Quaternifications and extensions of current algebras on S^3

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Abstract

Let $\mathbf{H}$ be the quaternion algebra. Let $\mathfrak{g}$ be a complex Lie algebra and let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. The quaternification $\mathfrak{g}^{\mathbf{H}} = (\,\mathbf{H}\otimes U(\mathfrak{g}),\,[\quad,\quad]_{\mathfrak{g}^{\mathbf{H}}}\,)$ of $\mathfrak{g}$ is defined by the bracket

$$\left[\,\mathbf{z}\otimes X\,,\,\mathbf{w}\otimes Y\,\right]_{\mathfrak{g}^{\mathbf{H}}} = (\mathbf{w}\cdot\mathbf{z})\otimes(XY) - (\mathbf{z}\cdot\mathbf{w})\otimes(YX)\,,$$

for $\mathbf{z},\,\mathbf{w}\in\mathbf{H}$ and $X,\,Y\in U(\mathfrak{g})$. Let $S^3\mathbf{H}$ be the (non-commutative) algebra of $\mathbf{H}$-valued smooth mappings over S^3 and let $S^3\mathfrak{g}^{\mathbf{H}} = S^3\mathbf{H}\otimes U(\mathfrak{g})$. The Lie algebra structure on $S^3\mathfrak{g}^{\mathbf{H}}$ is induced naturally from that of $\mathfrak{g}^{\mathbf{H}}$. As a subalgebra of $S^3\mathbf{H}$ we have the algebra of Laurent polynomial spinors $\mathbf{C}[\phi^{\pm(m,l,k)}]$ spanned by a complete orthogonal system of eigen spinors $\{\phi^{\pm(m,l,k)}\}_{m,l,k}$ of the tangential Dirac operator on S^3. Then $\mathbf{C}[\phi^{\pm(m,l,k)}]\otimes U(\mathfrak{g})$ is a Lie subalgebra of $S^3\mathfrak{g}^{\mathbf{H}}$. We introduce a 2-cocycle on $\mathbf{C}[\phi^{\pm(m,l,k)}]\otimes U(\mathfrak{g})$ by the aid of the radial vector field $\frac{\partial}{\partial n}$ on $S^3\subset\mathbf{C}^2$. Then we have the corresponding central extension $\widehat{\mathfrak{g}}(a) = (\,\mathbf{C}[\phi^{\pm(m,l,k)}]\otimes U(\mathfrak{g})\,)\oplus(\mathbf{C}a)$. Finally we have a Lie algebra $\widehat{\mathfrak{g}}$ which is obtained by adding to $\widehat{\mathfrak{g}}(a)$ a derivation d which acts on $\widehat{\mathfrak{g}}(a)$ as the radial derivation $\frac{\partial}{\partial n}$. That is the $\mathbf{C}$-vector space $\widehat{\mathfrak{g}} = \left(\mathbf{C}[\phi^{\pm(m,l,k)}]\otimes U(\mathfrak{g})\right)\oplus(\mathbf{C}a)\oplus(\mathbf{C}d)$ endowed with the bracket

$$\begin{aligned}&\left[\,\phi_1\otimes X_1+\lambda_1 a+\mu_1 d\,,\,\phi_2\otimes X_2+\lambda_2 a+\mu_2 d\,\right]_{\widehat{\mathfrak{g}}} = (\phi_1\phi_2)\otimes(X_1\,X_2) - (\phi_2\phi_1)\otimes(X_2X_1)\\ &+\mu_1\frac{\partial}{\partial n}\phi_2\otimes X_2 - \mu_2\frac{\partial}{\partial n}\phi_1\otimes X_1 + (X_1|X_2)c(\phi_1,\phi_2)a\,.\end{aligned}$$

When $\mathfrak{g}$ is a simple Lie algebra with its Cartan subalgebra $\mathfrak{h}$ we shall investigate the weight space decomposition of $\widehat{\mathfrak{g}}$ with respect to the subalgebra $\widehat{\mathfrak{h}} = (\phi^{+(0,0,1)} \otimes \mathfrak{h}) \oplus (\mathbf{C}a) \oplus (\mathbf{C}d)$.

The previous versions (v1-v6) of this arXiv text contained many incorrect assertions and here we have corrected them.

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0 Introduction

The set of smooth mappings from a manifold to a Lie algebra has been a subject of investigation both from a purely mathematical standpoint and from quantum field theory. In quantum field theory they appear as a current algebra or an infinitesimal gauge transformation group. Loop algebras are the simplest example. Loop algebras and their representation theory have been fully worked out. A loop algebra valued in a simple Lie algebra or its complexification turned out to behave like a simple Lie algebra and the highly developed theory of finite dimensional Lie algebra was extended to such loop algebras. Loop algebras appear in the simplified model of quantum field theory where the space is one-dimensional and many important facts in the representation theory of loop algebra were first discovered by physicists. We aim the three-dimensional generalization of this theory. It turned out that in many applications to field theory one must deal with certain extensions of the associated loop algebra rather than the loop algebra itself. The central extension of a loop algebra is called an affine Lie algebra and the highest weight theory of finite dimensional Lie algebra was extended to this case. [K], [K-W], [P-S] and [W] are good references to study these subjects. In this paper we shall investigate a generalization of affine Lie algebras to the Lie algebra of mappings from three-sphere S^3 to a Lie algebra. As an affine Lie algebra is a central extension of the Lie algebra of smooth mappings from S^1 to the complexification of a Lie algebra, so our objective is an extension of the Lie algebra of smooth mappings from S^3 to the quaternification of a Lie algebra. As for the higher dimensional generalization of loop groups, J. Mickelsson introduced an abelian exension of current groups $Map(S^3, SU(N))$ for $N \geq 3$, [M]. It is related to the Chern-Simons function on the space of $SU(N)$-connections and the associated current algebra $Map(S^3, su(N))$ has a abelian extension $Map(S^3, su(N)) \oplus \mathcal{A}_3^*$ by the affine dual of the space $\mathcal{A}_3^*$ of connections over S^3 [Ko4]. There does not exist non-trivial central extension of a complex Lie algebra and we are led to consider the quaternification of Lie algebras. Now we shall give a brief explanation of each section.

Let $\mathbf{H}$ be the quaternion numbers. In this paper we shall denote a quaternion $a+jb \in \mathbf{H}$ by $\begin{pmatrix} a \\ b \end{pmatrix}$. This comes from the identification of $\mathbf{H}$ with the matrix algebra

$$\mathfrak{mj}(2,\mathbf{C}) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} : \quad a, b \in \mathbf{C} \right\}.$$

$\mathbf{H}$ becomes an associative algebra and the Lie algebra structure $(\mathbf{H}, [\,,\,]_{\mathbf{H}})$ is induced on it. The trace of $\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbf{H}$ is defined by $tr\,\mathbf{a} = a + \overline{a}$. For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}$ we have $tr\,([\mathbf{u}, \mathbf{v}]_{\mathbf{H}} \cdot \mathbf{w}) = tr\,(\mathbf{u} \cdot [\mathbf{v}, \mathbf{w}]_{\mathbf{H}})$.

Let $(\mathfrak{g}, [\quad,\quad]_{\mathfrak{g}})$ be a complex Lie algebra. Let $U(\mathfrak{g})$ be the enveloping algebra. The quaternification of $\mathfrak{g}$ is defined as the vector space $\mathfrak{g}^{\mathbf{H}} = \mathbf{H} \otimes U(\mathfrak{g})$ endowed with the bracket

$$\left[\mathbf{z} \otimes X\,,\, \mathbf{w} \otimes Y\right]_{\mathfrak{g}^{\mathbf{H}}} = (\mathbf{z} \cdot \mathbf{w}) \otimes (XY) - (\mathbf{w} \cdot \mathbf{z}) \otimes (YX)\,, \tag{0.1}$$

for $\mathbf{z}, \mathbf{w} \in \mathbf{H}$ and $X, Y \in U(\mathfrak{g})$. It extends the Lie algebra structure $(\mathfrak{g}, [\quad,\quad]_{\mathfrak{g}})$ to $\left(\mathfrak{g}^{\mathbf{H}}, [\quad,\quad]_{\mathfrak{g}^{\mathbf{H}}}\right)$. The quaternions $\mathbf{H}$ give also a half spinor representation of $Spin(4)$. That is, $\Delta = \mathbf{H} \otimes \mathbf{C} = \mathbf{H} \oplus \mathbf{H}$ gives an irreducible complex representation of the Clifford algebra $\mathrm{Clif}(\mathbf{R}^4)$: $\mathrm{Clif}(\mathbf{R}^4) \otimes \mathbf{C} \simeq \mathrm{End}(\Delta)$, and Δ decomposes into irreducible representations $\Delta^{\pm} = \mathbf{H}$ of $\mathrm{Spin}(4)$. Let $S^{\pm} = \mathbf{C}^2 \times \Delta^{\pm}$ be the trivial even (respectively odd) spinor bundle. A section of spinor bundle is called a spinor. The space of even half spinors $C^{\infty}(S^3, S^+)$ is identified with the space $S^3\mathbf{H} = Map(S^3, \mathbf{H})$. Now the space $S^3\mathfrak{g}^{\mathbf{H}} = S^3\mathbf{H} \otimes U(\mathfrak{g})$ becomes a Lie algebra with respect to the bracket:

$$[\,\phi \otimes X\,,\, \psi \otimes Y\,]_{S^3\mathfrak{g}^{\mathbf{H}}} = (\phi\psi) \otimes (XY) - (\psi\phi) \otimes (YX), \tag{0.2}$$

for $X, Y \in U(\mathfrak{g})$ and $\phi, \psi \in S^3\mathbf{H}$. In the sequel we shall abbreviate the Lie bracket $[\,,\,]_{\mathbf{H}}$ simply to $[\,,\,]$. Such an abbreviation will be often adopted for other Lie algebras.

In section 2 we shall review the theory of spinor analysis after [Ko2, Ko3]. Let $D: S^+ \longrightarrow S^-$ be the (half spinor) Dirac operator. Let $D = \gamma_+(\frac{\partial}{\partial n} - \partial\!\!\!/)$ be the polar decomposition on $S^3 \subset \mathbf{C}^2$ of the Dirac operator, where $\partial\!\!\!/$ is the tangential Dirac operator on S^3 and γ_+ is the Clifford multiplication of the unit normal derivative on S^3. The eigenvalues of $\partial\!\!\!/$ are given by $\{\frac{m}{2},\ -\frac{m+3}{2}\,;\, m=0,1,\cdots\}$, with multiplicity $(m+1)(m+2)$. We have an explicitly written formula for eigenspinors $\left\{\phi^{+(m,l,k)},\ \phi^{-(m,l,k)}\right\}_{0\le l\le m,\,0\le k\le m+1}$ corresponding to the eigenvalue $\frac{m}{2}$ and $-\frac{m+3}{2}$ respectively and they give rise to a complete orthogonal system in $L^2(S^3,S^+)$. A spinor ϕ on a domain $G\subset\mathbf{C}^2$ is called *harmonic spinor* on G if $D\phi=0$. Each $\phi^{+(m,l,k)}$ is extended to a harmonic spinor on $\mathbf{C}^2$, while each $\phi^{-(m,l,k)}$ is extended to a harmonic spinor on $\mathbf{C}^2\setminus\{0\}$. Every harmonic spinor φ on $\mathbf{C}^2\setminus\{0\}$ has a Laurent series expansion by the basis $\phi^{\pm(m,l,k)}$:

$$\varphi(z)=\sum_{m,l,k}C_{+(m,l,k)}\phi^{+(m,l,k)}(z)+\sum_{m,l,k}C_{-(m,l,k)}\phi^{-(m,l,k)}(z). \tag{0.3}$$

If only finitely many coefficients are non-zero it is called a spinor of Laurent polynomial type. The algebra of spinors of Laurent polynomial type is denoted by $\mathbf{C}[\phi^{\pm(m,l,k)}]$. $\mathbf{C}[\phi^{\pm(m,l,k)}]$ is a subspace of $S^3\mathbf{H}$ that is algebraically generated by $\phi^{+(0,0,1)}=\begin{pmatrix}1\\0\end{pmatrix}$, $\phi^{+(0,0,0)}=\begin{pmatrix}0\\-1\end{pmatrix}$, $\phi^{+(1,0,1)}=\begin{pmatrix}z_2\\-\overline{z}_1\end{pmatrix}$ and $\phi^{-(0,0,0)}=\begin{pmatrix}z_2\\\overline{z}_1\end{pmatrix}$.

Recall that the central extension of a loop algebra $L\mathfrak{g}=\mathbf{C}[z,z^{-1}]\otimes\mathfrak{g}$ is the Lie algebra $(L\mathfrak{g}\oplus\mathbf{C}a\,,[\ ,\ ]_c)$ given by the bracket

$$[P\otimes X,Q\otimes Y]_c=PQ\otimes[X,Y]+(X|Y)c(P,Q)a\,,$$

with the aid of the 2-cocycle $c(P,Q)=res_{z=0}(\frac{d}{dz}P\cdot Q)=\frac{1}{2\pi}\int_{S^1}\frac{d}{dz}P\cdot Q\,dz$. Here $(\cdot|\cdot)$ is a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$. We shall give in section 3 an analogous 2-cocycle on $\mathbf{C}[\phi^{\pm(m,l,k)}]$. For $\phi_1,\ \phi_2\in\mathbf{C}[\phi^{\pm(m,l,k)}]$, we put

$$c(\phi_1,\phi_2)\ =\ \frac{1}{2\pi^2}\int_{S^3}tr(\,\frac{\partial}{\partial\mathbf{n}}\phi_1\ \cdot\phi_2\,)d\sigma. \tag{0.4}$$

Then c defines a 2-cocycle on the algebra $\mathbf{C}[\phi^{\pm(m,l,k)}]$. That is, c satisfies the following equations:

$$c(\phi,\,\psi)=\,-c(\psi,\,\phi)$$

and

$$c(\,\phi_1\cdot\,\phi_2\,,\,\phi_3)+c(\,\phi_2\cdot\,\phi_3\,,\,\phi_1)+c(\,\phi_3\cdot\,\phi_1\,\,,\,\phi_2)=0\,.$$

Let a be an indefinite number. The $\mathbf{C}$-vector space $\widehat{\mathfrak{g}}(a)=\mathbf{C}[\phi^{\pm(m,l,k)}]\otimes U(\mathfrak{g})\oplus\mathbf{C}a$ is endowed with a Lie algebra structure by the following bracket: For $X,Y\in U(\mathfrak{g})$ and

$\phi,\ \psi\ \in \mathbf{C}[\phi^{\pm(m,l,k)}]$ we put

$$\begin{aligned}
[\,\phi\otimes X\,,\,\psi\otimes Y\,]^{\wedge} &= (\phi\cdot\psi)\otimes(X\,Y) - (\psi\cdot\phi)\otimes(Y\,X) + c(\phi,\psi)(X|Y)\cdot a\,,\\
[a\,,\phi\otimes X\,]^{\wedge} &= 0\,.
\end{aligned}$$

$(\,\widehat{\mathfrak{g}}(a)\,,\,[\ ,\ ]^{\wedge})$ becomes an extension of the Lie algebra $\widehat{\mathfrak{g}}(a)$ with 1-dimensional center $\mathbf{C}a$. In section 4 we shall construct the Lie algebra which is obtained by adding to $\widehat{\mathfrak{g}}(a)$ a derivation d which acts on $\widehat{\mathfrak{g}}(a)$ as radial derivation $d_0 = |z|\frac{\partial}{\partial n}$ on S^3. The radial derivation is defined by $d_0 = |z|\frac{\partial}{\partial n} = \frac{1}{2}(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + \overline{z}_1\frac{\partial}{\partial \overline{z}_1} + \overline{z}_2\frac{\partial}{\partial \overline{z}_2})$. We have the following fundamental property of the cocycle c.

$$c(\,d_0\phi_1\,,\phi_2\,) + c(\,\phi_1\,,d_0\phi_2\,) = 0.$$

Let $\widehat{\mathfrak{g}} = (\,\mathbf{C}[\phi^{\pm(m,l,k)}]\otimes U(\mathfrak{g})\,)\oplus(\mathbf{C}a)\oplus(\mathbf{C}d)$. We endow $\widehat{\mathfrak{g}}$ with the bracket defined by

$$\begin{aligned}
[\,\phi\otimes X\,,\,\psi\otimes Y\,]_{\widehat{\mathfrak{g}}} &= [\,\phi\otimes X\,,\,\psi\otimes Y\,]^{\wedge}\,, \qquad [\,a\,,\phi\otimes X\,]_{\widehat{\mathfrak{g}}} = 0\,,\\
[\,d,\,a\,]_{\widehat{\mathfrak{g}}} &= 0\,, \qquad [\,d,\phi\otimes X\,]_{\widehat{\mathfrak{g}}} = d_0\phi\otimes X\,.
\end{aligned}$$

Then $(\,\widehat{\mathfrak{g}}\,,\,[\ ,\ ]_{\widehat{\mathfrak{g}}}\,)$ is an extension of the Lie algebra $\widehat{\mathfrak{g}}(a)$ on which d acts as d_0.

In section 5, when $\mathfrak{g}$ is a simple Lie algebra with its Cartan subalgebra $\mathfrak{h}$, we shall investigate the weight space decomposition of $\widehat{\mathfrak{g}}$ with respect to the subalgebra $\widehat{\mathfrak{h}} = (\phi^{+(0,0,1)}\otimes\mathfrak{h})\oplus(\mathbf{C}a)\oplus(\mathbf{C}d)$, the latter is a commutative subalgebra such that $ad(\widehat{\mathfrak{h}})$ is diagoniable. For this purpose we look at the representation of the adjoint action of $\mathfrak{h}$ on the enveloping algebra $U(\mathfrak{g})$. Let $\mathfrak{g} = \sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}$. Let $\Pi = \{\alpha_i;\, i = 1,\cdots,r = \mathrm{rank}\,\mathfrak{g}\}\subset\mathfrak{h}^*$ be the set of simple roots and $\{\alpha_i^\vee\,;\, i = 1,\cdots,r\,\}\subset\mathfrak{h}$ be the set of simple coroots. The Cartan matrix $A = (\,a_{ij}\,)_{i,j=1,\cdots,r}$ is given by $a_{ij} = \langle\alpha_i^\vee,\,\alpha_j\rangle$. Fix a standard set of generators $H_i = \alpha_i^\vee$, $X_i = X_{\alpha_i}\in\mathfrak{g}_{\alpha_i}$, $Y_i = X_{-\alpha_i}\in\mathfrak{g}_{-\alpha_i}$, so that $[X_i,\,Y_j] = H_j\delta_{ij}$, $[H_i,\,X_j] = -a_{ji}X_j$ and $[H_i,\,Y_j] = a_{ji}Y_j$. We see that the set of weights of the representation $(\,U(\mathfrak{g}),\,ad(\mathfrak{h}))$ becomes

$$\Sigma = \{\,\sum_{i=1}^r k_i\alpha_i\in\mathfrak{h}^*\,;\quad k_i\in\mathbf{Z},\ i = 1,\cdots,r\,\} \tag{0.5}$$

The weight space of $\lambda\in\Sigma$ is by definition

$$\mathfrak{g}^U_\lambda\ =\ \{\xi\in U(\mathfrak{g})\,;\, ad(h)\xi = \lambda(h)\xi,\ \forall h\in\mathfrak{h}\}, \tag{0.6}$$

when $\mathfrak{g}_\lambda^U \neq 0$. Then, given $\lambda = \sum_{i=1}^r k_i\alpha_i$, we have

$$\mathfrak{g}_\lambda^U = \mathbf{C}[Y_1^{q_1}\cdots Y_r^{q_r}\,H_1^{l_1}\cdots H_r^{l_r}\,X_1^{p_1}\cdots X_r^{p_r}\,;\,p_i,\,q_i,\,l_i \in \mathbf{N}\cup 0,\;k_i = p_i - q_i\,,\,i = 1,\cdots,r\,]\,. \tag{0.7}$$

The weight space decomposition becomes

$$U(\mathfrak{g}) = \Sigma_{\lambda\in\Sigma}\,\mathfrak{g}_\lambda^U, \quad \mathfrak{g}_0^U \supset U(\mathfrak{h}). \tag{0.8}$$

Now we proceed to the representation $(\,\widehat{\mathfrak{g}},\,ad(\widehat{\mathfrak{h}})\,)$. The dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ can be regarded naturally as a subspace of $\widehat{\mathfrak{h}}^*$. So $\Sigma \subset \mathfrak{h}^*$ is seen to be a subset of $\widehat{\mathfrak{h}}^*$. we define $\delta, \Lambda_0 \in \widehat{\mathfrak{h}}^*$ by putting $\langle \Lambda_0, h_i\,\rangle = \langle \delta, h_i\,\rangle = \langle \Lambda_0, d\rangle = \langle \delta, a\rangle = 0$, $1 \leqq i \leqq r$ and $\langle \Lambda_0, a\rangle = \langle \delta, d\rangle = 1$. Then the set of weights $\widehat{\Sigma}$ of $(\,\widehat{\mathfrak{g}},\,ad(\widehat{\mathfrak{h}})\,)$ is

$$\begin{aligned}\widehat{\Sigma} \;&=\; \left\{\frac{m}{2}\delta + \lambda; \quad \lambda \in \Sigma\,,\,m \in \mathbf{Z},\,m \neq -1,-2\right\}\\ &\qquad \bigcup \left\{\frac{m}{2}\delta; \quad m \in \mathbf{Z},\,m \neq -1,-2\right\}.\end{aligned} \tag{0.9}$$

The weight space decomposition of $\widehat{\mathfrak{g}}$ is given by

$$\widehat{\mathfrak{g}} = \bigoplus_{m\neq -1,-2} \widehat{\mathfrak{g}}_{\frac{m}{2}\delta} \bigoplus_{\lambda\in\Sigma,\,m\neq -1,-2} \widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\lambda} \bigoplus (\mathbf{C}a) \bigoplus (\mathbf{C}d)\,. \tag{0.10}$$

Each weightt space is given as follows:

$$\widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\lambda} = \mathbf{C}[\phi^{+(m,l,k)};\,0 \leq l \leq m,\,0 \leq k \leq m+1\,] \otimes \mathfrak{g}_\lambda^U\,, \tag{0.11}$$

$$\widehat{\mathfrak{g}}_{-\frac{m+3}{2}\delta+\lambda} = \mathbf{C}[\phi^{-(m,l,k)};\,0 \leq l \leq m,\,0 \leq k \leq m+1\,] \otimes \mathfrak{g}_\lambda^U\,, \tag{0.12}$$

$$\widehat{\mathfrak{g}}_{\frac{m}{2}\delta} = \mathbf{C}[\phi^{+(m,l,k)};\,0 \leq l \leq m,\,0 \leq k \leq m+1\,] \otimes \mathfrak{g}_0^U\,, \tag{0.13}$$

$$\widehat{\mathfrak{g}}_{-\frac{m+3}{2}\delta} = \mathbf{C}[\phi^{-(m,l,k)};\,0 \leq l \leq m,\,0 \leq k \leq m+1\,] \otimes \mathfrak{g}_0^U\,, \tag{0.14}$$

for $m \geq 0$.

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1 Quaternification of a Lie algebra

1.1 Quaternion algebra

The quaternions $\mathbf{H}$ are formed from the real numbers $\mathbf{R}$ by adjoining three symbols $i,\ j,\ k$ satisfying the identities:

$$\begin{aligned} i^2 &= j^2 = k^2 = -1\,, \\ ij &= -ji = k, \quad jk = -kj = i, \quad ki = -ik = j\,. \end{aligned} \tag{1.1}$$

A general quaternion is of the form $x = x_1 + x_2 i + x_3 j + x_4 k$ with $x_1, x_2, x_3, x_4 \in \mathbf{R}$. By taking $x_3 = x_4 = 0$ the complex numbers $\mathbf{C}$ are contained in $\mathbf{H}$ if we identify i as the usual complex number. Every quaternion x has a unique expression $x = z_1 + jz_2$ with $z_1, z_2 \in \mathbf{C}$. This identifies $\mathbf{H}$ with $\mathbf{C}^2$ as $\mathbf{C}$-vector spaces. The quaternion multiplication will be from the right $x \longrightarrow xy$ where $y = w_1 + jw_2$ with $w_1,\ w_2 \in \mathbf{C}$:

$$xy = (z_1 + jz_2\,)(w_1 + jw_2\,) = (z_1 w_1 - \overline{z}_2 w_2) + j(\overline{z}_1 w_2 + z_2 w_1). \tag{1.2}$$

The multiplication of a $g = a + jb \in \mathbf{H}$ to $\mathbf{H}$ from the left yields an endomorphism in $\mathbf{H}$: $\{x \longrightarrow gx\} \in End_{\mathbf{H}}(\mathbf{H})$. If we look on it under the identification $\mathbf{H} \simeq \mathbf{C}^2$ mentioned above we have the $\mathbf{C}$-linear map

$$\mathbf{C}^2 \ni \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longrightarrow \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbf{C}^2\,. \tag{1.3}$$

This establishes the $\mathbf{R}$- linear isomorphism

$$\mathbf{H} \ni a + jb \xrightarrow{\simeq} \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \in \mathfrak{mj}(2, \mathbf{C}), \tag{1.4}$$

where we defined

$$\mathfrak{mj}(2, \mathbf{C}) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \;:\quad a, b \in \mathbf{C} \right\}. \tag{1.5}$$

The complex matrices corresponding to i, j, $k \in \mathbf{H}$ are

$$e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

These are the basis of the Lie algebra $\mathfrak{su}(2)$. Thus we have the identification of the following objects

$$\mathbf{H} \simeq \mathfrak{mj}(2,\mathbf{C}) \simeq \mathbf{R} \oplus \mathfrak{su}(2). \tag{1.6}$$

The correspondence between the elements is given by

$$a + jb \equiv \begin{pmatrix} a \\ b \end{pmatrix} \longleftrightarrow \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \longleftrightarrow s + pe_1 + qe_2 + re_3\,, \tag{1.7}$$

where $a = s + ir$, $b = q + ip$.

$\mathbf{H}$ becomes an associative algebra with the multiplication law defined by

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} z_1 w_1 - \overline{z}_2 w_2 \\ \overline{z}_1 w_2 + z_2 w_1 \end{pmatrix}, \tag{1.8}$$

which is the rewritten formula of (1.2) and the right-hand side is the first row of the matrix multiplication

$$\begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix} \begin{pmatrix} w_1 & -\overline{w}_2 \\ w_2 & \overline{w}_1 \end{pmatrix}.$$

It implies the Lie bracket of two vectors in $\mathbf{H}$, that becomes

$$\left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] = \begin{pmatrix} z_2 \overline{w}_2 - \overline{z}_2 w_2 \\ (w_1 - \overline{w}_1) z_2 - (z_1 - \overline{z}_1) w_2 \end{pmatrix}. \tag{1.9}$$

These expressions are very convenient to develop the analysis on $\mathbf{H}$, and give an interpretation on the quaternion analysis by the language of spinor analysis.

Proposition 1.1. *Let* $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbf{H}$. *Then the trace of* $\mathbf{z} \cdot \mathbf{w} \in \mathbf{H} \simeq \mathfrak{mj}(2,\mathbf{C})$ *is given by*

$$tr\,(\mathbf{z} \cdot \mathbf{w}) = 2\mathrm{Re}(z_1 w_1 - \overline{z}_2 w_2), \tag{1.10}$$

and we have, for $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \in \mathbf{H}$,

$$tr\,(\,[\,\mathbf{z}_1,\,\mathbf{z}_2\,]\cdot\mathbf{z}_3\,)\;=\;tr\,(\,\mathbf{z}_1\cdot[\,\mathbf{z}_2\,,\,\mathbf{z}_3\,]\,)\,. \tag{1.11}$$

The center of the Lie algebra $\mathbf{H}$ is $\left\{\begin{pmatrix} t \\ 0 \end{pmatrix} \in \mathbf{H};\, t \in \mathbf{R}\right\} \simeq \mathbf{R}$, and (1.6) says that $\mathbf{H}$ is the trivial central extension of $\mathfrak{su}(2)$.

$\mathbf{R}^3$ being a vector subspace of $\mathbf{H}$:

$$\mathbf{R}^3 \ni \begin{pmatrix} p \\ q \\ r \end{pmatrix} \Longleftrightarrow \begin{pmatrix} ir \\ q+ip \end{pmatrix} = ir + j(q+ip) \in \mathbf{H}, \tag{1.12}$$

we have the action of $\mathbf{H}$ on $\mathbf{R}^3$.

1.2 Lie algebra structure on $\mathbf{H} \otimes U(\mathfrak{g})$

Let $(\,\mathfrak{g}\,,\,[\quad,\quad]_{\mathfrak{g}}\,)$ be a complex Lie algebra. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Let $\mathfrak{g}^{\mathbf{H}} = \mathbf{H} \otimes U(\mathfrak{g})$ and define the following bracket on $\mathfrak{g}^{\mathbf{H}}$:

$$\left[\,\cdot\,,\,\cdot\,\right]_{\mathfrak{g}^{\mathbf{H}}} : (\mathbf{H}\otimes\mathfrak{g})\;\times\;(\mathbf{H}\otimes\mathfrak{g})\;\longrightarrow\;\mathbf{H}\otimes U(\mathfrak{g})\,. \tag{1.13}$$

by

$$\left[\,\mathbf{z}\otimes X\,,\,\mathbf{w}\otimes Y\,\right]_{\mathfrak{g}^{\mathbf{H}}}\;=\;(\mathbf{z}\cdot\mathbf{w})\otimes\,[X,Y]_{\mathfrak{g}}\,+\,[\mathbf{z}\,,\,\mathbf{w}]\,\otimes(YX)\,, \tag{1.14}$$

for $\mathbf{z}$, $\mathbf{w} \in \mathbf{H}$ and X, $Y \in \mathfrak{g}$. $\left[\,\cdot\,,\,\cdot\,\right]_{\mathfrak{g}^{\mathbf{H}}}$ is extended naturally to $\mathbf{H}\otimes U(\mathfrak{g})$. Thus, for X, $Y \in U(\mathfrak{g})$ and $\mathbf{z}$, $\mathbf{w} \in \mathbf{H}$, we have

$$[\,\mathbf{z}\otimes X\,,\,\mathbf{w}\otimes Y\,]_{\mathfrak{g}^{\mathbf{H}}} = (\mathbf{z}\cdot\mathbf{w})\otimes\,(XY) - (\mathbf{w}\cdot\mathbf{z})\otimes(YX)\,. \tag{1.15}$$

By the quaternion number notation every element of $\mathbf{H}\otimes\mathfrak{g}$ may be written as $X+jY$ with $X, Y \in \mathfrak{g}$. Then the above definition is equivalent to

$$\begin{aligned}\left[\,X_1+jY_1\,,\,X_2+jY_2\,\right]_{\mathfrak{g}^{\mathbf{H}}} \;&=\; [X_1,X_2]_{\mathfrak{g}} - \left(\overline{Y}_1Y_2\,-\,\overline{Y}_2Y_1\,\right) \\ &\quad + j\left(\overline{X}_1Y_2 - Y_2X_1 + Y_1X_2 - \overline{X}_2Y_1\right)\,,\end{aligned} \tag{1.16}$$

where $\overline{X}$ is the complex conjugate of X.

Proposition 1.2. *The bracket* $[\cdot\,,\,\cdot]_{\mathfrak{g}^{\mathbf{H}}}$ *defines a Lie algebra structure on* $\mathbf{H}\otimes U(\mathfrak{g})$ *.*

In fact the bracket defined in (1.15) satisfies the antisymmetry equation and the Jacobi identity.

Definition 1.3. The Lie algebra $\left(\mathfrak{g}^{\mathbf{H}} = \mathbf{H}\otimes U(\mathfrak{g})\,,\,[\quad,\quad]_{\mathfrak{g}^{\mathbf{H}}}\right)$ is called the *quaternification of the Lie algebra* $\mathfrak{g}$ *.*

2 Analysis on H

In this section we shall review the analysis of the Dirac operator on $\mathbf{H}\simeq\mathbf{C}^2$. The general references are [B-D-S] and [G-M], and we follow the calculations developed in [Ko1], [Ko2] and [Ko3].

2.1 Harmonic polynomials

The Lie group SU(2) acts on $\mathbf{C}^2$ both from the right and from the left. Let dR(g) and dL(g) denote respectively the right and the left infinitesimal actions of the Lie algebra $\mathfrak{su}(2)$. We define the following vector fields on $\mathbf{C}^2$:

$$\theta_i = dR\left(\frac{1}{2}e_i\right), \quad \tau_i = dL\left(\frac{1}{2}e_i\right), \qquad i=1,2,3, \tag{2.1}$$

where $\{e_i;\, i=1,2,3\}$ is the normal basis of $\mathbf{R}^3$. Each of the triple $\theta_i(z)$, $i=1,2,3$, and $\tau_i(z)$, $i=1,2,3$, gives a basis of the vector fields on the three sphere $\{|z|=1\}\simeq S^3$.

It is more convenient to introduce the following vector fields:

$$e_+ = -z_2\frac{\partial}{\partial\bar{z}_1} + z_1\frac{\partial}{\partial\bar{z}_2} = \theta_1 - \sqrt{-1}\theta_2, \tag{2.2}$$

$$e_- = -\bar{z}_2\frac{\partial}{\partial z_1} + \bar{z}_1\frac{\partial}{\partial z_2} = \theta_1 + \sqrt{-1}\theta_2, \tag{2.3}$$

$$\theta = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - \bar{z}_1\frac{\partial}{\partial\bar{z}_1} - \bar{z}_2\frac{\partial}{\partial\bar{z}_2} = 2\sqrt{-1}\,\theta_3. \tag{2.4}$$

$$\hat{e}_+ = -\bar{z}_1\frac{\partial}{\partial\bar{z}_2} + z_2\frac{\partial}{\partial z_1} = \tau_1 - \sqrt{-1}\tau_2, \tag{2.5}$$

$$\hat{e}_- = \bar{z}_2\frac{\partial}{\partial\bar{z}_1} - z_1\frac{\partial}{\partial z_2} = \tau_1 + \sqrt{-1}\tau_2, \tag{2.6}$$

$$\hat{\theta} = z_2\frac{\partial}{\partial z_2} + \bar{z}_1\frac{\partial}{\partial\bar{z}_1} - \bar{z}_2\frac{\partial}{\partial\bar{z}_2} - z_1\frac{\partial}{\partial z_1} = 2\sqrt{-1}\,\tau_3. \tag{2.7}$$

We have the commutation relations;

$$[\theta, e_+] = 2e_+, \quad [\theta, e_-] = -2e_-, \quad [e_+, e_-] = -\theta. \tag{2.8}$$

$$[\hat{\theta}, \hat{e}_+] = 2\hat{e}_+, \quad [\hat{\theta}, \hat{e}_-] = -2\hat{e}_-, \quad [\hat{e}_+, \hat{e}_-] = -\hat{\theta}. \tag{2.9}$$

Both Lie algebras spanned by (e_+, e_-, θ) and $(\hat{e}_+, \hat{e}_-, \hat{\theta})$ are isomorphic to $\mathfrak{sl}(2, \mathbf{C})$.

In the following we denote a function $f(z, \bar{z})$ of variables $z, \bar{z}$ simply by $f(z)$. For $m = 0, 1, 2, \cdots$, and $l, k = 0, 1, \cdots, m$, we define the polynomials:

$$v^k_{(l,m-l)} = (e_-)^k z_1^l z_2^{m-l}. \tag{2.10}$$

$$w^k_{(l,m-l)} = (\hat{e}_-)^k z_2^l \bar{z}_1^{m-l}. \tag{2.11}$$

Then $v^k_{(l,m-l)}$ and $w^k_{(l,m-l)}$ are harmonic polynomials on $\mathbf{C}^2$;

$$\Delta v^k_{(l,m-l)} = \Delta w^k_{(l,m-l)} = 0\,,$$

where $\Delta = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2}$.

$\left\{ \frac{1}{\sqrt{2\pi}} v^k_{(l,m-l)}\,;\, m = 0, 1, \cdots, 0 \le k, l \le m \right\}$ forms a complete orthonormal basis of the space of harmonic polynomials, as well as $\left\{ \frac{1}{\sqrt{2\pi}} w^k_{(l,m-l)}\,;\, m = 0, 1, \cdots, 0 \le k, l \le m \right\}$.

Proposition 2.1.

$$\begin{aligned} e_+ v^k_{(l,m-l)} &= -k(m-k+1) v^{k-1}_{(l,m-l)}, \\ e_- v^k_{(l,m-l)} &= v^{k+1}_{(l,m-l)}, \\ \theta v^k_{(l,m-l)} &= (m-2k) v^k_{(l,m-l)}\,. \end{aligned} \tag{2.12}$$

$$\begin{aligned} \hat{e}_+ w^k_{(l,m-l)} &= -k(m-k+1) w^{k-1}_{(l,m-l)}, \\ \hat{e}_- w^k_{(l,m-l)} &= w^{k+1}_{(l,m-l)}, \\ \hat{\theta} w^k_{(l,m-l)} &= (m-2k) w^k_{(l,m-l)}. \end{aligned} \tag{2.13}$$

Therefore the space of harmonic polynomials on $\mathbf{C}^2$ is decomposed by the right action of SU(2) into $\sum_m \sum_{l=0}^m H_{m,l}$. Each $H_{m,l} = \sum_{k=0}^m C v^k_{(l,m-l)}$ gives an (m+1) dimensional irreducible representation of $SU(2)$ with the highest weight $\frac{m}{2}$.

We have the following relations.

$$w^k_{(l,m-l)} \;=\; (-1)^k \frac{l!}{(m-k)!}\, v^{m-l}_{(k,m-k)}\,, \tag{2.14}$$

$$\overline{v^k_{(l,m-l)}} \;=\; (-1)^{m-l-k} \frac{k!}{(m-k)!} v^{m-k}_{(m-l,l)}\,. \tag{2.15}$$

2.2 Harmonic spinors

$\Delta = \mathbf{H}\otimes\mathbf{C} = \mathbf{H}\oplus\mathbf{H}$ gives an irreducible complex representation of the Clifford algebra $\mathrm{Clif}(\mathbf{R}^4)$:

$$\mathrm{Clif}(\mathbf{R}^4)\otimes\mathbf{C} \;\simeq\; \mathrm{End}(\Delta)\,.$$

Δ decomposes into irreducible representations $\Delta^{\pm} = \mathbf{H}$ of $\mathrm{Spin}(4)$. Let $S = \mathbf{C}^2\times\Delta$ be the trivial spinor bundle on $\mathbf{C}^2$. The corresponding bundle $S^+ = \mathbf{C}^2\times\Delta^+$ (resp. $S^- = \mathbf{C}^2\times\Delta^-$) is called the even (resp. odd) spinor bundle and the sections are called even (resp. odd) spinors. The set of even spinors or odd spinors on a set $M\subset\mathbf{C}^2$ is nothing but the smooth functions on M valued in $\mathbf{H}$:

$$Map(M,\mathbf{H}) \;=\; C^{\infty}(M,S^+)\,. \tag{2.16}$$

The Dirac operator is defined by

$$\mathcal{D} = c\circ d \tag{2.17}$$

where $d: S\to S\otimes T^*\mathbf{C}^2 \simeq S\otimes T\mathbf{C}^2$ is the exterior differential and $c: S\otimes T\mathbf{C}^2 \to S$ is the bundle homomorphism coming from the Clifford multiplication. By means of the decomposition $S = S^+\oplus S^-$ the Dirac operator has the chiral decomposition:

$$\mathcal{D} = \begin{pmatrix} 0 & D^{\dagger} \\ D & 0 \end{pmatrix} : C^{\infty}(\mathbf{C}^2, S^+\oplus S^-) \to C^{\infty}(\mathbf{C}^2, S^+\oplus S^-). \tag{2.18}$$

We find that D and $D^{\dagger}$ have the following coordinate expressions;

$$D = \begin{pmatrix} \frac{\partial}{\partial z_1} & -\frac{\partial}{\partial \bar{z_2}} \\ \\ \frac{\partial}{\partial z_2} & \frac{\partial}{\partial \bar{z_1}} \end{pmatrix}, \quad D^{\dagger} = \begin{pmatrix} \frac{\partial}{\partial \bar{z_1}} & \frac{\partial}{\partial \bar{z_2}} \\ \\ -\frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_1} \end{pmatrix}. \tag{2.19}$$

An even (resp. odd) spinor φ is called a *harmonic spinor* if $D\varphi = 0$ (resp. $D^{\dagger}\varphi = 0$).

We shall introduce a set of harmonic spinors which forms a complete orthonormal basis of $L^2(S^3, S^+)$.

Let ν and μ be vector fields on $\mathbf{C}^2$ defined by

$$\nu = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \ , \quad \mu = z_2 \frac{\partial}{\partial z_2} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \ . \tag{2.20}$$

Then the radial vector field is defined by

$$\frac{\partial}{\partial n} = \frac{1}{2|z|}(\nu + \bar{\nu}) = \frac{1}{2|z|}(\mu + \bar{\mu}). \tag{2.21}$$

The vector field θ in (2.4) is also written by $\theta = \frac{1}{2\sqrt{-1}}(\nu - \bar{\nu})$.

We shall denote by γ the Clifford multiplication of the radial vector $\frac{\partial}{\partial n}$, (2.21). γ changes the chirality:

$$\gamma : S^+ \oplus S^- \longrightarrow S^- \oplus S^+; \quad \gamma^2 = 1.$$

The matrix expression of γ becomes as follows:

$$\gamma|S^+ = \frac{1}{|z|}\begin{pmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{pmatrix}, \quad \gamma|S^- = \frac{1}{|z|}\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}. \tag{2.22}$$

In the sequel we shall write γ_+ (resp. γ_-) for $\gamma|S^+$ (resp. $\gamma|S^+$).

Proposition 2.2. *The Dirac operators D and $D^\dagger$ have the following polar decompositions:*

$$\begin{aligned} D &= \gamma_+ \left(\frac{\partial}{\partial n} - \partial\!\!\!/ \right), \\ D^\dagger &= \left(\frac{\partial}{\partial n} + \partial\!\!\!/ + \frac{3}{2|z|} \right) \gamma_- \, , \end{aligned}$$

where the tangential (nonchiral) Dirac operator $\partial\!\!\!/$ is given by

$$\partial\!\!\!/ = -\left[\sum_{i=1}^{3} \left(\frac{1}{|z|}\theta_i \right) \cdot \nabla_{\frac{1}{|z|}\theta_i} \right] = \frac{1}{|z|}\begin{pmatrix} -\frac{1}{2}\theta & e_+ \\ -e_- & \frac{1}{2}\theta \end{pmatrix}.$$

Proof. In the matrix expression (2.19) of D and $D^\dagger$, we have $\frac{\partial}{\partial z_1} = \frac{1}{|z|^2}(\bar{z}_1\nu - z_2 e_-)$ etc., and we have the desired formulas. $\square$

The tangential Dirac operator on the sphere $S^3 = \{|z| = 1\}$;

$$\partial\!\!\!/ |S^3 : C^\infty(S^3, S^+) \longrightarrow C^\infty(S^3, S^+)$$

is a self adjoint elliptic differential operator.
We put, for $m = 0, 1, 2, \cdots ; l = 0, 1, \cdots , m$ and $k = 0, 1, \cdots , m + 1$,

$$\phi^{+(m,l,k)}(z) = \sqrt{\frac{(m+1-k)!}{k!l!(m-l)!}} \begin{pmatrix} k v^{k-1}_{(l,m-l)} \\ \\ -v^k_{(l,m-l)} \end{pmatrix}, \tag{2.23}$$

$$\phi^{-(m,l,k)}(z) = \sqrt{\frac{(m+1-k)!}{k!l!(m-l)!}} \left(\frac{1}{|z|^2}\right)^{m+2} \begin{pmatrix} w^k_{(m+1-l,l)} \\ \\ w^k_{(m-l,l+1)} \end{pmatrix}. \tag{2.24}$$

From Proposition 2.1 we have the following

Proposition 2.3. *On $S^3 = \{|z| = 1\}$ we have:*

$$\partial\!\!\!/ \phi^{+(m,l,k)} = \frac{m}{2}\phi^{+(m,l,k)}, \tag{2.25}$$

$$\partial\!\!\!/ \phi^{-(m,l,k)} = -\frac{m+3}{2}\phi^{-(m,l,k)}. \tag{2.26}$$

The eigenvalues of $\partial\!\!\!/$ are

$$\frac{m}{2}, \quad -\frac{m+3}{2}; \quad m = 0, 1, \cdots , \tag{2.27}$$

and the multiplicity of each eigenvalue is equal to $(m + 1)(m + 2)$.
The set of eigenspinors

$$\left\{ \frac{1}{\sqrt{2}\pi}\phi^{+(m,l,k)}, \quad \frac{1}{\sqrt{2}\pi}\phi^{-(m,l,k)}; \quad m = 0, 1, \cdots , 0 \le l \le m, 0 \le k \le m + 1 \right\} \tag{2.28}$$

forms a complete orthonormal system of $L^2(S^3, S^+)$.

The constant for normalization of $\phi^{\pm(m,l,k)}$ is determined by the integral:

$$\int_{S^3} |z_1^a z_2^b|^2 d\sigma = 2\pi^2 \frac{a!b!}{(a+b+1)!}, \tag{2.29}$$

where σ is the surface measure of the unit sphere $S^3 = \{|z| = 1\}$:

$$\int_{S^3} d\sigma_3 = 2\pi^2. \tag{2.30}$$

$\phi^{+(m,l,k)}$ is a harmonic spinor on $\mathbf{C}^2$ and $\phi^{-(m,l,k)}$ is a harmonic spinor on $\mathbf{C}^2\setminus\{0\}$ that is regular at infinity. If φ is a harmonic spinor on $\mathbf{C}^2 \setminus \{0\}$ then we have the expansion

$$\varphi(z) = \sum_{m,l,k} C_{+(m,l,k)}\phi^{+(m,l,k)}(z) + \sum_{m,l,k} C_{-(m,l,k)}\phi^{-(m,l,k)}(z), \tag{2.31}$$

that is uniformly convergent on any compact subset of $\mathbf{C}^2 \setminus \{0\}$. The coefficients $C_{\pm(m,l,k)}$ are given by the formula:

$$C_{\pm(m,l,k)} = \frac{1}{3\pi^2}\int_{S^3} \langle \varphi,\, \phi^{\pm(m,l,k)}\rangle \, d\sigma, \tag{2.32}$$

where $\langle\, ,\, \rangle$ is the inner product of S^+. In particular, since $\phi^{+(0,0,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $J = \phi^{+(0,0,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$,

$$\begin{aligned} \int_{S^3} tr\, \varphi \, d\sigma &= 4\pi^2 Re.C^{+(0,0,1)}, \\ \int_{S^3} tr\, J\varphi \, d\sigma &= 4\pi^2 Re.C^{+(0,0,0)}. \end{aligned} \tag{2.33}$$

Definition 2.4. 1. We call the series (2.31) *a spinor of Laurent polynomial type* if only finitely many coefficients $C_{\pm(m,l,k)}$ are non-zero . The vector space of spinors of Laurent polynomial type is denoted by $\mathbf{C}[\phi^{\pm(m,l,k)}]$.

2. For a spinor of Laurent polynomial type φ we call the vector $\mathrm{res}\,\varphi = \begin{pmatrix} -C_{-(0,0,1)} \\ C_{-(0,0,0)} \end{pmatrix}$ the residue at 0 of φ.

We shall see later that $\mathbf{C}[\phi^{\pm(m,l,k)}]$ with the multiplication law coming from (1.8) becomes an associative algebra.

We have the residue formula. See, for example, Proposition 4.2 of [Ko3].

$$\mathrm{res}\,\varphi = \frac{1}{2\pi^2}\int_{S^3} \gamma_+(z)\varphi(z)\sigma(dz). \tag{2.34}$$

2.3 Algebraic generators of $\mathbf{C}[\phi^{\pm(m,l,k)}]$

We investigate the generators of the algebra $\mathbf{C}[\phi^{\pm(m,l,k)}]$. First we observe the following facts.

1. We have the following product formula for the harmonic polynomials $v^k_{(a.b)}$, (2.10).

$$v^{k_1}_{(a_1,b_1)} v^{k_2}_{(a_2,b_2)} = \sum_{j=0}^{a_1+a_2+b_1+b_2} C_j |z|^{2j}\, v^{k_1+k_2-j}_{(a_1+a_2-j,\, b_1+b_2-j)} \tag{2.35}$$

   for some rational numbers $C_j = C_j(a_1, a_2, b_1, b_2, k_1, k_2)$, see Lemma 4.1 of [Ko1].

2. Let $k = k_1 + k_2$, $a = a_1 + a_2$ and $b = b_1 + b_2$. The harmonic polynomial $v^k_{(a,b)}$ is equal to a constant multiple of $v^{k_1}_{(a_1,b_1)} v^{k_2}_{(a_2,b_2)}$ modulo a linear combination of polynomials $v^{k-j}_{(a-j,b-j)}$, $1 \le j \le min(k, a, b)$.

3. $\begin{pmatrix} v^k_{(l,m-l)} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ v^{k+1}_{(l,m-l)} \end{pmatrix}$ are written by linear combinations of $\phi^{+(m,l,k+1)}$ and $\phi^{-(m-1,k,l)}$.

4. Therefore the product of two spinors $\phi^{\pm(m_1,l_1,k_1)} \cdot \phi^{\pm(m_2,l_2,k_2)}$ belongs to $\mathbf{C}[\phi^{\pm(m,l,k)}]$. $\mathbf{C}[\phi^{\pm(m,l,k)}]$ becomes an associative algebra and the Lie algebra structure follows from it.

5. $\phi^{\pm(m,l,k)}$ is written by a linear combination of the products $\phi^{\pm(m_1,l_1,k_1)} \cdot \phi^{\pm(m_2,l_2,k_2)}$ for $0 \le m_1 + m_2 \le m - 1$, $0 \le l_1 + l_2 \le l$ and $0 \le k_1 + k_2 \le k$.

Hence we find that the algebra $\mathbf{C}[\phi^{\pm(m,l,k)}]$ is generated by the following I, J, κ, μ:

$$I = \phi^{+(0,0,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad J = \phi^{+(0,0,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

$$\kappa = \phi^{+(1,0,1)} = \begin{pmatrix} z_2 \\ -\overline{z}_1 \end{pmatrix}, \quad \mu = \phi^{-(0,0,0)} = \begin{pmatrix} z_2 \\ \overline{z}_1 \end{pmatrix}. \tag{2.36}$$

The others are generated by these basis, For example,

$$\lambda=\phi^{+(1,1,1)}=\begin{pmatrix} z_1 \\ \overline{z}_2 \end{pmatrix}=-\kappa J\,, \quad \nu=\phi^{-(0,0,1)}=\begin{pmatrix} -z_1 \\ \overline{z}_2 \end{pmatrix}=-\mu J,$$

$$\phi^{+(1,0,0)}=\sqrt{2}\begin{pmatrix} 0 \\ -z_2 \end{pmatrix}=\frac{1}{\sqrt{2}}J(\kappa+\mu), \quad \phi^{+(1,0,2)}=\sqrt{2}\begin{pmatrix} \overline{z}_1 \\ 0 \end{pmatrix}=\frac{1}{\sqrt{2}}J(\mu-\kappa)\,,$$

$$\phi^{+(1,1,2)}=\sqrt{2}\begin{pmatrix} -\overline{z}_2 \\ 0 \end{pmatrix}=-\frac{1}{\sqrt{2}}J(\lambda+\nu), \quad \phi^{+(1,1,0)}=\sqrt{2}\begin{pmatrix} 0 \\ -z_1 \end{pmatrix}=\frac{1}{\sqrt{2}}J(\lambda-\nu),$$

$$\phi^{-(1,0,0)}=\sqrt{2}\begin{pmatrix} z_2^2 \\ z_2\overline{z}_1 \end{pmatrix}=\frac{1}{\sqrt{2}}\nu J(\kappa+\mu), \quad \phi^{-(1,1,0)}=\sqrt{2}\begin{pmatrix} z_2\overline{z}_1 \\ \overline{z}_1^2 \end{pmatrix}=\frac{1}{\sqrt{2}}\mu J(\mu-\kappa),$$

$$\phi^{-(1,1,2)}=\sqrt{2}\begin{pmatrix} -z_1\overline{z}_2 \\ \overline{z}_2^2 \end{pmatrix}=\frac{1}{\sqrt{2}}\nu J(\lambda+\nu), \quad \phi^{-(1,0,2)}=\sqrt{2}\begin{pmatrix} z_1^2 \\ -z_1\overline{z}_2 \end{pmatrix}=\frac{1}{\sqrt{2}}\mu J(\lambda-\nu)$$

$$\phi^{-(1,0,1)}=\begin{pmatrix} -2z_1z_2 \\ |z_2|^2-|z_1|^2 \end{pmatrix}=\frac{\nu}{2}(\kappa+\mu+J(\lambda-\nu)),$$

$$\phi^{-(1,1,1)}=\begin{pmatrix} |z_2|^2-|z_1|^2 \\ 2\overline{z}_1\overline{z}_2 \end{pmatrix}=\frac{\mu}{2}(-\kappa+\mu+J(\lambda+\nu))\,.$$

2.4 2-cocycle on $\mathbf{C}[\phi^{\pm(m,l,k)}]$

Let $S^3\mathbf{H}=Map(S^3,\mathbf{H})=C^\infty(S^3,S^+)$ be the set of smooth even spinors on S^3. We define the Lie algebra structure on $S^3\mathbf{H}$ after (1.9), that is, for even spinors $\phi_1=\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and $\phi_2=\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$, we have the Lie bracket

$$[\,\phi_1\,,\,\phi_2\,]=\begin{pmatrix} v_1\bar{v}_2-\bar{v}_1v_2 \\ (u_2-\bar{u}_2)v_1-(u_1-\bar{u}_1)v_2 \end{pmatrix}. \tag{2.37}$$

Let $\frac{\partial}{\partial n}$ be the radial vector field (2.21). For $\varphi = \begin{pmatrix} u \\ v \end{pmatrix} \in S^3\mathbf{H}$ we put

$$\frac{\partial}{\partial \mathbf{n}}\varphi = \begin{pmatrix} \frac{\partial}{\partial n}u \\ \frac{\partial}{\partial n}v \end{pmatrix}.$$

We have the following Leibnitz rule.

$$\frac{\partial}{\partial \mathbf{n}}(\phi_1 \cdot \phi_2) = (\frac{\partial}{\partial \mathbf{n}}\phi_1) \cdot \phi_2 + \phi_1 \cdot (\frac{\partial}{\partial \mathbf{n}}\phi_2)\,, \tag{2.38}$$

Lemma 2.5. *1.*

$$\frac{\partial}{\partial \mathbf{n}}\phi^{+(m,l,k)} = \frac{m}{2|z|}\,\phi^{+(m,l,k)}, \quad \frac{\partial}{\partial \mathbf{n}}\phi^{-(m,l,k)} = -\frac{m+3}{2|z|}\,\phi^{-(m,l,k)}. \tag{2.39}$$

2. Let φ be a Laurent polynomial spinor:

$$\varphi(z) = \sum_{m,l,k} C_{+(m,l,k)}\phi^{+(m,l,k)}(z) + \sum_{m,l,k} C_{-(m,l,k)}\phi^{-(m,l,k)}(z).$$

Then

$$\int_{S^3} tr\,\frac{\partial}{\partial \mathbf{n}}\varphi\, d\sigma = 0. \tag{2.40}$$

The formula (2.39) follows from the definition (2.23). If φ is a spinor of Laurent Polynomial type then $|z|\frac{\partial}{\partial \mathbf{n}}\varphi$ is also a spinor of Laurent polynomial type and, since the coefficient of $\phi^{+(0,0,1)}$ in the Laurent expansion of $|z|\frac{\partial}{\partial \mathbf{n}}\varphi$ vanishes, the formula (2.40) follows from (2.33).

Definition 2.6. For ϕ_1 and $\phi_2 \in \mathbf{C}[\phi^{\pm(m,l,k)}]$, we put

$$c(\phi_1, \phi_2) \;=\; \frac{1}{2\pi^2}\int_{S^3} tr[\,(\frac{\partial}{\partial \mathbf{n}}\phi_1) \cdot \phi_2\,]d\sigma. \tag{2.41}$$

Proposition 2.7. *c defines a 2-cocycle on the algebra $\mathbf{C}[\phi^{\pm(m,l,k)}]$. That is, c satisfies the following equations:*

$$c(\phi_1\,\phi_2) = -\,c(\phi_2,\,\phi_1)\,, \tag{2.42}$$

$$c(\phi_1 \cdot \phi_2\,,\,\phi_3) + c(\phi_2 \cdot \phi_3\,,\,\phi_1\,) + c(\phi_3 \cdot \phi_1\,,\,\phi_2\,) = 0. \tag{2.43}$$

In fact (2.40) and the Leibnitz rule (2.38) imply (2.42). The following calculation proves (2.43).

$$\begin{aligned} c(\phi_1\cdot\phi_2\,,\,\phi_3) &= \frac{1}{2\pi^2}\int_{S^3} tr(\frac{\partial}{\partial\mathbf{n}}(\phi_1\cdot\phi_2)\cdot\phi_3\,)d\sigma \\ &= \frac{1}{2\pi^2}\int_{S^3} tr(\frac{\partial}{\partial\mathbf{n}}\phi_1\cdot(\phi_2\cdot\phi_3)\,)d\sigma \,+\, \frac{1}{2\pi^2}\int_{S^3} tr(\frac{\partial}{\partial\mathbf{n}}\phi_2\cdot(\phi_3\cdot\phi_1)\,)d\sigma \\ &= c(\phi_1\,,\,\phi_2\cdot\phi_3\,)+c(\phi_2\,,\,\phi_3\cdot\phi_1\,) = -c(\phi_2\cdot\phi_3\,,\phi_1\,)\,-\,c(\phi_3\cdot\phi_1\,,\,\phi_2). \end{aligned}$$

The last equality follows from (2.42). □

Example.

$$c(\,\phi^{+(1,1,1)}\,,\,\phi^{+(1,0,2)}-\frac{1}{\sqrt{2}}\phi^{+(1,1,1)}-\frac{1}{\sqrt{2}}\phi^{-(0,0,1)}\,)=2\sqrt{2}. \tag{2.44}$$

We introduce the derivation $d_0=\frac{1}{2}(\,\nu+\overline{\nu}\,)=|z|\frac{\partial}{\partial\mathbf{n}}$ acting on the algebra $\mathbf{C}[\phi^{\pm(m,l,k)}]$:

$$d_0(\,\phi_1\cdot\,\phi_2\,)\;=\;d_0\phi_1\cdot\,\phi_2\,+\,\phi_1\cdot\,d_0\phi_2\;. \tag{2.45}$$

Lemma 2.8.

$$c(\,d_0\phi_1\,,\phi_2\,)+c(\,\phi_1\,,d_0\phi_2\,)=0\,. \tag{2.46}$$

In fact, we have $d_0\phi|_{S^3}=\frac{\partial}{\partial n}\phi|_{S^3}$. Hence

$$c(\,\phi_1\,,\,d_0\phi_2\,)=\frac{1}{2\pi^2}\int_{S^3} tr\,(\,\frac{\partial}{\partial n}\phi_1\cdot d_0\phi_2\,)d\sigma=c(\,\phi_2\,,\,d_0\phi_1\,)\,.$$

3 Extensions of the Lie algebra $\mathbf{C}[\phi^{\pm(m,l,k)}]\otimes U(\mathfrak{g})$

3.1 Extension of $\mathbf{C}[\phi^{\pm(m,l,k)}]\otimes U(\mathfrak{g})$ by a 1-dimensional center

From Proposition 1.2 we see that $S^3\mathfrak{g}^{\mathbf{H}}=S^3\mathbf{H}\otimes U(\mathfrak{g})$ endowed with the following bracket $[\,,\,]_{S^3\mathfrak{g}^{\mathbf{H}}}$ becomes a Lie algebra.

$$[\,\phi\otimes X\,,\,\psi\otimes Y\,]_{S^3\mathfrak{g}^{\mathbf{H}}}=(\phi\cdot\psi)\otimes(XY)\,-(\psi\cdot\phi)\otimes(YX), \tag{3.1}$$

for $X,Y\in U(\mathfrak{g})$ and $\phi,\,\psi\,\in S^3\mathbf{H}$. And $\mathbf{C}[\phi^{\pm(m,l,k)}]\otimes U(\mathfrak{g})$ is a Lie subalgebra of $S^3\mathfrak{g}^{\mathbf{H}}$.

We take the non-degenerate invariant symmetric bilinear $\mathbf{C}$-valued form $(\,\cdot\,|\,\cdot\,)$ on $\mathfrak{g}$ and extend it to $U(\mathfrak{g})$: for $X=X_1^{l_1}\cdots X_n^{l_n}$ and $Y=Y_1^{k_1}\cdots Y_m^{k_m}$ written by the basisi of $\mathfrak{g}$,

$(X|Y)$ is defined by $(X|Y) = tr(ad(X_1^{l_1})\cdots ad(X_n^{l_n})ad(Y_1^{k_1})\cdots ad(Y_m^{k_m}))$. We extend this form by linearity to a $\mathbf{C}[\phi^{\pm(m,l,k)}]$-valued bilinear form on the Lie algebra $\mathbf{C}[\phi^{\pm(m,l,k)}] \otimes U(\mathfrak{g})$. Then we can define a $\mathbf{C}$-valued 2-cocycle on the Lie algebra $\mathbf{C}[\phi^{\pm(m,l,k)}] \otimes U(\mathfrak{g})$ by

$$c(\,\phi_1 \otimes X_1\,,\,\phi_2 \otimes X_2\,) = \,(X|Y)\,c(\phi_1,\phi_2). \tag{3.2}$$

The 2-cocycle property follows from the fact $(XY|Z) = (YZ|X)$ and Proposition 2.7.

Let a be an indefinite element. Denote by $\widehat{\mathfrak{g}}(a)$ the extension of the Lie algebra $\mathbf{C}[\phi^{\pm(m,l,k)}] \otimes U(\mathfrak{g})$ by a 1-dimensional center, associated to the cocycle c. Explicitly we have the following theorem.

Theorem 3.1.

$$\widehat{\mathfrak{g}}(a) = \,(\,\mathbf{C}[\phi^{\pm(m,l,k)}] \otimes U(\mathfrak{g})\,) \oplus (\mathbf{C}a), \tag{3.3}$$

and the bracket is given by

$$\begin{aligned}
[\,\phi \otimes X\,,\,\psi \otimes Y\,]\widehat{\ } &= (\phi\cdot\psi) \otimes\,(XY) - (\psi\cdot\phi)\otimes(YX) + (X|Y)\,c(\phi,\psi)\,a\,, &(3.4)\\
[\,a\,,\,\phi\otimes X\,]\widehat{\ } &= 0\,, &(3.5)
\end{aligned}$$

for $X, Y \in U(\mathfrak{g})$ *and* $\phi,\,\psi\,\in \mathbf{C}[\phi^{\pm(m,l,k)}]$.

3.2 Extension of $\widehat{\mathfrak{g}}(a)$ by a derivation

The derivation d_0 on $\mathbf{C}[\phi^{\pm(m,l,k)}]$ is extended to a derivation of the Lie algebra $\widehat{\mathfrak{g}}(a)$. In fact , if we define the action of d_0 on $\widehat{\mathfrak{g}}(a)$ by

$$\begin{aligned}
[\,d_0\,,\phi\otimes X\,]\widehat{\ } &= (d_0\phi\,)\otimes X\,, &(3.6)\\
[\,d_0\,,a\,]\widehat{\ } &= 0\,, &(3.7)
\end{aligned}$$

then we have from (2.45)

$$d_0\left(\left[\,\phi_1\otimes X_1 + t_1 a\,,\,\phi_2\times X_2 + t_2 a\,\right]\widehat{\ }\,\right) = d_0\left(\,(\phi_1\phi_2)\,\otimes(X_1X_2)\,-\,(\phi_2\phi_1)\otimes(X_2X_1)\,\right)$$

$$= \,(d_0\phi_1\cdot\phi_2)\otimes(X_1X_2)\,-\,(\phi_2\cdot d_0\phi_1)\otimes(X_2X_1)\,+\,(\phi_1\cdot d_0\phi_2)\otimes(X_1X_2)\,-\,(d_0\phi_2\cdot\phi_1)\otimes(X_2X_1)\,,$$

for $\phi_1 \otimes X_1 + t_1 a \in \widehat{\mathfrak{g}}(a)$ and $\phi_2 \otimes X_2 + t_2 a \in \widehat{\mathfrak{g}}(a)$. On the other hand

$$
\begin{aligned}
& [\, d_0(\phi_1 \otimes X_1 + t_1 a)\,,\, \phi_2 \otimes X_2 + t_2 a\,]^{\widehat{}} + [\,\phi_1 \otimes X_1 + t_1 a\,,\, d_0(\phi_2 \times X_2 + t_2 a)\,]^{\widehat{}} \\
& = (d_0\phi_1 \cdot \phi_2) \otimes (X_1 X_2) - (\phi_2 \cdot d_0\phi_1) \otimes (X_2 X_1) + (\phi_1 \cdot d_0\phi_2) \otimes (X_1 X_2) - (d_0\phi_2 \cdot \phi_1) \otimes (X_2 X_1) \\
& + (X_1|X_2)\,(\, c(d_0\phi_1,\, \phi_2) + c(\phi_1,\, d_0\phi_2))\, a\,.
\end{aligned}
$$

Since $c(d_0\phi_1,\, \phi_2) + c(\phi_1,\, d_0\phi_2) = 0$ from Lemma 2.8 we have

$$
\begin{aligned}
& d_0\left([\,\phi_1 \otimes X_1 + t_1 a\,,\, \phi_2 \times X_2 + t_2 a\,]^{\widehat{}}\, \right) \\
& = [\, d_0(\phi_1 \otimes X_1 + t_1 a)\,,\, \phi_2 \otimes X_2 + t_2 a\,]^{\widehat{}} + [\,\phi_1 \otimes X_1 + t_1 a\,,\, d_0(\phi_2 \times X_2 + t_2 a)\,]^{\widehat{}}\,.
\end{aligned}
$$

Thus d_0 is a derivation that acts on the Lie algebra $\widehat{\mathfrak{g}}(a)$.

We denote by $\widehat{\mathfrak{g}}$ the Lie algebra that is obtained by adjoining a derivation d to $\widehat{\mathfrak{g}}(a)$ which acts on $\widehat{\mathfrak{g}}(a)$ as d_0 and which kills a. More explicitly we have the following

Theorem 3.2. *Let a and d be indefinite numbers. We consider the $\mathbf{C}$ vector space:*

$$
\widehat{\mathfrak{g}} = \left(\mathbf{C}[\phi^{\pm(m,l,k)}] \otimes U(\mathfrak{g})\right) \oplus (\mathbf{C}\, a) \oplus (\mathbf{C} d)\,, \tag{3.8}
$$

and define the following bracket on $\widehat{\mathfrak{g}}$. For $X, Y \in U(\mathfrak{g})$ and $\phi,\, \psi \in \mathbf{C}[\phi^{\pm(m,l,k)}]$, we put

$$
\begin{aligned}
[\,\phi \otimes X\,,\, \psi \otimes Y\,]_{\widehat{\mathfrak{g}}} &= [\,\phi \otimes X\,,\, \psi \otimes Y\,]^{\widehat{}} \qquad (3.9)\\
&= (\phi \cdot \psi) \otimes (XY) - (\psi \cdot \phi) \otimes (YX) + (X|Y)\, c(\phi, \psi) \cdot a\,, \\
[\, a\,,\, \phi \otimes X\,]_{\widehat{\mathfrak{g}}} &= 0, \qquad [\, d, \phi \otimes X\,]_{\widehat{\mathfrak{g}}} = d_0\phi \otimes X\,, \qquad (3.10)\\
[\, a\,,\, d\,]_{\widehat{\mathfrak{g}}} &= 0\,. \qquad (3.11)
\end{aligned}
$$

Then $(\widehat{\mathfrak{g}}\,,\, [\,\cdot, \cdot\,]_{\widehat{\mathfrak{g}}})$ becomes a Lie algebra.

Proof

It is enough to prove the following Jacobi identity:

$$
[\,[\, d\,,\, \phi_1 \otimes X_1\,]_{\widehat{\mathfrak{g}}}\,,\, \phi_2 \otimes X_2\,]_{\widehat{\mathfrak{g}}} + [\,[\phi_1 \otimes X_1, \phi_2 \otimes X_2\,]_{\widehat{\mathfrak{g}}}\,,\, d\,]_{\widehat{\mathfrak{g}}} + [\,[\phi_2 \otimes X_2,\, d\,]_{\widehat{\mathfrak{g}}},\, \phi_1 \otimes X_1\,]_{\widehat{\mathfrak{g}}} = 0.
$$

In the following we shall abbreviate the bracket $[\,,\,]_{\widehat{\mathfrak{g}}}$ simply to $[\,,\,]$. We have

$$\begin{aligned}
[\,[\,d\,,\,\phi_1\otimes X_1\,]\,,\phi_2\otimes X_2\,] =& [\,d_0\phi_1\otimes X_1,\ \phi_2\otimes X_2\,]\\
=& (\,d_0\phi_1\cdot\phi_2)\otimes\ (X_1X_2) - \ (\,\phi_2\cdot d_0\phi_1\,)\otimes(X_2X_1)\\
&+(X_1|X_2)c(\,d_0\phi_1\,,\,\phi_2)\cdot a\,.
\end{aligned}$$

Similarly

$$\begin{aligned}
[\,[\,\phi_2\otimes X_2,\,d\,],\phi_1\otimes X_1\,] =& (\phi_1\cdot d_0\phi_2)\otimes(X_1X_2) - \ (\,d_0\phi_2\cdot\phi_1)\otimes(X_2X_1)\\
&+(X_1|X_2)\,c(\phi_1,\,d_0\phi_2\,)\,a\,.\\
[\,[\phi_1\otimes X_1,\phi_2\otimes X_2\,]\,,\,d\,] =& -\Big[\,d\,,\,(\phi_1\cdot\phi_2)\otimes(X_1X_2)-(\phi_2\cdot\phi_1)\otimes(X_2X_1)+(X_1|X_2)c(\phi_1\,,\phi_2)\,a\,\Big]\\
=& -\ d_0(\phi_1\cdot\phi_2)\otimes\ (X_1X_2)+d_0(\phi_2\cdot\phi_1)\otimes(X_2X_1)\,.
\end{aligned}$$

The sum of three equations vanishes by virtue of (2.45) and Lemma 2.8. □

4 Structure of $\widehat{\mathfrak{g}}$

4.1 The weight space decomposition of $U(\mathfrak{g})$

Let $(\,\mathfrak{g}\,,\,[\,,\,]_{\mathfrak{g}}\,)$ be a simple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{h}\oplus\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$ be the root space decomposition with the root space $\mathfrak{g}_\alpha=\{X\in\mathfrak{g};\,ad(h)X = <\alpha,h>X,\quad \forall h\in\mathfrak{h}\}$. $\Delta=\Delta(\mathfrak{g},\mathfrak{h})$ is the set of roots and dim $\mathfrak{g}_\alpha=1$. In the following we summarize the known results on the representation $(ad(\mathfrak{h}),\,U(\mathfrak{g})\,)$, [D, Ma]. Let $\Pi=\{\alpha_i;\,i=1,\cdots,r=\mathrm{rank}\,\mathfrak{g}\}\subset\mathfrak{h}^*$ be the set of simple roots and $\{\alpha_i^\vee\,;\,i=1,\cdots,r\,\}\subset\mathfrak{h}$ be the set of simple coroots. The Cartan matrix $A=(\,a_{ij}\,)_{i,j=1,\cdots,r}$ is given by $a_{ij}=\langle\alpha_i^\vee,\,\alpha_j\rangle$. Fix a standard set of generators $H_i=\alpha_i^\vee$, $X_i\equiv X_{\alpha_i}\in\mathfrak{g}_{\alpha_i}$, $Y_i\equiv X_{-\alpha_i}\in\mathfrak{g}_{-\alpha_i}$, so that $[X_i,\,Y_j]=H_j\delta_{ij}$, $[H_i,\,X_j]=a_{ji}X_j$ and $[H_i,\,Y_j]=-a_{ji}Y_j$. Let $\Delta\pm$ be the set of positive (respectively negative) roots of $\mathfrak{g}$ and put

$$\mathfrak{n}_\pm=\sum_{\alpha\in\Delta_\pm}\mathfrak{g}_\alpha\,.$$

Then $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. The enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ has the direct sum decomposition:

$$U(\mathfrak{g}) = U(\mathfrak{n}_-) \cdot U(\mathfrak{h}) \cdot U(\mathfrak{n}_+) \,. \tag{4.1}$$

The set

$$\{ \, Y_{m_1} \cdots Y_{m_q} H_1^{l_1} \cdots H_r^{l_r} X_{n_1} \cdots X_{n_p} \, ; \; 1 \leq m_i, \, n_i \leq r \, , \, l_i \geq 0 \, \}.$$

forms a basis of the enveloping algebra $U(\mathfrak{g})$. The adjoint action of $\mathfrak{h}$ is extended to that on $U(\mathfrak{g})$:

$$ad(h)(x \cdot y) = (\, ad(h)x \,) \cdot y \, + \, x \cdot (\, ad(h)y \,) \,.$$

$\lambda \in \mathfrak{h}^*$ is called a weight of the representation $(U(\mathfrak{g}), \, ad(\mathfrak{h})\,)$ if there exists a non-zero $x \in U(\mathfrak{g})$ such that $ad(h)x = hx - xh = \lambda(h)x$ for all $h \in \mathfrak{h}$. Let Σ be the set of weights of the representation $(U(\mathfrak{g}), \, ad(\mathfrak{h})\,)$. The weight space for the weight λ is by definition

$$\mathfrak{g}_\lambda^U \; = \; \{ x \in U(\mathfrak{g}) \, ; \quad ad(h)x = \lambda(h)x, \quad \forall h \in \mathfrak{h} \}.$$

Let $\lambda = \alpha_{n_1} + \cdots + \alpha_{n_p} - \alpha_{m_1} - \cdots - \alpha_{m_q} \in \mathfrak{h}^*$, $1 \leq n_i, m_i \leq \, r$. We can easily verify that

$$X_\lambda = \, Y_{m_1} \cdots Y_{m_q} H_1^{l_1} \cdots H_r^{l_r} X_{n_1} \cdots X_{n_p} \, \in \mathfrak{g}_\lambda^U$$

Therefore $\lambda = \sum_{i=1}^p \, \alpha_{n_i} - \sum_{i=1}^q \, \alpha_{m_i}$ is a weight of the representation $(U(\mathfrak{g}), ad(\mathfrak{h}))$ with the weight vetor X_λ. Conversely any weight λ may be written in the form $\lambda = \sum_{i=1}^p \, \alpha_{n_i} - \sum_{i=1}^q \, \alpha_{m_i}$. We note that this λ may be written also in the form $\lambda = (\sum_{i=1}^p \, \alpha_{n_i} + \alpha_k) - (\sum_{i=1}^q \, \alpha_{m_i} + \alpha_k)$ for an $\alpha_k \in \Pi$, and that it is written in the form $\lambda = \sum_{i=1}^r \, p_i \alpha_i - \sum_{i=1}^r \, q_i \alpha_i$, though $p_i \, q_i$ are not uniquely determined.

Lemma 4.1. *1. The set of weights of the adjoint representation* $(U(\mathfrak{g}), \, ad(\mathfrak{h}))$ *is*

$$\Sigma = \{ \sum \, k_i \alpha_i \, ; \quad \alpha_i \in \Pi, \, k_i \in \mathbf{Z} \, \}. \tag{4.2}$$

If we denote

$$\Sigma_\pm = \{ \, \pm \sum \, n_i \alpha_i \, \in \Sigma \, ; \quad n_i > 0 \, \} \tag{4.3}$$

then $\Sigma_\pm \cap \Delta = \Delta_\pm$.

2. If $\lambda \in \Sigma$ *then* $-\lambda \in \Sigma$.

3. *For each* $\lambda = \sum_{i=1}^{q} k_i\alpha_i \in \Sigma$, $\mathfrak{g}_\lambda^U$ *is generated by the basis*

$$X_\lambda = Y_1^{q_1} \cdots Y_r^{q_r} H_1^{l_1} \cdots H_r^{l_r} X_1^{p_1} \cdots X_r^{p_r}$$

with $p_i, q_i, l_i \in \mathbf{N} \cup 0$ *such that* $k_i = p_i - q_i$, $i = 1, \cdots, r$.
In particular $\mathfrak{g}_0^U$ *is generated by the basis*

$$X_\lambda = Y_1^{p_1} \cdots Y_r^{p_r} H_1^{l_1} \cdots H_r^{l_r} X_1^{p_1} \cdots X_r^{p_r}$$

with $p_i, l_i \in \mathbf{N} \cup 0$, $i = 1, \cdots, r$, *and*

$$U(\mathfrak{h}) \subset \mathfrak{g}_0^U .$$

4.

$$[\, \mathfrak{g}_\lambda^U \,,\, \mathfrak{g}_\mu^U \,] \subset \mathfrak{g}_{\lambda+\mu}^U , \tag{4.4}$$

4.2 Weight space decomposition of $\widehat{\mathfrak{g}}$

In the following we shall investigate the Lie algebra structure of

$$\widehat{\mathfrak{g}} = \big(\mathbf{C}[\phi^{\pm(m,l,k)}] \otimes U(\mathfrak{g}) \big) \oplus (\mathbf{C}\, a) \oplus (\mathbf{C} d) . \tag{4.5}$$

Recall that the Lie bracket was given by the formulas:

$$\begin{aligned}
[\, \phi \otimes X \,,\, \psi \otimes Y \,]_{\widehat{\mathfrak{g}}} &= (\phi\, \psi) \otimes (XY) - (\psi \phi) \otimes (YX) + (X|Y)\, c(\phi, \psi)\, a , \\
[\, a \,, \phi \otimes X \,]_{\widehat{\mathfrak{g}}} = 0 , &\qquad [\, a, \, d \,] = 0 , \\
[\, d, \phi \otimes X \,]_{\widehat{\mathfrak{g}}} &= d_0 \phi \otimes X ,
\end{aligned}$$

for $X, Y \in U(\mathfrak{g})$. Since $\phi^{+(0,0,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we identify $X \in U(\mathfrak{g})$ with $\phi^{+(0,0,1)} \otimes X$. Thus we look $\mathfrak{g}$ as a Lie subalgebra of $\widehat{\mathfrak{g}}$:

$$\big[\phi^{+(0,0,1)} \otimes X, \; \phi^{+(0,0,1)} \otimes Y\big]_{\widehat{\mathfrak{g}}} = [X, Y]_{\mathfrak{g}} , \tag{4.6}$$

and we shall write $\phi^{+(0,0,1)} \otimes X$ simply as X.

Let

$$\widehat{\mathfrak{h}} = ((\mathbf{C}\,\phi^{+(0,0,1)})\otimes\mathfrak{h}) \oplus (\mathbf{C}\,a) \oplus (\mathbf{C}\,d) = \mathfrak{h} \oplus (\mathbf{C}\,a) \oplus (\mathbf{C}d)\,. \tag{4.7}$$

$\widehat{\mathfrak{h}}$ is an abelian subalgebra of $\widehat{\mathfrak{g}}$ and $ad(\hat{h})$ is diagonalizable for any $\hat{h} \in \widehat{\mathfrak{h}}$.

We write $\hat{h} = h + sa + td \in \widehat{\mathfrak{h}}$ with $h \in \mathfrak{h}$ and $s,\, t \in \mathbf{C}$. An element λ of the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ can be regarded as an element of $\widehat{\mathfrak{h}}^*$ by putting

$$\langle \lambda, a\rangle = \langle \lambda, d\rangle = 0. \tag{4.8}$$

So $\Delta \subset \mathfrak{h}^*$ is seen to be a subset of $\widehat{\mathfrak{h}}^*$. We define the elements $\delta\,,\,\Lambda_0 \in \widehat{\mathfrak{h}}^*$ by

$$\langle \delta, \alpha_i^\vee \rangle = \langle \Lambda_0, \alpha_i^\vee \rangle = 0, \qquad (1 \leqq i \leqq r), \tag{4.9}$$

$$\langle \delta, a\rangle = 0\,, \qquad \langle \delta, d \rangle = 1, \tag{4.10}$$

$$\langle \Lambda_0, a\rangle = 1\,, \qquad \langle \Lambda_0, d \rangle = 0. \tag{4.11}$$

Then the set $\{\alpha_1, \cdots, \alpha_r,\, \delta\,,\, \Lambda_0\}$ forms a basis of $\widehat{\mathfrak{h}}^*$. Similarly Σ is a subset of $\widehat{\mathfrak{h}}^*$.

For any $h \in \mathfrak{h}$, $\phi \in \mathbf{C}[\phi^{\pm(m,l,k)}]$ and $X \in U(\mathfrak{g})$, it holds that

$$\begin{aligned}
[\,\phi^{+(0,0,1)} \otimes h,\, \phi \otimes X\,]_{\widehat{\mathfrak{g}}} &= \phi \otimes (\,hX - Xh\,)\,,\\
[\,d,\, \phi \otimes X\,]_{\widehat{\mathfrak{g}}} &= (d_0\phi\,) \otimes X\,,\\
[\,\phi^{+(0,0,1)} \otimes h,\, a\,]_{\widehat{\mathfrak{g}}} &= [\,\phi^{+(0,0,1)} \otimes h,\, d\,]_{\widehat{\mathfrak{g}}} = [\,d, a\,]_{\widehat{\mathfrak{g}}} = 0\,.
\end{aligned}$$

The adjoint actions of $\hat{h} = h + sa + td \in \widehat{\mathfrak{h}}$ on $\widehat{\mathfrak{g}}$ is written as follows.

$$ad(\hat{h})\,(\phi \otimes X + \mu a + \nu d) = \phi \otimes (hX - Xh) + td_0\phi \otimes X\,, \tag{4.12}$$

for $\xi = \phi \otimes X + \mu a + \nu d \in \widehat{\mathfrak{g}}$.

Since $\widehat{\mathfrak{h}}$ is a commutative subalgebra of $\widehat{\mathfrak{g}}$, $\widehat{\mathfrak{g}}$ is decomposed into a direct sum of the simultaneous eigenspaces of $ad\,(\hat{h})$, $\hat{h} \in \widehat{\mathfrak{h}}$. For $\lambda = \gamma + k_0\delta \in \widehat{\mathfrak{h}}^*$, $\gamma = \sum_{i=1}^r k_i\alpha_i \in \Sigma$, $k_i \in \mathbf{Z}$, $i = 0, 1, \cdots, r$, we put,

$$\widehat{\mathfrak{g}}_\lambda = \Big\{ \xi \in \widehat{\mathfrak{g}}\,; \quad [\,\hat{h},\, \xi\,] = \langle \lambda, \hat{h}\rangle\, \xi \quad \text{for } \forall \hat{h} \in \widehat{\mathfrak{h}} \Big\}. \tag{4.13}$$

λ is called a weight of $\widehat{\mathfrak{g}}$ if $\widehat{\mathfrak{g}}_\lambda \neq 0$. $\widehat{\mathfrak{g}}_\lambda$ is called the weight space of λ. It holds that

$$\widehat{\mathfrak{g}}_\lambda \subset \widehat{\mathfrak{g}}(a)\,. \tag{4.14}$$

Let $\widehat{\Sigma}$ denote the set of weights of the representation $\left(\widehat{\mathfrak{g}}\,, ad(\widehat{\mathfrak{h}})\right)$.

Theorem 4.2. *1.*

$$\begin{aligned}\widehat{\Sigma} \;&=\; \left\{\frac{m}{2}\delta + \lambda;\quad \lambda \in \Sigma\,,\, m \in \mathbf{Z},\, m \neq -1, -2\right\} \\ &\quad \bigcup \left\{\frac{m}{2}\delta;\quad m \in \mathbf{Z},\, m \neq -1, -2\right\}\,.\end{aligned}$$

2. Let $\lambda \in \Sigma$ and $\lambda \neq 0$.

$$\begin{aligned}\widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\lambda} \;&=\; \mathbf{C}[\phi^{+(m,l,k)};\, 0 \le l \le m,\, 0 \le k \le m+1\,] \otimes \mathfrak{g}^U_\lambda\,, \qquad \textit{for } m \ge 0, \\ \widehat{\mathfrak{g}}_{-\frac{m+3}{2}\delta+\lambda} \;&=\; \mathbf{C}[\phi^{-(m,l,k)};\, 0 \le l \le m,\, 0 \le k \le m+1\,] \otimes \mathfrak{g}^U_\lambda\,, \qquad \textit{for } m \ge 0.\end{aligned}$$

3.

$$\begin{aligned}\widehat{\mathfrak{g}}_{\frac{m}{2}\delta} \;&=\; \mathbf{C}[\phi^{+(m,l,k)};\, 0 \le l \le m,\, 0 \le k \le m+0\,] \otimes \mathfrak{g}^U_0\,, \quad \textit{for } m \ge 0, \\ \widehat{\mathfrak{g}}_{-\frac{m+3}{2}\delta} \;&=\; \mathbf{C}[\phi^{-(m,l,k)};\, 0 \le l \le m,\, 0 \le k \le m+1\,] \; \otimes \mathfrak{g}^U_0\,, \quad \textit{for } m \ge 0\,.\end{aligned}$$

4. $\widehat{\mathfrak{g}}$ has the following decomposition:

$$\widehat{\mathfrak{g}} = \bigoplus_{m \neq -1,-2} \widehat{\mathfrak{g}}_{\frac{m}{2}\delta} \bigoplus_{\lambda \in \Sigma,\, m \neq -1,-2} \widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\lambda} \bigoplus(\mathbf{C}a) \bigoplus(\mathbf{C}d) \tag{4.15}$$

Proof

First we prove the second assertion. Let $X \in \mathfrak{g}^U_\lambda$ for a $\lambda \in \Sigma$, $\lambda \neq 0$. We have, for any $h \in \mathfrak{h}$ and $m \ge 0$,

$$\begin{aligned}[][\phi^{+(0,0,1)} \otimes h, \phi^{\pm(m,l,k)} \otimes X]_{\widehat{\mathfrak{g}}} \;&=\; \phi^{\pm(m,l,k)} \otimes (hX - Xh) \,=\, \langle \lambda, h\rangle\, \phi^{\pm(m,l,k)} \otimes X, \\ [d, \phi^{+(m,l,k)} \otimes X]_{\widehat{\mathfrak{g}}} \;&=\; \frac{m}{2}\phi^{+(m,l,k)} \otimes X, \\ [d, \phi^{-(m,l,k)} \otimes X]_{\widehat{\mathfrak{g}}} \;&=\; -\frac{m+3}{2}\,\phi^{-(m,l,k)} \otimes X\,,\end{aligned}$$

that is, for every $\hat{h} \in \widehat{\mathfrak{h}}$, we have

$$[\hat{h}\,,\phi^{+(m,l,k)} \otimes X]_{\widehat{\mathfrak{g}}} \;=\; \left\langle \frac{m}{2}\delta + \lambda\,,\,\hat{h}\right\rangle (\phi^{+(m,l,k)} \otimes X)\,, \tag{4.16}$$

$$[\hat{h}\,,\phi^{-(m,l,k)} \otimes X]_{\widehat{\mathfrak{g}}} \;=\; \left\langle -\frac{m+3}{2}\delta + \lambda\,,\,\hat{h}\right\rangle (\phi^{-(m,l,k)} \otimes X)\,. \tag{4.17}$$

Therefore, for $m \geq 0$, we have $\phi^{+(m,l,k)} \otimes X \in \widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\lambda}$ and $\phi^{-(m,l,k)} \otimes X \in \widehat{\mathfrak{g}}_{-\frac{m+3}{2}\delta+\lambda}$.

Conversely we shall show that any $\xi \in \widehat{\mathfrak{g}}_{-\frac{m+3}{2}\delta+\lambda}$ is written by a linear combination of $\{\,\phi^{-(m,l,k)} \otimes X\,;\, 0 \leq l \leq m, 0 \leq k \leq m+1,\; X \in \mathfrak{g}^U_\lambda\}$. Let $\xi = \phi \otimes X + \mu a + \nu d$ for $\phi \in \mathbf{C}[\phi^{\pm(n,l,k)}]$, $X \in U(\mathfrak{g})$ and μ, $\nu \in \mathbf{C}$, where ϕ is a Laurent polynomial spinor;

$$\phi = \sum_{n,l,k} C_{-(n,l,k)}\phi^{-(n,l,k)} + \sum_{n,l,k} C_{+(n,l,k)}\phi^{+(n,l,k)}\,.$$

We have

$$\begin{aligned}
[\hat{h},\xi] = [\,\phi^{+(0,0,1)} \otimes h + td\,,\phi \otimes X \;+\mu a\,] = [\,h\,,\,\phi \otimes X\,] \\
+\,t\left(\sum_{n,l,k}(-\frac{n+3}{2})C_{-(n,l,k)}\phi^{-(n,l,k)} + \frac{n}{2}C_{+(n,l,k)}\phi^{+(n,l,k)}\right) \otimes X
\end{aligned}$$

for any $\hat{h} = \phi^{+(0,0,1)} \otimes h + sa + td \in \widehat{\mathfrak{h}}$. From the assumption we have $[\hat{h},\xi] = \langle -\frac{m+3}{2}\delta + \lambda\,,\hat{h}\,\rangle\,\xi$, that is,

$$\begin{aligned}
\langle\, -\frac{m+3}{2}\delta + \lambda\,,\,\hat{h}\rangle\xi \;=< \lambda, h > \phi \otimes X \;+ (-\frac{m+3}{2}t+ < \lambda, h >)(\mu a + \nu d) \\
+\,(-\frac{m+3}{2})t\left(\sum_{n,l,k} C_{-(n,l,k)}\phi^{-(n,l,k)} + C_{+(n,l,k)}\phi^{+(n,l,k)}\right) \otimes X.
\end{aligned}$$

Comparing the above two equations we have $C_{+(n,l,k)} = 0$ for all $n \geq 0$, while $C_{-(n,l,k)} = 0$ except for $n = m$, and $\mu = \nu = 0$. Therefore $\phi = \sum_{0\leq l\leq m, 0\leq k\leq m+1} C_{-(m,l,k)}\phi^{-(m,l,k)}$, and $[\hat{h},\xi] = \phi \otimes [h,X] = \langle\lambda,\,h\rangle\,\phi \otimes X$ for all $\hat{h} = \phi^{+(0,0,1)} \otimes h + sa + td \in \widehat{\mathfrak{h}}$. Hence $X \in \mathfrak{g}^U_\lambda$ and $\xi = \left(\sum_{0\leq l\leq m, 0\leq k\leq m+1} C_{-(m,l,k)}\phi^{-(m,l,k)}\right) \otimes X \in \widehat{\mathfrak{g}}_{-\frac{m+3}{2}\delta+\lambda}$. We have proved

$$\widehat{\mathfrak{g}}_{-\frac{m+3}{2}\delta+\lambda} = \mathbf{C}[\phi^{-(m,l,k)};\, 0 \leq l \leq m,\, 0 \leq k \leq m+1\,] \;\otimes \mathfrak{g}^U_\lambda\,, \quad \text{for } m \geq 0\;.$$

The same argument proves also

$$\widehat{\mathfrak{g}}_{-\frac{1}{2}\delta+\lambda} = \widehat{\mathfrak{g}}_{-\frac{2}{2}\delta+\lambda} = 0. \tag{4.18}$$

Similarly we have

$$\widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\lambda} = \mathbf{C}[\phi^{+(m,l,k)};\, 0 \leq l \leq m,\, 0 \leq k \leq m+1\,] \,\otimes \mathfrak{g}^U_\lambda\,, \quad \text{for } m \geq 0\ .$$

The proof of the third assertion is also carried out by the same argument as above if we revise it for the case $\lambda = 0$. Always the same argument as above yields

$$\begin{aligned} \widehat{\mathfrak{g}}_{-\frac{1}{2}\delta+\alpha} &= \widehat{\mathfrak{g}}_{-\frac{2}{2}\delta+\alpha} = 0, \\ \widehat{\mathfrak{g}}_{-\frac{1}{2}\delta} &= \widehat{\mathfrak{g}}_{-\frac{2}{2}\delta} = 0. \end{aligned} \tag{4.19}$$

From the above discussion follow the first and the fourth assertions.

Proposition 4.3. *We have the following relations:*

1.

$$\left[\,\widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\alpha}\,,\,\widehat{\mathfrak{g}}_{\frac{n}{2}\delta+\beta}\,\right]_{\widehat{\mathfrak{g}}} \subset \widehat{\mathfrak{g}}_{\frac{m+n}{2}\delta+\alpha+\beta}\ , \tag{4.20}$$

$$\left[\,\widehat{\mathfrak{g}}_{\frac{m}{2}\delta}\,,\,\widehat{\mathfrak{g}}_{\frac{n}{2}\delta}\,\right]_{\widehat{\mathfrak{g}}} \subset \widehat{\mathfrak{g}}_{\frac{m+n}{2}\delta}\,. \tag{4.21}$$

2.

$$\left[\,\widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\alpha}\,,\,\widehat{\mathfrak{g}}_{-\frac{n+3}{2}\delta+\beta}\,\right]_{\widehat{\mathfrak{g}}} \subset \widehat{\mathfrak{g}}_{\frac{m-n-3}{2}\delta+\alpha+\beta}\,, \qquad \textit{for } m \leq n \textit{ or } m \geq n+3\,, \tag{4.22}$$

and for $n = m-1$ or $n = m-2$,

$$\left[\,\widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\alpha}\,,\,\widehat{\mathfrak{g}}_{-\frac{n+3}{2}\delta+\beta}\,\right]_{\widehat{\mathfrak{g}}} = 0\,. \tag{4.23}$$

3.

$$\left[\,\widehat{\mathfrak{g}}_{\frac{m}{2}\delta}\,,\,\widehat{\mathfrak{g}}_{-\frac{n+3}{2}\delta}\,\right]_{\widehat{\mathfrak{g}}} \subset \widehat{\mathfrak{g}}_{\frac{m-n-3}{2}\delta}\,, \qquad \textit{for } m \leq n \textit{ or } m > n+3, \tag{4.24}$$

$$\left[\,\widehat{\mathfrak{g}}_{\frac{m}{2}\delta}\,,\,\widehat{\mathfrak{g}}_{-\frac{m}{2}\delta}\,\right]_{\widehat{\mathfrak{g}}} \subset \widehat{\mathfrak{g}}_{0\delta} \oplus \widehat{\mathfrak{h}}\,, \tag{4.25}$$

and for $n = m-1$ or $n = m-2$,

$$\left[\,\widehat{\mathfrak{g}}_{\frac{m}{2}\delta}\,,\,\widehat{\mathfrak{g}}_{-\frac{n+3}{2}\delta}\,\right]_{\widehat{\mathfrak{g}}} \;=\; 0\,. \tag{4.26}$$

4.

$$\left[\,\widehat{\mathfrak{g}}_{-\frac{m+3}{2}\delta+\alpha}\,,\,\widehat{\mathfrak{g}}_{-\frac{n+3}{2}\delta+\beta}\,\right]_{\widehat{\mathfrak{g}}} \quad\subset\quad \widehat{\mathfrak{g}}_{(-\frac{m+n}{2}-3)\delta+\alpha+\beta}\;, \tag{4.27}$$

$$\left[\,\widehat{\mathfrak{g}}_{-\frac{m+3}{2}\delta}\,,\,\widehat{\mathfrak{g}}_{-\frac{n+3}{2}\delta}\,\right]_{\widehat{\mathfrak{g}}} \quad\subset\quad \widehat{\mathfrak{g}}_{(-\frac{m+n}{2}-3)\delta}\,. \tag{4.28}$$

Proof

Let $\phi\otimes X \in \widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\alpha}$ and $\psi\otimes Y \in \widehat{\mathfrak{g}}_{\frac{n}{2}\delta+\beta}$. Then we have, for $h\in\mathfrak{h}$,

$$\begin{aligned}[\,h,[\,\phi\otimes X,\psi\otimes Y]\,] &= -[\,\phi\otimes X,[\,\psi\otimes Y,h\,]\,]-[\,\psi\otimes Y,[\,h,\phi\otimes X]\,]\\ &=<\beta,h>[\,\phi\otimes X,\psi\otimes Y\,]+<\alpha,h>[\,\phi\otimes X,\psi\otimes Y\,]\\ &=<\alpha+\beta,h>[\,\phi\otimes X,\psi\otimes Y\,].\end{aligned}$$

On the other hand,

$$\begin{aligned}[\,d,[\,\phi\otimes X,\psi\otimes Y\,]\,] &= -[\,\phi\otimes X,[\psi\otimes Y,d]\,]-[\,\psi\otimes Y,[\,d,\phi\otimes X]\,]\\ &=\frac{m+n}{2}[\,\phi\otimes X,\psi\otimes Y\,]\,.\end{aligned}$$

Hence

$$[\,\widehat{h},[\,\phi\otimes X,\psi\otimes Y\,]\,] = \left\langle\frac{m+n}{2}\delta+\alpha+\beta\,,\,\widehat{h}\right\rangle[\,\phi\otimes X,\psi\otimes Y\,] \tag{4.29}$$

for any $\widehat{h}\in\widehat{\mathfrak{h}}$. Therefore

$$\left[\,\widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\alpha}\,,\,\widehat{\mathfrak{g}}_{\frac{n}{2}\delta+\beta}\,\right]_{\widehat{\mathfrak{g}}} \subset \widehat{\mathfrak{g}}_{\frac{m+n}{2}\delta+\alpha+\beta}\;, \tag{4.30}$$

The same calculation for $\phi\otimes H\in\widehat{\mathfrak{g}}_{\frac{m}{2}\delta}$ and $\psi\otimes H'\in\widehat{\mathfrak{g}}_{\frac{n}{2}\delta}$ yields

$$\left[\,\widehat{\mathfrak{g}}_{\frac{m}{2}\delta}\,,\,\widehat{\mathfrak{g}}_{\frac{n}{2}\delta}\,\right]_{\widehat{\mathfrak{g}}} \subset \widehat{\mathfrak{g}}_{\frac{m+n}{2}\delta}\,. \tag{4.31}$$

The rests are proved in the same way. The second assertion for the commutativity in case of $n=m-1$ or $n=m-2$ is proved by virtue of (4.19). $\square$

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