Singularities arising at conjugate loci in ellipsoids and certain Liouville manifolds

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"The last geometric statement of Jacobi" asserts:

The conjugate locus of a general point on the two-dimensional ellipsoid contains just four cusps (Vorlesungen über Dynamik).

Aim of this talk: To discuss singularities of the conjugate loci on higher dimensional ellipsoids and certain Liouville manifolds.

In particular, we shall show:

(1) *The conjugate locus of a general point contains just three connected components of singularities, each of which is a cuspidal edge* ($\dim > 2$).

(2) *At the end of the above cuspidal edges there appear* D_4^+ *Lagrangian singularities (of Arnold).*

Plan of this talk

1. Conjugate points, conjugate locus (review)
2. Two-dim case
3. Ellipsoids
4. Results
5. D_4^+ Langrangian singularity
6. Liouville manifold and its geodesics

1. Conjugate point, conjugate locus

M: a riemannian manifold, $\dim M = n$.

$\gamma_v(t)$: the geodesic with $\gamma_v(0) = p$, $\dot{\gamma}_v(0) = v \quad (p \in M,\ v \in U_pM)$.

$\mathrm{Exp} : T_pM \to M$ is defined by $\mathrm{Exp}\,(tv) = \gamma_v(t)$.

$\gamma_v(t_0)$ $(t_0 > 0)$ is called a *conjugate point* of p along γ_v if the differential

$$d\,\mathrm{Exp}_{\,t_0 v} : T_pM \to T_{\gamma_v(t_0)}M$$

is singular. In other words,

- There is a Jacobi field $Y(t) \not\equiv 0$ along $\gamma_v(t)$ such that $Y(0) = 0$ and $Y(t_0) = 0$.

Conjugate points of $p = \gamma_v(0)$ along $\gamma_v(t)$ are discrete; $\gamma_v(t_1)$, $\gamma_v(t_2)$, $\ldots (0 < t_1 \leq t_2 \leq \ldots)$, called the first conjugate point, the second $\ldots$, etc.. The multiplicity is less than or equal to $n - 1$.

The i-th *conjugate locus* of $p \in M$, denoted by $C_i(p)$, is the set of all i-th conjugate point of p along the geodesics emanating from p.

The term "conjugate locus" is usually used with the meaning of the first conjugate locus.

Example: The sphere of constant curvature S^n

$C_1(x) = \{-x\}$ for any $x \in S^n$; the multiplicity is $n - 1$ along any geodesic. In particular,

$$C_1(x) = C_2(x) = \cdots = C_{n-1}(x) \ .$$

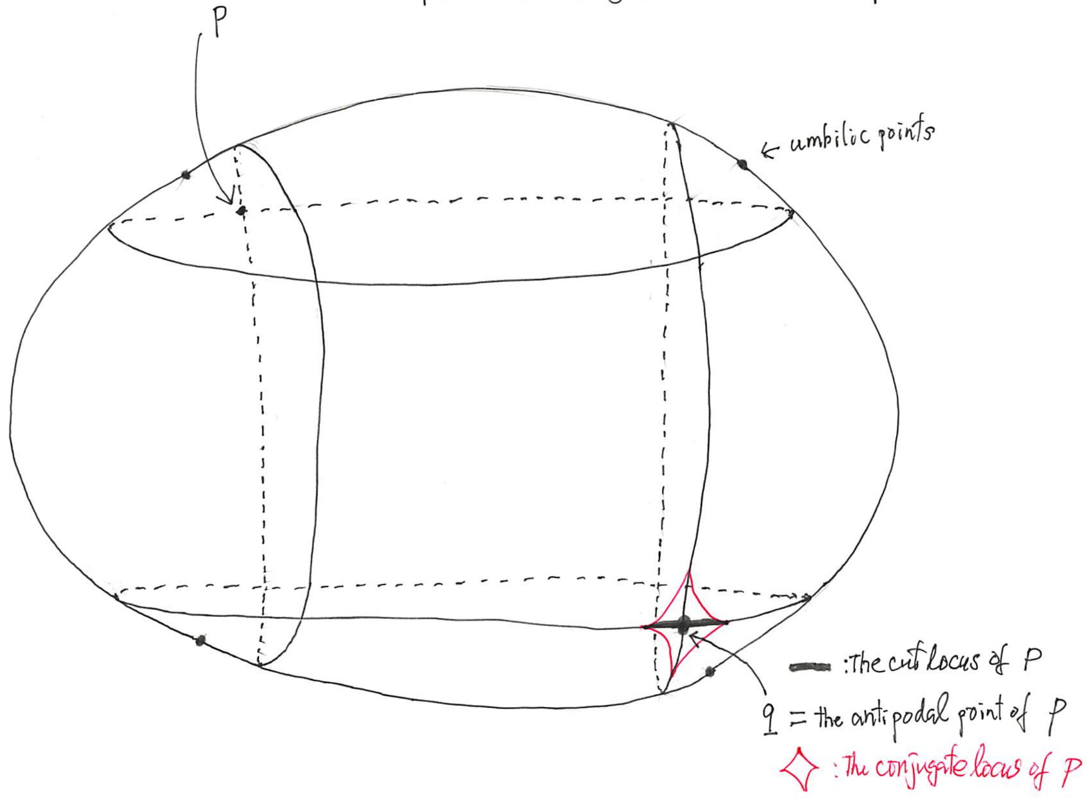
2. Two-dimensional case

- Two-dim. tri-axial ellipsoid S

For each non-umbilic point $p \in S$*, the first conjugate locus of* p *contains exactly four cusps, which are located on the coordinate lines of the elliptic coordinate system passing through the antipodal point of* p*.* (The cut locus of p is a segment of the curvature line whose end points are two of the above four conjugate points, see the next pictures.)

cf. J. Itoh, K. Kiyohara, The cut loci and the conjugate loci on ellipsoids, Manuscripta Math., **114** (2004), 247–264.

Two-dim. (tri-axial) Ellipsoid



3. Ellipsoids

Ellipsoid $M : \sum_{i=0}^{n} \frac{u_i^2}{a_i} = 1 \quad (0 < a_n < \cdots < a_0)$

Submanifolds N_k and J_k:

$$N_k = \{u = (u_0, \ldots, u_n) \in M \mid u_k = 0\} \qquad (0 \leq k \leq n)$$

$$J_k = \{u \in M \mid u_k = 0,\ \textstyle\sum_{i \neq k} \frac{u_i^2}{a_i - a_k} = 1\} \qquad (1 \leq k \leq n-1)$$

Then: (1) N_k is totally geodesic, codimension 1;

(2) $J_k \subset N_k$, J_k is diffeomorphic to $S^{k-1} \times S^{n-k-1}$;

(3) $\bigcup_k J_k$ is the set of points where some principal curvature with respect to the inclusion $M \subset \mathbb{R}^{n+1}$ has multiplicity ≥ 2;

(4) Denoting by $(\lambda_1, \dots, \lambda_n)$ the elliptic coordinate system on M such that $a_k \le \lambda_k \le a_{k-1}$,

$$N_k = \{\lambda_k = a_k \quad \text{or} \quad \lambda_{k+1} = a_k \},$$
$$J_k = \{\lambda_k = \lambda_{k+1} = a_k \}.$$

The elliptic coordinate system $(\lambda_1, \dots, \lambda_n)$ on M $(\lambda_n \le \cdots \le \lambda_1)$ is defined by the following identity in λ:

$$\sum_{i=0}^{n} \frac{u_i^2}{a_i - \lambda} - 1 = \frac{\lambda \prod_{k=1}^{n}(\lambda_k - \lambda)}{\prod_i (a_i - \lambda)}.$$

For a fixed $u \in M$, λ_k are determined by n "confocal quadrics" passing through u. From λ_k's, u_i are explicitly described as:

$$u_i^2 = \frac{a_i \prod_{k=1}^{n}(\lambda_k - a_i)}{\prod_{j \ne i}(a_j - a_i)}.$$

- **Elliptic coordinate system on a unit tangent space**

Let $p \in M$ be a general point, i.e., $p \notin N_i$ $(i \le i \le n-1)$. Let

$$v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial \lambda_i} \in T_p M.$$

Then, putting

$$\tilde{v}_i = \sqrt{\frac{(-1)^{n-i} \lambda_i \prod_{l \neq i} (\lambda_l - \lambda_i)}{(-1)^i \, 4 \prod_{j=0}^{n} (\lambda_i - a_j)}} \, v_i \, ,$$

we have an Euclidean coordinate system $(\tilde{v}_1, \ldots, \tilde{v}_n)$ on $T_p M$, i.e.,

$$g(v, v) = 1 \quad \text{if and only if} \quad \sum_{i=1}^{n} \tilde{v}_i^2 = 1 \, .$$

Define an elliptic coordinate system $(\mu_1, \ldots, \mu_{n-1})$ on the unit tangent space $U_pM \subset T_pM$ by the following identity in μ:

$$\sum_{i=1}^{n} \frac{\tilde{v}_i^2}{\mu - \lambda_i(p)} = \frac{\prod_{k=1}^{n-1}(\mu - \mu_k)}{\prod_{j=1}^{n}(\mu - \lambda_j(p))}, \quad \lambda_{i+1}(p) \le \mu_i \le \lambda_i(p)\,.$$

Then

$$\tilde{v}_i^2 = \frac{\prod_{k=1}^{n-1}(\lambda_i(p) - \mu_k)}{\prod_{j\neq i}(\lambda_i(p) - \lambda_j(p))}, \qquad \sum_{i=1}^{n-1} \tilde{v}_i^2 = 1\,.$$

Define the submanifolds (with boundary) $L_i^{\pm}$ $(1 \le i \le n-1)$ by

$$L_i^- = \{v \in U_pM \mid \mu_i(v) = \lambda_{i+1}(p)\}, \quad L_i^+ = \{v \in U_pM \mid \mu_i(v) = \lambda_i(p)\}\,.$$

They satisfy

$$L_{i-1}^- \cup L_i^+ = \text{ the great sphere } \{\tilde{v}_i = 0\}$$

$$L_i^- \simeq S^{i-1} \times \bar{D}^{n-1-i}, \quad L_i^+ \simeq \bar{D}^{i-1} \times S^{n-1-i}\,,$$

where S^k and $\bar{D}^k$ stand for k-sphere and closed k-disk respectively.

Also,

$$\partial L_i^+ = \partial L_{i-1}^- = L_i^+ \cap L_{i-1}^- \simeq S^{i-2} \times S^{n-1-i} \quad (2 \leq i \leq n-1)\,,$$

$$\partial L_{n-1}^- = \emptyset = \partial L_1^+\,.$$

• **Jacobi fields**

$V_i = \pm(\partial/\partial\mu_i)/\|\partial/\partial\mu_i\|$ $(1 \le i \le n-1)$, defined at each $v \in U_pM - \partial L_i^\pm$

$\gamma_v(t)$: the geodesic with $\dot\gamma_v(0) = v \in U_pM$

$Y_i(t,v)$ $(1 \le i \le n-1)$: the Jacobi field along $\gamma_v(t)$ with

$$Y_i(0,v) = 0, \quad Y_i'(0,v) = V_i(v)$$

Assume first that $v \notin \partial L_j^\pm$ for any j. Then the Jacobi field $Y_i(t,v)$ is of the form

$$Y_i(t,v) = y_i(t,v)\tilde{V}_i(t,v),$$

where $y_i(t,v)$ is a function and $\tilde{V}_i(t,v)$ is the parallel vector field along the geodesic $\gamma_v(t)$ such that $\tilde{V}_i(0,v) = V_i(v)$. (Actually, we may say $\tilde{V}_i(t,v) = V_i(\dot\gamma_v(t))$.)

$t = r_i(v)$: the first zero of $\quad t \mapsto y_i(t, v)$ for $t > 0$.

$r_i(v)$ can be continuously extended to all over U_pM and is of C^∞ outside $\partial L_i^{\pm}$.

4. Results

Our first result is:

Theorem A.

1. $r_{n-1}(v) \leq r_{n-2}(v) \leq \cdots \leq r_1(v)$ for any $v \in U_pM$.
2. $r_{i-1}(v) = r_i(v)$ if and only if $v \in \partial L_{i-1}^- = \partial L_i^+ \quad (2 \leq i \leq n-1)$

Put

$$\tilde{K}_i(p) = \{r_i(v)v \mid v \in U_pM\}, \quad K_i(p) = \{\gamma_v(r_i(v)) \mid v \in U_pM\}\,.$$

As a consequence of the above theorem:

Theorem B.

1. $K_{n-1}(p)$ is the (first) conjugate locus of p.
2. If M is close to the round sphere in an appropriate sense, then $K_{n-i}(p)$ is the i-th conjugate locus of p for $2 \leq i \leq n-1$.

The assumption in (2) of the above theorem is actually given as follows: "if the second zero, say $r^2_{n-1}(v)$, of $y_{n-1}(t, v)$ is greater than $r_1(v)$ for any $v \in U_pM$".

- **Singularities on the conjugate locus**

Define the mapping $\Phi : U_pM \to M$ by

$$\Phi(v) = \mathrm{Exp}_p(r_{n-1}(v)v) = \gamma_v(r_{n-1}(v))\,,$$

whose image is the conjugate locus $K_{n-1}(p)$ of p. Then:

Theorem C.

1. Φ is an immersion outside $L^-_{n-1} \cup L^+_{n-1}$.
2. The germ of Φ is a cuspidal edge at each point of L^-_{n-1} and each interior point of L^+_{n-1}; the restriction of Φ to (the interior of) $L^{\pm}_{n-1}$ is immersions to the edges of the vertices.

Remark. The restriction of Φ to L^-_{n-1} is actually an embedding and the image bounds the cut locus of p.

As for the singularities arising on the boundary ∂L^+_{n-1}, we need to treat them as the singularities of the mapping $\mathrm{Exp}_p : T_pM \to M$, since the mapping Φ is not differentiable at points on ∂L^+_{n-1} (and the tangential conjugate locus $\tilde{K}_{n-1}(p)$ is not smooth at $r_{n-1}(v)v$, $v \in \partial L^+_{n-1}$). However, it should be noted that the function r_{n-1} restricted to ∂L^+_{n-1} is smooth. Thus $\Lambda = \{r_{n-1}(v)v \,|\, v \in \partial L^+_{n-1}\}$ is a submanifold of T_pM diffeomorphic to $S^{n-3} \times S^0$.

Theorem D. The germ of the mapping $\mathrm{Exp}_p : T_pM \to M$ at each point $w \in \Lambda$ is a D_4^+ Lagrangian singularity.

Under the situation of **Theorem B** (2), we have the similar result for $K_{n-i}(p)$.

Theorem E. Suppose $K_{n-i}(p)$ $(2 \leq i \leq n-1)$ is the i-th conjugate locus of p. Then, defining the mapping $\Phi_{n-i} : U_pM \to M$ by

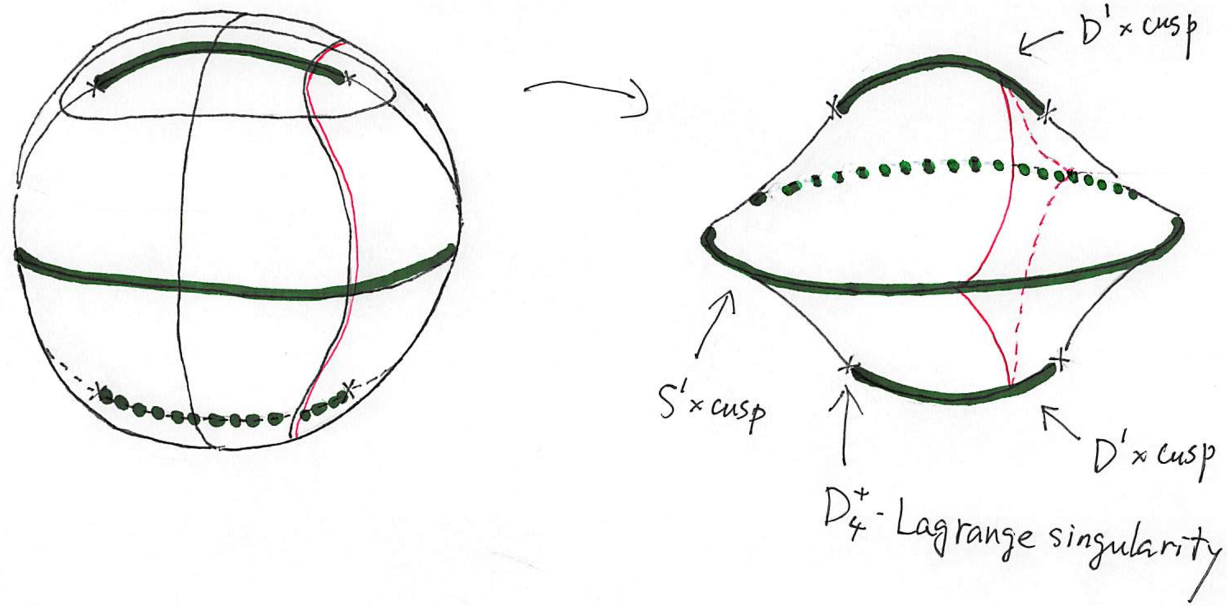
$$\Phi_{n-i}(v) = \mathrm{Exp}_p(r_{n-i}(v)v)\,,$$

we have:

1. Φ_{n-i} is an immersion outside $L^-_{n-i} \cup L^+_{n-i}$.
2. The germ of Φ_{n-i} is a cuspidal edge at each interior point of L^-_{n-i} and L^+_{n-i}; the restriction of Φ_{n-i} to the interior of $L^\pm_{n-i}$ is immersions to the edges of the vertices.
3. The germ of the mapping $\mathrm{Exp}_p : T^*_pM \to M$ at each point $r_{n-i}(v)v$, $v \in \partial L^\pm_{n-i}$, is a D^+_4 Lagrangian singularity and the restriction $\mathrm{Exp}_p|_{\partial L^\pm_{n-i}}$ is an immersion to the edge of vertices.

3-dim. case

$$U_pM \ni v \longmapsto \underset{\displaystyle \mathrm{Exp}_p(r(v)v)}{\overset{\displaystyle \gamma_v(r(v))}{\shortparallel}} \in C_1(p)$$

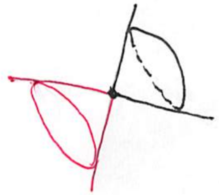


3-dim. case

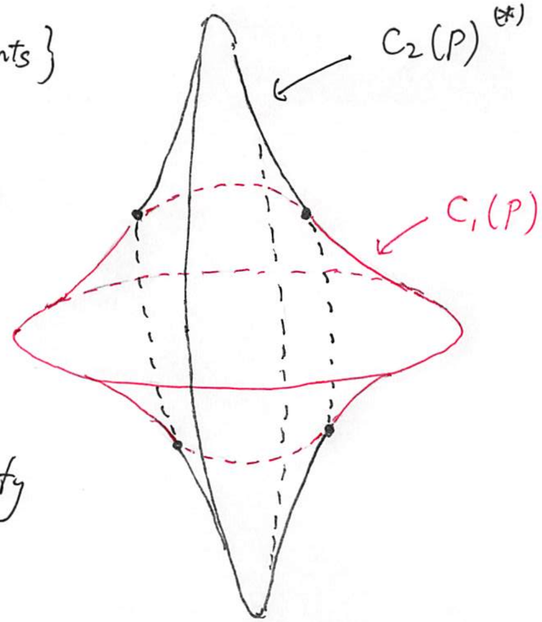
$M \supset C_1(p) \cup C_2(p)$

$T_pM \supset \widehat{C}_1(p) \cup \widehat{C}_2(p)$

around a point $\in \widehat{C}_1(p) \cap \widehat{C}_2(p) = \{4 \text{ points}\}$



$\xrightarrow{\mathrm{Exp}_p}$



- D_4^+ singularity

(*) if $M \approx$ round sphere

5. D_4^+ Lagrangian singularity

- **Lagrangian singularity**

  N: a manifold, $\quad L \subset T^*N$: a Lagrangian submanifold

  A *Lagrangian singularity* is a singularity of the map germ

$$(\pi \circ i) : (L, \lambda_0) \to (N, q_0)$$

Two such map-germs $(L, \lambda_0) \to (N, q_0)$ and $(\pi' \circ i') : (L', \lambda_0') \to (N', q_0')$ are said to be Lagrangian equivalent if

$$\exists \phi : (N, q_0) \to (N', q_0') \,, \quad \exists \Phi : (T^*N, \lambda_0) \to (T^*N', \lambda_0')$$

such that $\Phi(L, \lambda_0) = (L', \lambda'_0)$ and the diagram

$$\begin{array}{ccc} (T^*N, \lambda_0) & \xrightarrow{\Phi} & (T^*N', \lambda'_0) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ (N, q_0) & \xrightarrow[\phi]{} & (N', q'_0) \end{array}$$

is commutative. Actually, Φ is described as

$$\Phi(\lambda) = (\phi^*)^{-1}(\lambda) + dh_{\phi(\pi(\lambda))}\,, \quad \lambda \in T^*N$$

for some function h on N' in this case.

- **Generating family**

  Let $(L, \lambda_0) \subset T^*N$ and (N, q_0) be as above.

  $x = (x_1, \dots, x_n)$: a coordinate system on N; $\quad q_0 \leftrightarrow a$

  A function germ $F(u, x) = F(u_1, \dots, u_k, x_1, \dots, x_n)$ at $(b, a) \in \mathbb{R}^k \times \mathbb{R}^n$ is called a "generating family" for L at $\lambda_0 \in L$ if it satisfies

1. $0 \in \mathbb{R}^k$ is a regular value of the map

$$d_u F : (u, x) \mapsto (\partial F/\partial u_1, \dots, \partial F/\partial u_k)$$

   and $d_u F(b, a) = 0$. Thus $C = (d_u F)^{-1}(0)$ is a n-dimensional manifold and $(b, a) \in C$.

2. The map

$$d_xF : C \ni (u,x) \mapsto \sum_{l=1}^{n} (\partial F/\partial x_l)(u,x)\, dx_l \in T_x^* N \subset T^* N$$

gives an embedding of C into T^*N whose image is L and $d_xF(b,a) = \lambda_0$.

The number k satsifies

$$k \geq \dim \ker((\pi \circ i)_*)_{\lambda_0} : T_{\lambda_0} L \to T_{q_0} N\,.$$

If the equality holds, then the generating family is called *minimal*.

Let $G(v_1,\ldots,v_{k'},y_1,\ldots,y_n)$ with the base point (b',a') be another minimal generating family for a Lagrangian submanifold $(\tilde{L},\tilde{\lambda}_0)\subset T^*\tilde{N}$. Then those two minimal generating families are said to be $\mathcal{R}^+$-*equivalent* if $k'=k$ and there is a diffeomorphism $\Psi:\mathbb{R}^k\times\mathbb{R}^n\to\mathbb{R}^k\times\mathbb{R}^n$ $((b,a)\mapsto(b',a'))$ of the form

$$\Psi(u,x)=(\psi(u,x),\phi(x))$$

and a function $h(x)$ so that $F(u,x)=G(\Psi(u,x))+h(x)$. The following criterion is crucial

Theorem (Arnold) Two minimal generating families $F(u,x)$ and $G(v,y)$ are $\mathcal{R}^+$-equivalent if and only if the corresponding Lagrangian submanifolds $(L,\lambda_0)\subset T^*N$ and $(\tilde{L},\tilde{\lambda}_0)\subset T^*\tilde{N}$ are Lagrangian equivalent.

- **Versal deformation of a function germ**

  Let $F(u,x)$ be a function germ on $\mathbb{R}^k \times \mathbb{R}^n$ at (b,a) and put

$$f(u) = f(u_1, \ldots, u_k) = F(u,a).$$

Such F is called a *deformation* (or an *unfolding*) of the function germ $(f(u), b)$. We are interested in the case where $F(u,x)$ is a *versal* deformation of f.

Theorem (Mather) The function germ $(F(u,x),(b,a))$ is a versal deformation of the function germ $(f(u),b)$ if an only if the quotient space

$$\mathcal{E}_k \Big/ \left(\frac{\partial f}{\partial u_1},\ldots,\frac{\partial f}{\partial u_k}\right)$$

is spanned by elements represented by constant functions and

$$\frac{\partial F}{\partial x_j}(u,a) \quad (1 \leq j \leq n)$$

as a vector space.

Here $\mathcal{E}_k$ denotes the algebra of function germs in $(u_1,\ldots,u_k)$ at $u = b$ and $(\ldots,(\partial F/\partial x_j)(u,a),\ldots)$ stands for its ideal generated by $(\partial F/\partial x_j)(u,a)$ $(1 \leq j \leq n)$.

Theorem Let $(F(u,x),(b,a))$ and $(H(v,y),(b',a'))$ be two deformation germs on $\mathbb{R}^k \times \mathbb{R}^n$ of $f(u) = F(u,a)$ and $h(v) = H(u,a')$ respectively. Suppose F and H are versal deformations. Then the two deformation germs F and H are $\mathcal{R}^+$-equivalent if and only if the function germs $(f(u),b)$ and $(h(v),b')$ are equivalent, i.e., there is a diffeomorphism germ $\phi : (\mathbb{R}^k, b) \to (\mathbb{R}^k, b')$ and a constant $c \in \mathbb{R}$ such that $f = h \circ \phi + c$.

If $(F(u,x),(b,a))$ is a versal deformation of $(f(u),b)$, then it is known that the function germ $f(u)$ is *finitely determined*, i.e., there is a positive integer l such that any function germ $(h(u),b)$ whose l-jet is equal to the l-jet of $f(u)$ at b is equivalent to $(f(u),b)$. (In this case $(f(u),b)$ is said to be l-determined.) Therefore we have the following criterion for Lagrangian equivalence of Lagrangian singulaities.

Theorem Let $(F(u,x),(b,a))$, a function germ on $\mathbb{R}^k \times \mathbb{R}^n$, be a minimal generating family for a Lagrangian submanifold $(L,\lambda_0) \subset T^*N$. Suppose F is a versal deformation of $f(u) = F(u,a)$ at b and $f(u)$ is l-determined. Let $H(v,y),(b',a'))$ be another function germ on $\mathbb{R}^k \times \mathbb{R}^n$ and is a minimal generating family of a Lagrangian submanifold $(L',\lambda_0) \subset T^*N'$. Suppose also that H is a versal defromation of $h(v) = H(v,a')$ at b'. Then the Lagrangian singulality $\pi \circ i : (L,\lambda_0) \to (N,q_0)$ is Lagrangian equivalent to $\pi' \circ i : (L',\lambda_0') \to (N',q_0')$ if and only if there is a diffeomorphism germ $\phi : (\mathbb{R}^k, b) \to (\mathbb{R}^k, b')$ and a constant $c \in \mathbb{R}$ such that the l-jets of $h(\phi(u)) + c$ and $f(u)$ at b coincide.

- D_4^+ **singularity**

The equivalence class of the function germ $f(u_1, u_2) = u_1^3 + u_1 u_2^2$ at $0 \in \mathbb{R}^2$ is called the D_4^+-singularity. It is 3-determined and the quotient space

$$\mathcal{E}_2 / \left(\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2} \right)$$

is spanned by $1, u_1, u_2$, and u_2^2. Put

$$F(u, x_1, \ldots, x_n) = u_1^3 + u_1 u_2^2 + x_1 u_1 + x_2 u_2 + x_3 u_2^2 + \sum_{j=4}^{n} c_j x_j \,,$$

where $c_4, \ldots, c_n \in \mathbb{R}$. Then $(F(u, x), (0, 0))$ is a versal deformation of $(f(u), 0)$.

Putting

$$C = \{(u, x) \mid \partial F/\partial u_1 = \partial F/\partial u_2 = 0\},$$

we define a germ of a Lagrangian submanifold $(L, \lambda_0) \subset T^*\mathbb{R}^n$ as the imapge of the map

$$C \ni (u, x) \mapsto \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(u, x) dx_j \in T^*\mathbb{R}^n\,, \quad \lambda_0 = \frac{\partial F}{\partial x_j}(0, 0) dx_j\,.$$

Namely, $L \subset T^*\mathbb{R}^n = \{(x, \xi)\}$ is parametrized by $(u_1, u_2, x_3. \ldots, x_n)$ as

$$x_1 = -(3u_1^2 + u_2^2), \quad x_2 = -2(u_1 + x_3)u_2, \quad \xi = (u_1, u_2, u_2^2, c_4, \ldots, c_n)\,.$$

The Lagrangian equivalence class represented by

$$\pi \circ i : (L, \lambda_0) \to (\mathbb{R}^n, 0)$$

is called the D_4^+ Lagrangian singularity.

- D_4^+ singularity: $n = 3$, L_0: the image of $(x_3, \xi_1, \xi_2) \mapsto (x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) \in T^*\mathbb{R}^3$ (germ at $(x, \xi) = (0, 0)$)

such that

$$\begin{aligned} x_1 &= -(3\xi_1^2 + \xi_2^2)\,, \\ x_2 &= -(2\xi_1\xi_2 + 2x_3\xi_2)\,, \\ \xi_3 &= \xi_2^2\ . \end{aligned}$$

$F_0(x, \xi) = \xi_1^3 + \xi_1\xi_2^2 + x_1\xi_1 + x_2\xi_2 + x_3\xi_2^2$; a *generating function* of L_0

Then, for a Lagrangian submanifold germ $(L, (0, b)) \subset T^*\mathbb{R}^3$, $\pi : L \to \mathbb{R}^3$ *is called* D_4^+ *singularity* if there is a diffeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ ($\phi(0) = 0$) and a function germ $h(x)$ on $(\mathbb{R}^3, 0)$ ($(dh)_0 = b$) such that

L_0 is mapped to L by the diffeomorphism $\Phi : T^*\mathbb{R}^3 \to T^*\mathbb{R}^3$ given by

$$\Phi(x, \xi) = \phi^*(x, \xi) + (dh)_x$$

- Correspondence to the conjugate locus

  The singular locus of $\pi : L_0 \to \mathbb{R}^3 \Longleftrightarrow$ the conjugate locus

  Denote by f_0 the composed map

$$f_0 : (x_3, \xi_1, \xi_2) \mapsto (x, \xi) \in L_0 \xrightarrow{\pi} (x_1, x_2, x_3)$$

The Jacobian of f_0 is:

$\det(Df_0) = 3(2\xi_1 + x_3)^2 - 3x_3^2 - 4\xi_2^2$

Therefore the tangential conjugate locus in T_pM is represented by the cone:

$$3(2\xi_1 + x_3)^2 - 3x_3^2 - 4\xi_2^2 = 0$$

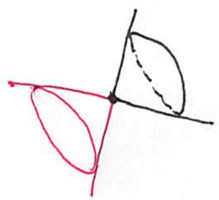
in the 3-dim. space $\{(x_3, \xi_1, \xi_2)\}$.

3-dim. case

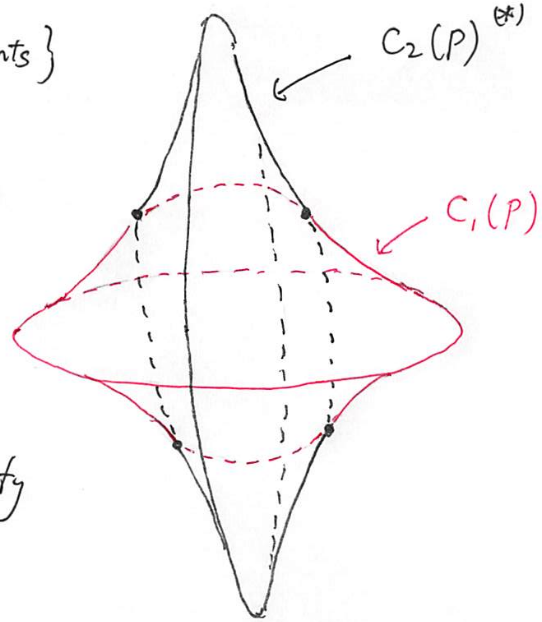
$T_pM \supset \widetilde{C}_1(p) \cup \widetilde{C}_2(p)$

$M \supset C_1(p) \cup C_2(p)$

around a point $\in \widetilde{C}_1(p) \cap \widetilde{C}_2(p) = \{4 \text{ points}\}$



$\xrightarrow{Exp_p}$



- D_4^+ singularity

(*) if $M \approx$ round sphere

6. Liouville manifold and its geodesics

M: n-dim. riemannian manifold

$\mathcal{F}$: n-dim. vector space of functions on T^*M

$(M, \mathcal{F})$ is called a Liouville manifold if it satisfies:

i) each $F \in \mathcal{F}$ is fiberwise a quadratic polynomial;

ii) those quadratic forms are simultaneously normalizable on each fiber;

iii) $\mathcal{F}$ is commutative with respect to the Poisson bracket;

iv) $\mathcal{F}$ contains the hamiltonian of the geodesic flow.

cf. K. Kiyohara, Two classes of Riemannian manifolds whose geodesic flows are integrable, Mem. Amer. Math. Soc., 130/619 (1997).

- Liouville manifold (restricted version, used here)

Constructed from

$$\begin{cases} \text{Constants} \quad a_0 > \cdots > a_n > 0 \\ \text{Function} \quad A(\lambda) > 0 \quad \text{on} \quad a_n \leq \lambda \leq a_0 \end{cases}$$

(1) Torus $R = \prod_{i=1}^{n} (\mathbb{R}/\alpha_i \mathbb{Z}) = \{(x_1, \ldots, x_n)\}$, where

$$\alpha_i = 2 \int_{a_i}^{a_{i-1}} \frac{A(\lambda)\, d\lambda}{\sqrt{(-1)^i \prod_{j=0}^{n} (\lambda - a_j)}}$$

$\tau_i : R \to R \ (1 \leq i \leq n-1)$

$\tau_i(x) = (x_1, \ldots, , -x_i, \frac{\alpha_{i+1}}{2} - x_{i+1}, \ldots, x_n)$

$G = < \tau_1, \ldots, \tau_{n-1} > \simeq \ (\mathbb{Z}/2\mathbb{Z})^{n-1}$

Branched cover $R \to R/G = M \simeq S^n$

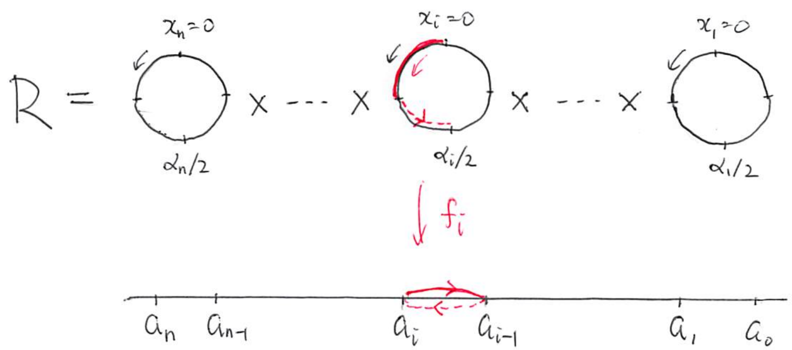
(2) Functions $f_i(x_i)$ on $\mathbb{R}/\alpha_i\mathbb{Z}$ are defined by

$$\left(\frac{df_i}{dx_i}\right)^2 = \frac{(-1)^i 4\prod_{j=0}^{n}(f_i - a_j)}{A(f_i)^2}$$

$$f_i(0) = a_i, \quad f_i(\frac{\alpha_i}{4}) = a_{i-1}$$

$$f_i(-x_i) = f_i(x_i) = f_i(\frac{\alpha_i}{2} - x_i)$$

The range of $f_i(x_i)$ is $a_i \leq f_i(x_i) \leq a_{i-1}$.



- Riemannian metric

$$g = \sum_{i=1}^{n} (-1)^{n-i} \prod_{l \neq i} (f_l(x_l) - f_i(x_i))\, dx_i^2$$

Examples:

$A(\lambda) = \sqrt{\lambda} \Rightarrow M$ is the ellipsoid $\displaystyle\sum_{i=0}^{n} \frac{u_i^2}{a_i} = 1$

$A(\lambda) =$ constant $\Rightarrow M$ is the sphere of constant curvature

• First integrals. Putting

$$b_{ij}(x_i) = \begin{cases} (-1)^i \displaystyle\prod_{\substack{1\le k\le n-1 \\ k\neq j}} (f_i(x_i) - a_k) & (1 \le j \le n-1) \\ (-1)^{i+1} \prod_{k=1}^{n-1} (f_i(x_i) - a_k) & (j = n), \end{cases}$$

we define functions $F_1, \ldots, F_n = 2E$ on the cotangent bundle by

$$\sum_{j=1}^{n} b_{ij}(x_i) F_j = \xi_i^2 \ ,$$

where ξ_i are the fiber coordinates.

F_i represent well-defined smooth functions on T^*M and E is the hamiltonian of the geodesic flow of (M, g).

Putting

$$\mathcal{F} = \mathrm{span}\{F_1, \ldots, F_n\} \ ,$$

$(M, \mathcal{F})$ becomes a Liouville manifold.

• The condition required in Theorems

The theorems hold for those Liouville manifolds such that, putting $C(\lambda) = (\lambda - a_n)A(\lambda)$,

$$(-1)^k C^{(k)}(\lambda) > 0 \quad \text{on } [a_n, a_0] \ \ (2 \leq k \leq n).$$

Clearly, ellipsoids ($A(\lambda) = \sqrt{\lambda}$) satisfy this condition.

Geodesics and Jacobi fields

- First integrals H_i $(1 \leq i \leq n-1)$: $H_i \leq H_{i-1}$,

$$-\sum_{j=1}^{n-1} \left(\prod_{\substack{1 \leq k \leq n-1 \\ k \neq j}} (\lambda - a_k) \right) F_j + \prod_{k=1}^{n-1} (\lambda - a_k)$$
$$= \prod_{l=1}^{n-1} (\lambda - H_l)$$

They are functions on U^*M (unit cotangent bundle) satisfying

$$a_{i+1} \leq H_i \leq a_{i-1}.$$

If

$$a_{i+1} < b_i < a_{i-1},\ b_i \neq a_i,\ b_i < b_{i-1}$$

for any i, then the subset of the unit cotangent bundle given by $H_i = b_i$ $(1 \leq i \leq n-1)$ is a disjoint union of smooth (Lagrange) tori.

Put

$$a_i^+ = \max\{a_i, b_i\}, \quad a_i^- = \min\{a_i, b_i\}$$
$$a_n^+ = a_n, \qquad a_0^- = a_0$$

Let $\gamma(t) = (x_1(t), \ldots, x_n(t))$ be a geodesic with $H_i = b_i$ $(1 \le i \le n-1)$. Then,

$$a_i^+ \le f_i(x_i(t)) \le a_{i-1}^- \quad (1 \le i \le n).$$

and $x_i(t) \in L_i$, where L_i is a connected component of $f_i^{-1}([a_i^+, a_{i-1}^-]) \subset \mathbb{R}/\alpha_i\mathbb{Z}$. Each L_i is an interval or the whole circle. Then

$$L = L_1 \times \cdots \times L_n \subset R$$

is injectively mapped to M by the quotient map $R \to M$, and is identical to the image of the Lagrange torus by the projection $\pi : T^*M \to M$.

• The geodesic equation for $(x_1(t), \ldots, x_n(t))$:

$$\sum_{i=1}^{n} \frac{(-1)^i G(f_i) A(f_i)\ |df_i(x_i(t))/dt|}{\sqrt{-\prod_{k=1}^{n-1}(f_i - b_k) \cdot \prod_{k=0}^{n}(f_i - a_k)}} = 0,$$

where $G(\lambda)$ is any polynomial of degree $\leq n-2$, and

$$\sum_{i=1}^{n} \frac{(-1)^i \tilde{G}(f_i) A(f_i)\ |df_i(x_i(t))/dt|}{\sqrt{-\prod_{k=1}^{n-1}(f_i - b_k) \cdot \prod_{k=0}^{n}(f_i - a_k)}} = 1,$$

where $\tilde{G}(\lambda)$ is a polynomial of degree $n-1$ and the coefficient of λ^{n-1} is equal to 1.

• Jacobi fields

Let $\gamma(t) = (x_1(t), \ldots, x_n(t))$ be a geodesic such that $a_{i+1} < b_i < a_{i-1}$ and $b_i < b_{i-1}$ for any i. Then:

Theorem *There is a unique (mod sign) parallel orthonormal frame $V_1(t), \ldots, V_{n-1}(t), \dot\gamma(t)$ of TM along $\gamma(t)$ such that any Jacobi field $Y(t)$ with $Y(t_0), Y'(t_0) \in \mathbb{R}V_i(t_0)$ at some t_0 is of the form $f(t)V_i(t)$ for any t.*

Namely, the vector space of Jacobi fields along $\gamma(t)$ which are orthogonal to $\dot\gamma(t)$ ($(2n-2)$-dim.) is a direct sum $\sum_{i=1}^{n-1} \mathcal{Y}_i$ of two-dim. subspaces, where any $Y(t) \in \mathcal{Y}_i$ is of the form $f(t)V_i(t)$.

$V_i(t)$ is proportional to $d\left(H_i|_{U^*_{\gamma(t)}M}\right)$.