K : S \longrightarrow (Lie\,\mathcal{G})^* \simeq \Omega^m(M,\, Lie\,G)\,,$$

is defined as follows:
Put
$K^{\xi}(A,\lambda) = \langle K(A,\lambda),\, \xi\,\rangle$ for $\xi \in Lie\,\mathcal{G}$.
Then K is the moment map $\overset{def}{\Longleftrightarrow}$

1. K^{ξ} is $Ad^*\mathcal{G}$-equivariant:

$$K^{Ad_g\xi}(g\cdot A,\, g\cdot\lambda\,) \;=\; K^{\xi}(A,\lambda)\,,$$

2. The Hamilton vector field K^{ξ} is equal to the fundamental vector field ξ_S:

$$X_{K^{\xi}} \;=\; \xi_S\,, \quad \text{that is,}$$

$$\widetilde{d}\,K^{\xi} \;=\; \omega\,(\,\cdot \quad, \xi_S\,)\,. \tag{3.9}$$

Proposition 3.3.
The action of the group of gauge transformations $\mathcal{G}(M)$ on the symplectic space (S, ω) is an hamiltonian action with the moment map given by

$$K(A,\lambda) = -d_A \lambda . \tag{3.10}$$

Proof

The equivariance of K^ξ follows easily. We shall verify the condition (3.9).
Stokes' theorem yields

$$K^\xi(A,\lambda) = \langle K(A,\lambda), \xi \rangle = -\int_M tr\,(d_A\lambda)\,\xi = \int_M tr\,(\,d_A\xi \wedge \lambda) .$$

Since

$$\lim_{t\longrightarrow 0} \frac{1}{t} \int_M tr\ (d_{A+ta}\xi \wedge (\lambda + t\alpha) - d_A\xi \wedge \lambda) = \int_M tr\,(a \wedge [\,\xi\,,\lambda] + d_A\xi \wedge \alpha\,,$$

(c.f. (**??**)), we have

$$\left(\widetilde{d}\,K^\xi\right)_{(A,\lambda)} \begin{pmatrix} a \\ \alpha \end{pmatrix} = \omega_{(A,\lambda)} \left(\begin{pmatrix} a \\ \alpha \end{pmatrix} , \begin{pmatrix} d_A\xi \\ [\lambda\,,\xi] \end{pmatrix} \right) .$$

$\square$

3.4 moment map on R

$\mathcal{G} = Aut_0(P) = \Omega^0(M, Ad_G P)$ acts on the symplectic manifold (R, ω):

$$g \cdot (A, p) \,=\, (\, A + g^{-1} d_A g \,,\, g^{-1} p g \,)\,, \quad g \in \mathcal{G}, \quad (A, p) \in R.$$

Proposition 3.4. *The action of $\mathcal{G}$ on the symplectic space $(\, R = T\mathcal{A},\, \sigma \,)$ is an hamiltonian action with the moment map $J :\, R \longrightarrow (Lie\, \mathcal{G})^* \simeq Lie\, \mathcal{G}$ given by*

$$J(A, p) \,=\, d_A^* \, p\,. \tag{3.11}$$

3.5 Duality

Proposition 3.5. *$(S\,,\,\omega\,)$ and $(R\,,\,\sigma\,)$ are isomorphic by the Hodge dual :*

$$* : T_A\mathcal{A} \simeq \Omega^1(M, su(n)) \longrightarrow T^*_A\mathcal{A} \simeq \Omega^{m-1}(M, su(n))\,, \quad A \in \mathcal{A}.$$

$\exists$ the symplectic isomorphism:

$$\sigma_{(A,p)}\left(\begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix}\right) = (\,b\,,\,x\,)_1 - (\,a\,,\,y\,)_1 = \omega_{(A,*p)}\left(\begin{pmatrix} a \\ *x \end{pmatrix}, \begin{pmatrix} b \\ *y \end{pmatrix}\right).$$

We have the following two dual descriptions:

Proposition 3.6. *Let $A \in \mathcal{A}$.*

1. (a) *Orthogonal decomposition of $T_A\mathcal{A}$?@:*

$$T_A\mathcal{A} \,=\, d_A\, Lie\, \mathcal{G} \,\oplus\, H_A\,,$$

with $H_A \,=\, \{x \in \Omega^1(M,\, Lie\, G);\; d_A^* x = 0\,\}$.

   (b) *Put*

$$R^0 \,=\, \cup_{A\in\mathcal{A}} H_A\,.$$

Then R^0 is isomorphic to the symplectic reduction of R by the moment map J,(3.11):

$$J^{-1}(0)/\mathcal{G} \,\simeq\, R^0\,. \tag{3.12}$$

2. (a) *Orthogonal decomposition of $T_A^*\mathcal{A}$:*

$$T_A^*\mathcal{A} \,=\, d_A^*(Lie\, \mathcal{G})^* \,\oplus\, H_A^*\,,$$

with $H_A^* \,=\, \{w \in \Omega^{m-1}(M,\, Lie\, G);\; d_A w = 0\,\}$.

   (b) *Put*

$$S^0 \,=\, \cup_{A\in\mathcal{A}} H_A^* \tag{3.13}$$

Then S^0 is isomorphic to the symplectic reduction of S by the moment map K,(3.9):

$$K^{-1}(0)/\mathcal{G} \,\simeq\, S^0\,. \tag{3.14}$$

4 Electronic Magnetic paradigm of Yang-Mills fields

4.1 Symplectic structure over $T\mathcal{A} \times_{\mathcal{A}} T^*\mathcal{A}$

$$R = T\mathcal{A}\,, \qquad S = T^*\mathcal{A}\,.$$

Whitteney's direct sum of tangent and cotangent bundles:

$$\mathbb{T} = R \times_{\mathcal{A}} S \longrightarrow \mathcal{A}\,.$$

$$\begin{array}{ccc} \mathbb{T} = R \times_{\mathcal{A}} S & \longrightarrow & (S\,,\omega) \\ \pi_* \downarrow & & \pi \downarrow \\ (R\,,\sigma) & \longrightarrow & \mathcal{A}\,. \end{array} \tag{4.1}$$

Denote by (A, E, B) any point in $\mathbb{T}$ (often abbreviated to (E, B)) :

$$\mathbb{T} \ni (A, E, B) \quad \text{with } A \in \mathcal{A},\ E \in T_A\mathcal{A} \text{ and } B \in T^*_A\mathcal{A}.$$

The tangent space to $\mathbb{T}$ at $(E, B) \in \mathbb{T}$ (over $A \in \mathcal{A}$) is

$$T_{(E,B)}\mathbb{T} = T_A\mathcal{A} \oplus T^*_A\mathcal{A}.$$

Every tangent vector in $T_{(E,B)}\mathbb{T}$ is written as

$$\mathbf{a} = \begin{pmatrix} e \\ \beta \end{pmatrix}$$

by $e \in T_A\mathcal{A}$ and $\beta \in T^*_A\mathcal{A}$.

Define the following inner product on each fiber $T_{(E,B)}\mathbb{T}$?F

$$\left(\begin{pmatrix} e_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ \beta_2 \end{pmatrix} \right)_{\mathbb{T}} = (e_2, d^*_A\beta_1)_1 + (e_1, d^*_A\beta_2)_1\,. \tag{4.2}$$

—————— Differential calculation on $\mathbb{T}$ ——————

The directional derivative $(\partial\Phi)_{(E,B)}\begin{pmatrix} e \\ 0 \end{pmatrix}$ of a function $\Phi = \Phi(E,B)$ on $\mathbb{T}$ to the direction $e \in T\mathcal{A}$ is defined by the Frechet derivative.

Then the partial derivative $\frac{\delta\Phi}{\delta E} \in T^*_A\mathcal{A}$ is defined by

$$(\partial\Phi)_{(E,B)}\begin{pmatrix} e \\ 0 \end{pmatrix} = \left(\begin{pmatrix} e \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{\delta\Phi}{\delta E} \end{pmatrix} \right)_{\mathbb{T}} = (e, d_A^*\frac{\delta\Phi}{\delta E})_1, \quad \forall e \in T_A\mathcal{A},$$

where $A = \pi_*(E,B)$.

Similarly the partial derivative $\frac{\delta\Phi}{\delta B} \in T_A\mathcal{A}$ is defined by

$$(\partial\Phi)_{(E,B)}\begin{pmatrix} 0 \\ \beta \end{pmatrix} = (\frac{\delta\Phi}{\delta B}, d_A^*\beta)_1, \quad \forall\beta \in T^*_A\mathcal{A}.$$

Where the left hand side is the Frechet directional derivative to the direction $\beta \in T^*\mathcal{A}$.

The exterior differentiation is given by

$$(\widetilde{d}\Phi)_{(E,B)}\begin{pmatrix} e \\ \beta \end{pmatrix} = \left(\begin{pmatrix} \frac{\delta\Phi}{\delta B} \\ \frac{\delta\Phi}{\delta E} \end{pmatrix}, \begin{pmatrix} e \\ \beta \end{pmatrix} \right)_{\mathbb{T}} \tag{4.3}$$

Example

Let $H = H(E,B)$; Hamiltonian function on $\mathbb{T}$ be given by

$$H(E,B) = \frac{1}{2}\left(\begin{pmatrix} E \\ B \end{pmatrix}, \begin{pmatrix} d_A^*B \\ d_AE \end{pmatrix} \right)_{\mathbb{T}} = \frac{1}{2}(d_AE, d_AE)_1 + \frac{1}{2}(d_A^*B, d_A^*B)_1.$$

Then

$$\frac{\delta H}{\delta B} = d_A^*B, \quad \frac{\delta H}{\delta E} = d_AE. \tag{4.4}$$

□

The symplectic structure on $\mathbb{T}$ and $\mathcal{F}$

Definition 4.1. The 2-form Ω on $\mathbb{T}$ is defined by the following formula:

$$\Omega_{(E,B)}\left(\begin{pmatrix} e_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ \beta_2 \end{pmatrix}\right) = (\, e_2 \,,\, d_A^*\beta_1 \,)_1 \; - \; (\, e_1 \,,\, d_A^*\beta_2 \,)_1 \tag{4.5}$$

for any $\begin{pmatrix} e_i \\ \beta_i \end{pmatrix} \in T_{(E,B)}\mathbb{T}\,,\; i = 1, 2.$

Ω is a non-degenerate skew-symmetric and
$\widetilde{d}\,\Omega = 0$.

Theorem 4.1.
$(\, \mathbb{T} = T\mathcal{A} \times_{\mathcal{A}} T^*\mathcal{A} \,,\, \Omega \,)$ *is a symplectic manifold.*

Proposition 4.2.
Let $\Phi = \Phi(E, B)$ be a function on the fields $\mathbb{T}$. Then the Hamiltonian vector field X_Φ of Φ is given by

$$X_\Phi(E, B) \; = \; \begin{pmatrix} -\frac{\delta\Phi}{\delta B} \\ \frac{\delta\Phi}{\delta E} \end{pmatrix} \; . \tag{4.6}$$

The formulae (4.3) and (4.5) imply (4.6) yield the proposition.

Definition 4.2.
The Poisson bracket on $\mathbb{T}$ is defined by the formula

$$\{\, \Phi \,,\, \Psi \,\}^{\mathbb{T}}_{(E,B)} \;=\; \Omega_{(E,B)}\,(\, X_\Phi \,,\, X_\Psi \,)\;, \tag{4.7}$$

for Φ, $\Psi \in C^\infty(\mathbb{T})$.

The following formula is our counterpart to the Marsden-Weinsrein's vortex formula for the Poisson baracket of Maxwell's fields .

Proposition 4.3.

$$\begin{aligned} \{\, \Phi \,,\, \Psi \,\}^{\mathbb{T}}_{(E,B)} \;&=\; \left(\frac{\delta\Phi}{\delta B} \,,\, d^*_A \frac{\delta\Psi}{\delta E} \right)_1 \;-\; \left(\frac{\delta\Psi}{\delta B} \,,\, d^*_A \frac{\delta\Phi}{\delta E} \right)_1 \\ &=\; \left(d_A \frac{\delta\Phi}{\delta B} \,,\, \frac{\delta\Psi}{\delta E} \right)_2 \;-\; \left(d_A \frac{\delta\Psi}{\delta B} \,,\, \frac{\delta\Phi}{\delta E} \right)_2 \end{aligned} \tag{4.8}$$

Proposition follows from (4.6).

The equation of motion for the Hamiltonian H is written in the form

$$\dot{\Phi} \;=\; \{\, \Phi \,,\, H \}^{\mathbb{T}}_{(E,B)}\;. \tag{4.9}$$

Let $H = H(E,B)$ be the Hamiltonian function of (4.1):

$$H(E,B) \;= \frac{1}{2}\left(\left(\begin{array}{c} E \\ B \end{array}\right), \left(\begin{array}{c} d_A^* B \\ d_A E \end{array}\right)\right)_{\mathbb{T}} = \frac{1}{2}(d_A E, d_A E)_1 + \frac{1}{2}(d_A^* B, d_A^* B)_1.$$

Then the Hamiltonian equation of motion of H is

$$\dot{E} \;=\; -d_A^* B\,, \qquad \dot{B} \;=\; d_A E\,. \tag{4.10}$$

This is the Maxwell electro-Magnetic aspect of our Yang-Mills field.

These equations correpond to the Faraday's quation (2.2) and Ampère's equation (2.4) respectively.

The equations for the nonexsistence of magnetic monopole and for the Gauss's law are deduced from the symplectic reduction by the group of gauge transformations $\mathcal{G}$.
These are already observed:

$$J^{-1}(0)/\mathcal{G} \simeq R^0 = \{(A, E) \in R = T\mathcal{A}; \quad E \in T_A\mathcal{A} \quad d^*E = 0\}$$

$$K^{-1}(0)/\mathcal{G} \simeq S^0 = \{(A, B) \in S = T^*\mathcal{A}; \quad B \in T_A^*\mathcal{A} \quad dB = 0\}$$

Writing these in $(\mathbb{T}, \Omega)$, we have the definiiton of the following Yang-Mills fields.

4.2 Yang-Mills fields

Definition 4.3.

The Yang-Mills field is the subspace of $\mathbb{T}$ defined by

$$\mathcal{F} \;=\; \{(E, B) \in \mathbb{T} \;: \quad d_A B = 0,\; d^*_A E = 0 \quad \text{with } \pi_*(E, B) = A\,\} \tag{4.11}$$

The Yang-Mills field $\mathcal{F}$ is a symplectic subspace of $(\mathbb{T}, \Omega)$ and $\mathcal{G}$-invariant because of

$$d_{g \cdot A}(g \cdot B) \;=\; g \cdot (d_A B)\,, \quad d_{g \cdot A}(g \cdot E) \;=\; g \cdot (d_A E)\,.$$

$$\begin{array}{ccc} \mathcal{F} = R^0 \times_{\mathcal{A}/\mathcal{G}} S^0 & \longrightarrow & S^0 \\ \pi_* \downarrow & & \pi \downarrow \\ R^0 & \longrightarrow & \mathcal{A}/\mathcal{G}\,. \end{array} \tag{4.12}$$

On Yang-Mills field $\mathcal{F}$ hold the counterpart of Maxwell's equations:

$$d^*_A B \,+\, \dot{E} \;=\; 0 \quad , \qquad d^*_A E \;=0\,, \tag{4.13}$$

$$d_A E \,-\, \dot{B} \;=\; 0 \quad , \qquad d_A B \;=\; 0\,. \tag{4.14}$$

The group of gauge transformations $\mathcal{G}$ acts on $\mathcal{F}$ by

$$
\begin{aligned}
g \cdot (A, E, B) &= (g \cdot A,\, Ad_g\, E\,,\, Ad^*_{g^{-1}} B\,) &(4.15)\\
&= (g^{-1} A g + g^{-1} dg,\, g^{-1} E g\,,\, g^{-1} B g\,)\,. &(4.16)
\end{aligned}
$$

It is a symplectic action because of

$$
\big(\, g \cdot e\,,\, d^*_{g\cdot A}(g \cdot \beta)\,\big)_1 \;=\; (\, g \cdot e\,,\, g \cdot (d^*_A \beta\,)_1 \;=\; (e\,,\, d^*_A \beta\,)\,,
$$

for any $(e, \beta) \in T_{(E,B)}\mathbb{T}$.

The Lie algebra of infinitesimal gauge transformations $Lie\,\mathcal{G} = \Omega^0(M, Lie\,G)$ acts on $\mathbb{T}$ by

$$
\xi \cdot \begin{pmatrix} A \\ E \\ B \end{pmatrix} = \begin{pmatrix} d_A \xi \\ [\,E, \xi\,] \\ [\,B, \xi\,] \end{pmatrix}, \tag{4.17}
$$

that is, the fundamental vector field on $\mathbb{T}$ corresponding to $\xi \in Lie\,\mathcal{G}$ becomes

$$
\xi_{\mathbb{T}}\,(E, B) \;=\; \begin{pmatrix} d_A \xi \\ [\,E, \xi\,] \\ [\,B, \xi\,] \end{pmatrix}. \tag{4.18}
$$

Now we shall investigate the Hamiltonian action of $\mathcal{G}$ on the Yang-Mills field $\mathcal{F}$.

A map

$$\mathbb{J} \,:\, \mathcal{F} \longrightarrow (Lie\,\mathcal{G})^* = \Omega^3(M,\, Lie\,G) \tag{4.19}$$

is by definition a moment map for the symplectic action of $\mathcal{G}$ on $\mathcal{F}$ provided

1. If we put $\mathbb{J}^\xi(E,B) = \langle \mathbb{J}(E,B)\,,\xi\rangle$, the Hamiltonian vector fields of $\mathbb{J}^\xi$ coincides with the fundamental vector field $\xi_{\mathbb{T}}$, (4.2).

2. $\mathbb{J}$ is Ad^*-equivariant:

$$\mathbb{J}^\xi(g^{-1}Eg\,,\, g^{-1}Bg\,) \;=\; \mathbb{J}^{Ad_{g^{-1}}\xi}(E,B)\,.$$

In this case we say that the action of $\mathcal{G}$ is Hamiltonian.

Proposition 4.4. *The action of $\mathcal{G}$ on $\mathcal{F}$ is Hamiltonian with the moment map*

$$\mathbb{J}(\,E,\,B\,) \;=\; [\,d_A E\,,\, *B\,]\,. \tag{4.20}$$

Proof

We have

$$\mathbb{J}^\xi(E,B) \;=\; (\,d_A E\,,\,[\,B,\,\xi\,]\,)_2 \;=\; -\,(\,d_A^* B\,,\,[\,E,\,\xi\,]\,)_1, \quad \forall \xi \in Lie\,\mathcal{G}\,, \tag{4.21}$$

and

$$(\tilde{d}\,\mathbb{J}^\xi)_{(E,B)} \begin{pmatrix} e \\ \beta \end{pmatrix} \;=\; (e,\, d_A^*[B,\xi]\,)_1 \,-\, (d_A^*\beta,\,[E,\xi]\,)_1$$

$$=\; \left(\begin{pmatrix} -[E,\xi] \\ [B,\xi] \end{pmatrix} ,\, \begin{pmatrix} e \\ \beta \end{pmatrix} \right)_{\mathbb{T}} .$$

By (4.6) the Hamiltonian vector field of $\mathbb{J}^\xi$ becomes

$$X_{\mathbb{J}^\xi} = \begin{pmatrix} [E,\xi] \\ \\ [B,\xi] \end{pmatrix} = \xi_{\mathbb{T}}(E,B)\,.$$

The equivariance of $\mathbb{J}$ is easy to verify. $\square$

Corollary 4.5. *On the $\mathcal{G}$-orbit passing through a solution of equation (4.13) we have* $\mathbb{J}(E,B) = [\dot{B}, *B]$.

4.3 Symplectic variable $\gamma : R \longrightarrow \mathcal{F}$

Since any $A \in \mathcal{A}$ is an irreducible connection we have the Green operator G_A defined on $\Omega^k((M, Lie\, G)$, $k = 1,2$,

$$(\,d_A d_A^* + d_A^* d_A\,)\, G_A \alpha \;=\; \alpha\,, \quad \forall \alpha \in \Omega^k((M, Lie\, G).$$

G_A is a self adjoint operator; $(G_A u, v)_k = (u, G_A v)_k$ for any u, $v \in \Omega^k(M, Lie\, G)$, $k = 1,2$. We note also the fact that G_A commutes with d_A and d_A^* :

$$d_A G_A = G_A d_A\,, \qquad d_A^* G_A = G_A d_A^*\,.$$

Restricted to the space $\mathcal{F}$ we have

$$d_A d_A^* G_A \beta = \beta, \qquad d_A^* d_A G_A e = e, \tag{4.22}$$

for $e \in T_A\mathcal{A}$ and $\beta \in T_A^*\mathcal{A}$.

Definition 4.4.

1. $\phi :\, R \longrightarrow \mathcal{F} \subset \mathbb{T}$ is the map defined by

$$\phi\,(A\,,\, p) \;=\; (E = -p\,,\, B = F_A\,)\,. \tag{4.23}$$

2. Let $\phi_* : TR \longrightarrow T\mathbb{T}$ be the tangent map of ϕ:

$$(\phi_*)_{(A,p)} \begin{pmatrix} a \\ x \end{pmatrix} = \begin{pmatrix} -x \\ d_A a \end{pmatrix},$$

and let $G_A : T_A\mathcal{A} \longrightarrow \mathcal{T}_A\mathcal{A}$ be the Green operator.
We define the modified tangent map $\gamma : TR \longrightarrow T\mathbb{T}$ of ϕ as follows

$$\gamma = \phi_* \circ \begin{pmatrix} 1 & 0 \\ 0 & G_A \end{pmatrix} = \begin{pmatrix} 0 & -G_A \\ d_A & 0 \end{pmatrix}, \tag{4.24}$$

that is,

$$T_{(A,p)}R \ni \begin{pmatrix} a \\ x \end{pmatrix} \longrightarrow \gamma_{(A,p)} \begin{pmatrix} a \\ x \end{pmatrix} = \begin{pmatrix} -G_A x \\ d_A a \end{pmatrix} \in T_{\phi(A,p)}\mathbb{T}\,.$$

Lemma 4.6.

$$\gamma^*\Omega \;=\; \sigma\,. \tag{4.25}$$

In fact, we have, for any $\begin{pmatrix} a_i \\ x_i \end{pmatrix} \in T_{(A,p)}R$, $i = 1, 2$,

$$(\gamma^*\Omega)_{(A,p)} \left(\begin{pmatrix} a_1 \\ x_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ x_2 \end{pmatrix} \right) \;=\; \Omega_{(E,B)} \left(\gamma \begin{pmatrix} a_1 \\ x_1 \end{pmatrix}, \gamma \begin{pmatrix} a_2 \\ x_2 \end{pmatrix} \right) =$$

$$\Omega_{(E,B)} \left(\begin{pmatrix} -G_A x_1 \\ d_A a_1 \end{pmatrix}, \begin{pmatrix} -G_A x_2 \\ d_A a_2 \end{pmatrix} \right) \;=\; (-G_A x_2\,,\, d_A^* d_A a_1\,)_1 \;-\; (-G_A x_1\,,\, d$$

$$= (x_1\,,\, G_A d_A^* d_A a_2\,)_1 \;-\; (x_2\,,\, G_A d_A^* d_A a_1\,)_1 \;=\; (x_1, a_2)_1 \;-\; (x_2, a_1)$$

$$= \sigma_{(A,p)} \left(\begin{pmatrix} a_1 \\ x_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ x_2 \end{pmatrix} \right).$$

□

Let $R^0 = \cup_{A\in\mathcal{A}} H^0_A$ be the reduction of R, (3.12). Remenber that the symplectic reduction of R by the moment map J is isomorphic to R^0, Proposition 3.6 .

Theorem 4.7. *(R^0, σ) is symplectomorph to $(\mathcal{F}, \Omega)$.*

Proof

Since $d_A^*(-p) = 0$ and $d_A F_A = 0$ for $(A,p) \in R^0$, ϕ maps the subspace R^0 into $\mathcal{F}$. The tangent space of H^0_A consist of those vectors $\begin{pmatrix} a \\ x \end{pmatrix} \in T_{(A,p)}R$ such that $d_A^* x = 0$, and the tangent space $T\mathcal{F}$ consists of those vectors $\begin{pmatrix} e \\ \beta \end{pmatrix} \in T\mathbb{T}$ such that $d_A^* e = 0$ and $d_A\beta = 0$. If $\begin{pmatrix} a \\ x \end{pmatrix}$ is tangent to H^0_A then $d_A^* G_A x = 0$ and $d_A(d_A a) = 0$, (the latter follows from the derivation of $d_A F_A = 0$). So γ maps TR^0 into $T\mathcal{F}$. Moreover γ is a bijective map of TR^0 onto $T\mathcal{F}$. In fact we have the inverse map given by

$$T\mathcal{F} \ni \begin{pmatrix} e \\ \beta \end{pmatrix} \longrightarrow (-d_A^* \circ \gamma) \begin{pmatrix} e \\ \beta \end{pmatrix} = \begin{pmatrix} d_A^* G_A \beta \\ -d_A^* d_A e \end{pmatrix} \in TR^0.$$

By virtue of the implicit function theorem in Banach space the vector spaces R and $\mathcal{F}$ are diffeomorphic. Let $\tilde{\gamma}: R^0 \longrightarrow \mathcal{F}$ be the diffeomorphism. Lemma 4.6 implies that $\tilde{\gamma}$ is a symplectomorphism.

□