Universality of Determinantal Point Processes on Riemannian Manifolds

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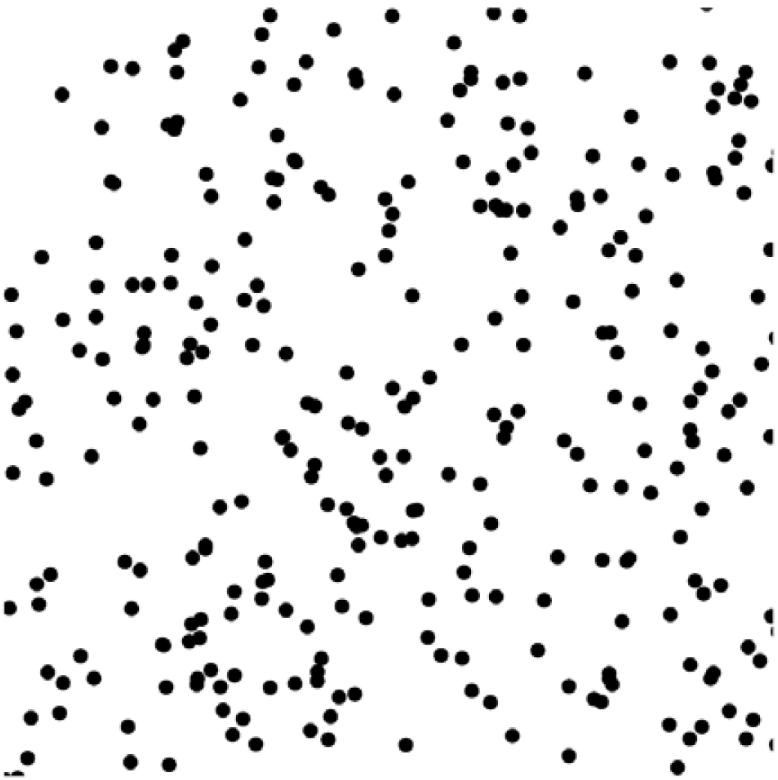


図１：Poisson 点過程．平面上の一様ランダムな点の配置であるが，実現した配置には粒子分布の空間的な粗密が見られる．

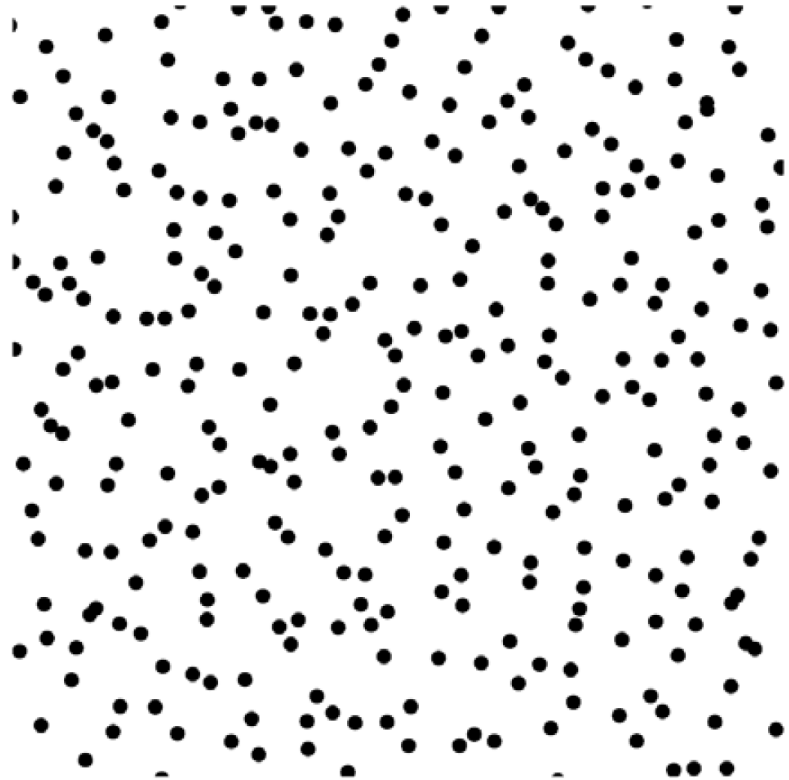
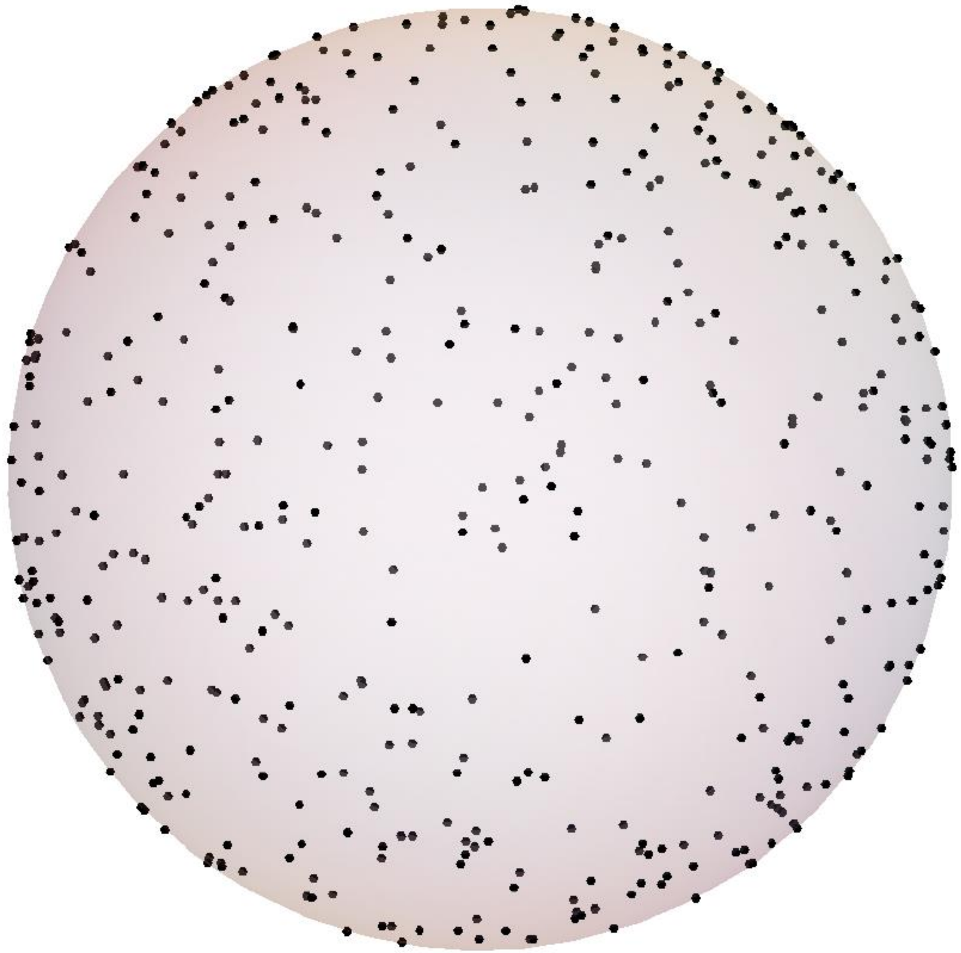
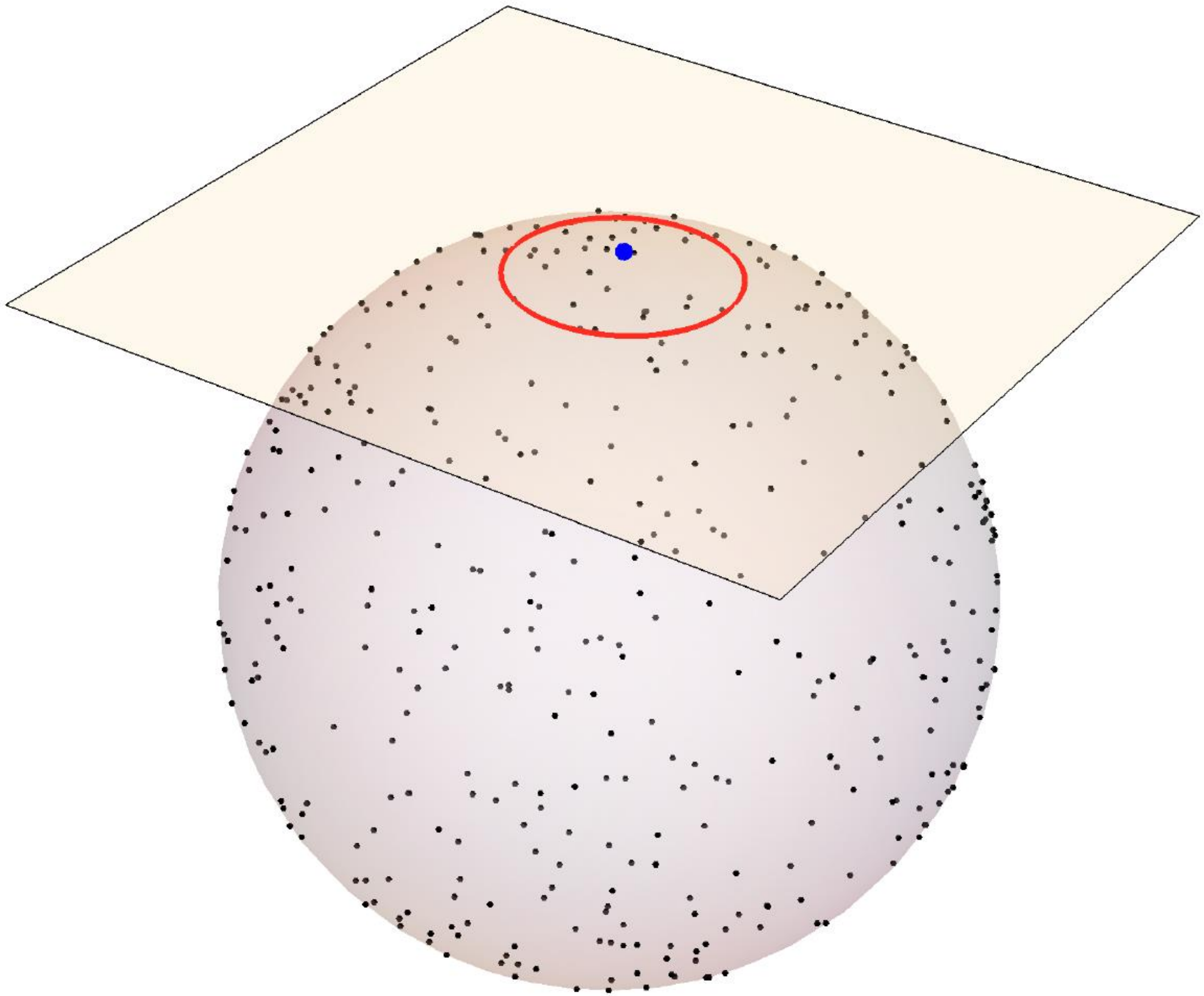
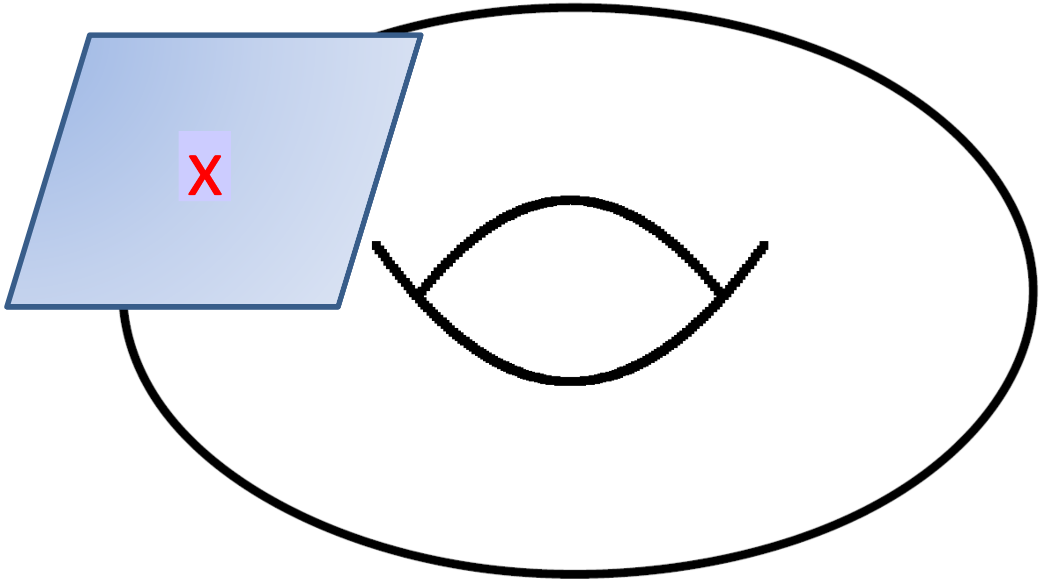


図２：複素ランダム行列の複素平面上の固有値分布として実現される Ginibre 点過程．粒子間に斥力相互作用が働き，棲み分けが実現している．（粒子数密度は図１と同じ．）



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Plan

1. Introduction
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3. DPPs on Sphere and Torus
4. Two Families of Universal DPPs in Arbitrary Dimensions

1. Introduction

- Let S be a base space, which is locally compact Hausdorff space with countable basis, and λ be a Radon measure on S.

- The configuration space over S is given by the set of nonnegative-integer-valued Radon measures;

$$\mathrm{Conf}(S) = \left\{ \xi = \sum_j \delta_{x_j} : x_j \in S,\ \Xi(\Lambda) < \infty \text{ for all bounded set } \Lambda \subset S \right\}.$$

- $\mathrm{Conf}(S)$ is equipped with the topological Borel σ-fields with respect to the vague topology. We say $\xi_n, n \in \mathbb{N} := \{1, 2, \dots\}$ converges to ξ in the vague topology, if

$$\int_S f(x)\xi_n(dx) \to \int_S f(x)\xi(dx), \quad \forall f \in \mathcal{C}_c(S),$$

where $\mathcal{C}_c(S)$ is the set of all continuous real-valued functions with compact support.

- A point process on S is a $\mathrm{Conf}(S)$-valued random variable $\Xi = \Xi(\cdot, \omega)$ on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. If $\Xi(\{x\}) \in \{0, 1\}$ for any point $x \in S$, then the point process is said to be simple.

- **Assume that $\Lambda_j, j = 1, 2, \ldots, m$, $m \in \mathbb{N}$ are disjoint bounded sets in S and $k_j \in \mathbb{N}_0, j = 1, 2, \ldots, m$ satisfy $\sum_{j=1}^{m} k_j = n \in \mathbb{N}_0$.**

- **A symmetric measure λ^n on S^n is called the n-th correlation measure, if it satisfies**

$$\mathbf{E}\left[\prod_{j=1}^{m} \frac{\Xi(\Lambda_j)!}{(\Xi(\Lambda_j) - k_j)!}\right] = \lambda^n(\Lambda_1^{k_1} \times \cdots \times \Lambda_m^{k_m}),$$

  where if $\Xi(\Lambda_j) - k_j \leq 0$, we interpret $\Xi(\Lambda_j)!/(\Xi(\Lambda_j) - k_j)! = 0$.

- **If λ^n is absolutely continuous with respect to the n-product measure $\lambda^{\otimes n}$, the Radon-Nikodym derivative $\rho^n(x_1, \ldots, x_n)$ is called the n-point correlation function with respect to the background measure λ;**

$$\lambda^n(dx_1, \ldots, dx_n) = \rho^n(x_1, \ldots, x_n)\lambda^{\otimes n}(dx_1, \ldots, dx_n).$$

- **Determinantal point process (DPP) is defined as follows.**

Definition 1.1 *A simple point process* Ξ *on* (S, λ) *is said to be a determinantal point process (DPP) with correlation kernel* $K : S \times S \mapsto \mathbb{C}$, *if it has correlation functions* $\{\rho^n\}_{n \geq 1}$, *and they are given by*

$$\rho^n(x_1, \ldots, x_n) = \det_{1 \leq j,k \leq n} [K(x_j, x_k)] \quad \textit{for every } n = 1, 2, \ldots, \textit{ and } x_1, \ldots, x_n \in S.$$

The triplet $(\Xi, K, \lambda(dx))$ *denotes the DPP,* $\Xi \in \mathrm{Conf}(S)$, *specified by the correlation kernel* K *with respect to the measure* $\lambda(dx)$.

- **If the correlation kernel K is of rank $N \in \mathbb{N}$, then the number of points is N a.s. If $N < \infty$ (resp. $N = \infty$), we call the system a finite DPP (resp. an infinite DPP).**

- **The density of points with respect to the background measure $\lambda(dx)$ is given by**

$$\rho(x) = \rho^1(x) = K(x, x).$$

- **The DPP is negatively correlated as shown by**

$$\rho^2(x, x') = \det \begin{bmatrix} K(x, x) & K(x, x') \\ K(x', x) & K(x', x') \end{bmatrix}$$
$$= K(x, x)K(x', x') - |K(x, x')|^2 \leq \rho(x)\rho(x'), \quad x, x' \in S,$$

provided that K is Hermitian.

- Let $L^2(S, \lambda)$ be an L^2-space.

- For operators $\mathcal{A}, \mathcal{B}$ on $L^2(S, \lambda)$, we write $\mathcal{A} \geq O$ if $\langle \mathcal{A}f, f \rangle_{L^2(S,\lambda)} \geq 0$ for any $f \in L^2(S, \lambda)$, and $\mathcal{A} \geq \mathcal{B}$ if $\mathcal{A} - \mathcal{B} \geq O$.

- For a compact subset $\Lambda \subset S$, the projection from $L^2(S, \lambda)$ to $L^2(\Lambda, \lambda)$ is denoted by $\mathcal{P}_\Lambda$.

- We say that the bounded Hermitian operator $\mathcal{A}$ on $L^2(S, \lambda)$ is said to be of locally trace class, if the restriction of $\mathcal{A}$ to each compact subset Λ, $\mathcal{A}_\Lambda := \mathcal{P}_\Lambda \mathcal{A} \mathcal{P}_\Lambda$, is of trace class; $\mathrm{Tr}\, \mathcal{A}_\Lambda < \infty$.

- The totality of locally trace class operators on $L^2(S, \lambda)$ is denoted by $\mathcal{I}_{1,\mathrm{loc}}(S, \lambda)$.

- It is known that [Soshnikov (2002), Shirai–Takahashi (2003)], if

$$\mathcal{K} \in \mathcal{I}_{1,\mathrm{loc}}(S, \lambda) \quad \text{and} \quad O \leq \mathcal{K} \leq I,$$

  where I is the identity operator, then we have a unique DPP on S with the determinantal correlation functions with the correlation kernel given by the integral kernel for $\mathcal{K}$.

- **In the present talk, we consider the case that**

$$\boxed{\mathcal{K}f = f \quad \textbf{for all } f \in (\ker \mathcal{K})^{\perp} \subset L^2(S, \lambda),}$$

where $(\ker \mathcal{K})^{\perp}$ denotes the orthogonal complement of the kernel of $\mathcal{K}$.

- **That is, $\mathcal{K}$ is an orthogonal projection.**
- **By definition, it is obvious that the condition $O \le \mathcal{K} \le I$ is satisfied.**
- **The purpose of the present talk is to propose useful methods to provide orthogonal projections $\mathcal{K}$ and DPPs whose correlation kernels are given by the integral kernels $K(x, x'), x, x' \in S$ of $\mathcal{K}$.**

- We consider a pair of Hilbert spaces, $H_\ell, \ell = 1, 2$, which are assumed to be realized as L^2-spaces, $L^2(S_\ell, \lambda_\ell)$, $\ell = 1, 2$.
- We introduce a linear operator $\mathcal{W}$ and its adjoint $\mathcal{W}^*$,

$$\mathcal{W} : H_1 \mapsto H_2, \quad \mathcal{W}^* : H_2 \mapsto H_1.$$

- We prove that if

(i) both of $\mathcal{W}$, $\mathcal{W}^*$ are partial isometries and

(ii) $\mathcal{W}^*\mathcal{W} \in \mathcal{I}_{1,\mathrm{loc}}(S_1, \lambda_1)$, $\mathcal{W}\mathcal{W}^* \in \mathcal{I}_{1,\mathrm{loc}}(S_2, \lambda_2)$.

then we have unique pair of DPPs, $(\Xi_\ell, K_\ell, \lambda_\ell)$, $\ell = 1, 2$.

- The pair of DPPs satisfies some useful duality relations.
- We assume that $\mathcal{W}$ admits an integral kernel W on $L^2(S_1, \lambda_1)$, and give practical setting of W which makes $\mathcal{W}$ and $\mathcal{W}^*$ satisfy the above two assumptions.

- **In order to demonstrate the class of DPPs obtained by our method is large enough to study a variety of DPPs and universal structures behind them, we show many examples of DPPs in one- and two-dimensional spaces.**

- **In particular, we use the symbols of classical and affine roots systems (*e.g.*, $A_{N-1}, B_N, C_N, D_N, N \in \mathbb{N}$) to classify finite DPPs.**

- **Several types of weak convergence theorems of finite DPPs to infinite DPPs are given.**

- **We will show that in the one-dimensional space, there are three universal DPPs with an infinite number of points specified by the correlation kernels,**

$$K_{\rm sinc}(x,x') = \frac{\sin(x-x')}{\pi(x-x')} = \frac{1}{2\pi}\int_{-1}^{1} e^{i\gamma(x-x')}d\gamma, \qquad x, x' \in \mathbb{R},$$

$$K_{\rm Bessel}^{(1/2)}(x,x') = \frac{\sin(x-x')}{\pi(x-x')} - \frac{\sin(x+x')}{\pi(x+x')} = \frac{1}{\pi}\int_{-1}^{1} \sin(\gamma x)\sin(\gamma x')d\gamma,$$

$$K_{\rm Bessel}^{(-1/2)}(x,x') = \frac{\sin(x-x')}{\pi(x-x')} + \frac{\sin(x+x')}{\pi(x+x')} = \frac{1}{\pi}\int_{-1}^{1} \cos(\gamma x)\cos(\gamma x')d\gamma, \qquad x, x' \in [0,\infty),$$

  where $i := \sqrt{-1}$.

- $K_{\rm sinc}$ **is usually called the sine kernel in random matrix theory, but it shall be called the sinc kernel.**

- $K_{\rm Bessel}^{(1/2)}$ **and** $K_{\rm Bessel}^{(-1/2)}$ **are special cases of the Bessel kernels** $K_{\rm Bessel}^{(\nu)}, \nu > -1$ **with indices** $\nu = 1/2$ **and** $-1/2$**, respectively.**

- **Note that** $K_{\rm sinc}(x,x') = \frac{1}{2}\left\{K_{\rm Bessel}^{(1/2)}(x,x') + K_{\rm Bessel}^{(-1/2)}(x,x')\right\}$, $x, x' \in [0,\infty)$.

- **Corresponding to the threefold, $K_{\rm sinc}$, $K_{\rm Bessel}^{(1/2)}$, $K_{\rm Bessel}^{(-1/2)}$, we also show the three universal DPPs on $\mathbb{C}$, whose correlation kernels are given by**

$$K_{\rm Ginibre}^{A}(x,x') = e^{x\overline{x'}} = \sum_{n=0}^{\infty} \frac{(x\overline{x'})^n}{n!},$$

$$K_{\rm Ginibre}^{C}(x,x') = \sinh(x\overline{x'}) = \sum_{n=0}^{\infty} \frac{(x\overline{x'})^{2n+1}}{(2n+1)!},$$

$$K_{\rm Ginibre}^{D}(x,x') = \cosh(x\overline{x'}) = \sum_{n=0}^{\infty} \frac{(x\overline{x'})^{2n}}{(2n)!}, \quad x, x' \in \mathbb{C},$$

where $\overline{x'}$ denotes the complex conjugate of x'.

- **$K_{\rm Ginibre}^{A}$ is known as the correlation kernel of the Ginibre ensemble in random matrix theory [Ginibre 1965] and $K_{\rm Ginibre}^{C}$ and $K_{\rm Ginibre}^{D}$ were studied in [K2019].**

- **Note that $K_{\rm Ginibre}^{A}(x,x') = K_{\rm Ginibre}^{C}(x,x') + K_{\rm Ginibre}^{D}(x,x')$, $x, x' \in \mathbb{C}$.**

- **Our method to generate DPPs is valid also in higher dimensional spaces.**
- **We will state that the DPP with the sinc kernel $K_{\rm sinc}$ is the lowest-dimensional ($d=1$) example of the one-parameter ($d \in \mathbb{N}$) family of DPPs on $\mathbb{R}^d$, whose correlation kernels are given by**

$$K^{(d)}_{\rm Euclid}(x,x') = \frac{1}{(2\pi)^{d/2}} \frac{J_{d/2}(||x-x'||_{\mathbb{R}^d})}{||x-x'||^{d/2}_{\mathbb{R}^d}} = \frac{1}{(2\pi)^d} \int_{\mathbb{B}^d} e^{i\gamma\cdot(x-x')} d\gamma, \quad x,x' \in \mathbb{R}^d,$$

where J_ν is the Bessel function of the first kind, $||x-x'||_{\mathbb{R}^d}$ is the Euclidean distance between x and x' in $\mathbb{R}^d$, and $\mathbb{B}^d$ is a unit ball in $\mathbb{R}^d$ centered at the origin.

- We also claim that the Ginibre ensemble is the lowest-dimensional example $(d=1)$ of another one-parameter $(d \in \mathbb{N})$ family of DPPs on $\mathbb{C}^d$, whose correlation kernel is given by

$$K^{(d)}_{\text{Heisenberg}}(x, x') = e^{x \cdot \overline{x'}}, \quad x, x' \in \mathbb{C}^d.$$

- We call these two families of DPPs the Euclidean family of DPPs and the Heisenberg family of DPPs, respectively, following the terminologies by Zelditch (2000).

2. Main Theorems

2.1 Isometry, partial isometry, and DPPs

- Let $H_\ell, \ell = 1, 2$ be separable Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_{H_\ell}$.
- For a linear operator $\mathcal{W} : H_1 \mapsto H_2$, the adjoint of $\mathcal{W}$ is defined as the operator $\mathcal{W}^* : H_2 \mapsto H_1$, such that

$$\langle \mathcal{W} f, g \rangle_{H_2} = \langle f, \mathcal{W}^* g \rangle_{H_1} \quad \text{for all } f \in H_1 \text{ and } g \in H_2.$$

- A linear operator $\mathcal{W}$ is called an isometry if

$$||\mathcal{W} f||_{H_2} = ||f||_{H_1} \quad \text{for all } f \in H_1.$$

- For $\mathcal{W}$ its kernel is denoted as $\ker \mathcal{W}$ and the orthogonal complement of $\ker \mathcal{W}$ is written as $(\ker \mathcal{W})^\perp$.
- A linear operator $\mathcal{W}$ is called a partial isometry, if

$$||\mathcal{W} f||_{H_2} = ||f||_{H_1} \quad \text{for all } f \in (\ker \mathcal{W})^\perp.$$

- For the partial isometry $\mathcal{W}$, $(\ker \mathcal{W})^\perp$ is called the initial space and the range of $\mathcal{W}$ is called the final space.

- **By definition,** $||\mathcal{W}f||^2_{H_2} = \langle \mathcal{W}f, \mathcal{W}f\rangle_{H_2} = \langle f, \mathcal{W}^*\mathcal{W}f\rangle_{H_1}$**. This implies the following.**

Lemma 2.1 *The linear operator* $\mathcal{W}$ *(resp.* $\mathcal{W}^*$*) is a partial isometry, if and only if* $\mathcal{W}^*\mathcal{W}$ *(resp.* $\mathcal{W}\mathcal{W}^*$*) is the identity on* $(\ker \mathcal{W})^\perp$ *(resp.* $(ker\mathcal{W}^*)^\perp$*).*

Assumption 1 **Both $\mathcal{W}$ and $\mathcal{W}^*$ are partial isometries.**

- **Under Assumption 1, the operator $\mathcal{W}^*\mathcal{W}$ (resp. $\mathcal{W}\mathcal{W}^*$) is the projection onto the initial space of $\mathcal{W}$ (resp. the final space of $\mathcal{W}$).**
- **Now we assume that H_1 and H_2 are realized as L^2-spaces, $L^2(S_1, \lambda_1)$ and $L^2(S_2, \lambda_2)$, respectively.**
- **We consider the case in which $\mathcal{W}$ admits an integral kernel $W : S_2 \times S_1 \mapsto \mathbb{C}$ such that**

$$(\mathcal{W}f)(y) = \int_{S_1} W(y,x)f(x)\lambda_1(dx), \quad f \in L^2(S_1, \lambda_1),$$

and then

$$(\mathcal{W}^*g)(x) = \int_{S_2} \overline{W(y,x)}g(y)\lambda_2(dy), \quad g \in L^2(S_2, \lambda_2).$$

- **We put the second assumption.**

Assumption 2 $\mathcal{W}^*\mathcal{W} \in \mathcal{I}_{1,\mathrm{loc}}(S_1, \lambda_1)$ **and** $\mathcal{W}\mathcal{W}^* \in \mathcal{I}_{1,\mathrm{loc}}(S_2, \lambda_2)$.

- **We have**

$$(\mathcal{W}^*\mathcal{W}f)(x) = \int_{S_1} K_{S_1}(x, x') f(x') \lambda_1(dx'), \qquad f \in L^2(S_1, \lambda_1),$$
$$(\mathcal{W}\mathcal{W}^*g)(y) = \int_{S_2} K_{S_2}(y, y') g(y') \lambda_2(dy'), \qquad g \in L^2(S_2, \lambda_2),$$

with the integral kernels,

$$K_{S_1}(x, x') = \int_{S_2} \overline{W(y, x)} W(y, x') \lambda_2(dy) = \langle W(\cdot, x'), W(\cdot, x) \rangle_{L^2(S_2, \lambda_2)},$$
$$K_{S_2}(y, y') = \int_{S_1} W(y, x) \overline{W(y', x)} \lambda_1(dx) = \langle W(y, \cdot), W(y', \cdot) \rangle_{L^2(S_1, \lambda_1)}.$$

- **We see that** $\overline{K_{S_1}(x', x)} = K_{S_1}(x, x')$ **and** $\overline{K_{S_2}(y', y)} = K_{S_2}(y, y')$.

- **The main theorem is the following.**

Theorem 2.2 *Under Assumptions 1 and 2, associated with* $\mathcal{W}^*\mathcal{W}$ *and* $\mathcal{W}\mathcal{W}^*$*, there exists unique pair of DPPs;* $(\Xi_1, K_{S_1}, \lambda_1(dx))$ *on* S_1 *and* $(\Xi_2, K_{S_2}, \lambda_2(dy))$ *on* S_2*. The correlation kernels* $K_{S_\ell}, \ell = 1, 2$ *are Hermitian and given by*

$$K_{S_1}(x, x') = \int_{S_2} \overline{W(y, x)} W(y, x') \lambda_2(dy) = \langle W(\cdot, x'), W(\cdot, x) \rangle_{L^2(S_2, \lambda_2)},$$
$$K_{S_2}(y, y') = \int_{S_1} W(y, x) \overline{W(y', x)} \lambda_1(dx) = \langle W(y, \cdot), W(y', \cdot) \rangle_{L^2(S_1, \lambda_1)}.$$

- **Note that the densities of the DPPs,** $(\Xi_1, K_{S_1}, \lambda_1(dx))$ **and** $(\Xi_2, K_{S_2}, \lambda_2(dy))$**, are given by**

$$\rho_1(x) = K_{S_1}(x, x) = \int_{S_2} |W(y, x)|^2 \lambda_2(dy) = ||W(\cdot, x)||_{L^2(S_2, \lambda_2)}, \quad x \in S_1,$$
$$\rho_2(y) = K_{S_2}(y, y) = \int_{S_1} |W(y, x)|^2 \lambda_1(dx) = ||W(y, \cdot)||_{L^2(S_1, \lambda_1)}, \quad y \in S_2,$$

with respect to the background measures $\lambda_1(dx)$ **and** $\lambda_2(dy)$**, respectively.**

2.2 Basic properties of DPPs

- **For** $v = (v^{(1)}, \dots, v^{(d)}) \in \mathbb{R}^d$, $y = (y^{(1)}, \dots, y^{(d)}) \in \mathbb{R}^d$, $d \in \mathbb{N}$, **the inner product of them is given by** $v \cdot y = y \cdot v := \sum_{a=1}^{d} v^{(a)} y^{(a)}$, **and** $|v|^2 := v \cdot v$.
- **When** $S \subset \mathbb{C}^d, d \in \mathbb{N}$, $x \in S$ **has** d **complex components;** $x = (x^{(1)}, \dots, x^{(d)})$ **with** $x^{(a)} = \Re x^{(a)} + i \Im x^{(a)}$, $a = 1, \dots, d$.
- **In order to describe clearly such a complex structure, we set** $x_{\mathrm{R}} = (\Re x^{(1)}, \dots, \Re x^{(d)}) \in \mathbb{R}^d$, $x_{\mathrm{I}} = (\Im x^{(1)}, \dots, \Im x^{(d)}) \in \mathbb{R}^d$, **and write** $x = x_{\mathrm{R}} + i x_{\mathrm{I}}$.
- **The Lebesgue measure is written as** $dx = dx_{\mathrm{R}} dx_{\mathrm{I}} := \prod_{a=1}^{d} d\Re x^{(a)} d\Im x^{(a)}$. **The complex conjugate of** $x = x_{\mathrm{R}} + i x_{\mathrm{I}}$ **is defined as** $\overline{x} = x_{\mathrm{R}} - i x_{\mathrm{I}}$.
- **For** $x = x_{\mathrm{R}} + i x_{\mathrm{I}}$, $x' = x'_{\mathrm{R}} + i x'_{\mathrm{I}} \in \mathbb{C}^d$, **we use the Hermitian inner product;**

$$x \cdot \overline{x'} := (x_{\mathrm{R}} + i x_{\mathrm{I}}) \cdot (x'_{\mathrm{R}} - i x'_{\mathrm{I}}) = (x_{\mathrm{R}} \cdot x'_{\mathrm{R}} + x_{\mathrm{I}} \cdot x'_{\mathrm{I}}) - i(x_{\mathrm{R}} \cdot x'_{\mathrm{I}} - x_{\mathrm{I}} \cdot x'_{\mathrm{R}})$$

  and define

$$|x|^2 := x \cdot \overline{x} = |x_{\mathrm{R}}|^2 + |x_{\mathrm{I}}|^2, \quad x \in \mathbb{C}^d.$$

- **For** $(\Xi, K, \lambda(dx))$ **on** $S = \mathbb{R}^d$ **or** $S = \mathbb{C}^d$**, we introduce the following operations.**

  (shift) **For** $u \in S$**,** $\tau_u \Xi := \sum_j \delta_{x_j+u}$**,**

$$\tau_u K(x, x') = K(x+u, x'+u),$$

  and $\tau_u \lambda(dx) = \lambda(u+dx)$**. We write** $(\tau_u \Xi, \tau_u K, \tau_u \lambda(dx))$ **simply as** $\tau_u(\Xi, K, \lambda(dx))$**.**

  (Dilatation) **For** $c > 0$**, we set** $c \circ \Xi := \sum_j \delta_{cx_j}$

$$c \circ K(x, x') := K\left(\frac{x}{c}, \frac{x'}{c}\right), \quad x, x' \in cS,$$

  and $c \circ \lambda(dx) := \lambda(dx/c)$**. We define** $c \circ (\Xi, K, \lambda(dx)) := (c \circ \Xi, c \circ K, c \circ \lambda(dx))$**.**

  (Squared) **For** $(\Xi, K, \lambda(dx)$ **on** $S = \mathbb{R}$**, we put** $\Xi^{\langle 2 \rangle} = \sum_j \delta_{x_j^2}$**,** $K^{\langle 2 \rangle}(x, x') = K(x^2, x'^2)$**, and** $\lambda^{\langle 2 \rangle}(dx) = \lambda(dx^2)$**. We define** $(\Xi, K, \lambda(dx))^{\langle 2 \rangle} := (\Xi^{\langle 2 \rangle}, K^{\langle 2 \rangle}, \lambda^{\langle 2 \rangle}(dx))$ **on** $[0, \infty)$**.**

  (Gauge transformation) **For** $u : S \mapsto \mathbb{C}$**, a gauge transformation of** K **by** u **is defined as**

$$K(x, x') \mapsto \widetilde{K}_u := u(x) K(x, x') u(x')^{-1}.$$

  In particular, when $u : S \mapsto \mathrm{U}(1)$**, the** $\mathrm{U}(1)$**-gauge transformation of** K **is given by**

$$K(x, x') \mapsto \widetilde{K}_u := u(x) K(x, x') \overline{u(x')}.$$

- **We will use the following basic properties of DPP.**

  [Gauge invariance] **For any $u : S \mapsto \mathbb{C}$, a gauge transformation does note change the probability law of DPP;**

$$(\Xi, K, \lambda(dx)) \stackrel{\text{(law)}}{=} (\Xi, \widetilde{K}_u, \lambda(dx)).$$

  [Measure change] **For a measurable function $g : S \mapsto [0, \infty)$,**

$$(\Xi, K(x, x'), g(x)\lambda(dx)) \stackrel{\text{(law)}}{=} (\Xi, \sqrt{g(x)}K(x, x')\sqrt{g(x')}, \lambda(dx)).$$

  [Mapping and Scaling] **For a one-to-one measurable mapping $h : S \mapsto \widehat{S}$, if we set**

$$\widehat{\Xi} = \sum_j \delta_{h(x_j)}, \quad \widehat{K}(x, x') = K(h^{-1}(x), h^{-1}(y)), \quad \widehat{\lambda}(dx) = \lambda(h^{-1}(dx)),$$

  then $(\widehat{\Xi}, \widehat{K}, \widehat{\lambda}(dx))$ is a DPP on $\widehat{S}$.
  In particular, when $h(x) = cx, c > 0$, $(\widehat{\Xi}, \widehat{K}, \widehat{\lambda}(dx)) = c\circ(\Xi, K, \lambda(dx))$. If $c\circ\lambda(dx) = c^{-d}\lambda(dx)$, then

$$c \circ (\Xi, K, \lambda(dx)) \stackrel{\text{(law)}}{=} (c \circ \Xi, K_c, \lambda(dx)), \quad c > 0,$$

  with

$$K_c(x, x') = \frac{1}{c^d} K\left(\frac{x}{c}, \frac{x'}{c}\right),$$

  where the base space is given by cS.

2.3 Orthogonal functions and correlation kernels

- **In addition to $L^2(S_\ell, \lambda_\ell)$, $\ell = 1, 2$, we introduce $L^2(\Gamma, \nu)$ as a parameter space for functions in $L^2(S_\ell, \lambda_\ell), \ell = 1, 2$.**
- **We put the following.**

Assumption 3 There are two sets of functions $\{\psi_\ell(\cdot, \gamma) \in L^2(S_\ell, \lambda_\ell) : \gamma \in \Gamma\}$, $\ell = 1, 2$, which satisfy the following.
(i) The orthonormality relations hold,

$$\langle \psi_\ell(\cdot, \gamma), \psi_\ell(\cdot, \gamma') \rangle_{L^2(S_\ell, \lambda_\ell)} \nu(d\gamma) = \delta(\gamma - \gamma') d\gamma, \quad \gamma, \gamma' \in \Gamma, \quad \ell = 1, 2.$$

(ii) $\psi_1(x, \cdot) \in L^2(\Gamma, \nu)$ for λ_1-a.e.$x \in S_1$, and $\psi_2(y, \cdot) \in L^2(\Gamma, \nu)$ for λ_2-a.e.$y \in S_2$.

- **Under Assumption 3, we set**

$$W(y,x) = \int_\Gamma \overline{\psi_1(x,\gamma)}\psi_2(y,\gamma)\nu(d\gamma) = \langle \psi_2(y,\cdot), \psi_1(x,\cdot)\rangle_{L^2(\Gamma,\nu)}.$$

- **The following is obtained as a corollary of Theorem 2.2.**

Corollary 2.3 *Under Assumption 3, if we set W as above, then there exist unique pair of DPPs; $(\Xi_1, K_{S_1}, \lambda_1(dx))$ on S_1 and $(\Xi_2, K_{S_2}, \lambda_2(dy))$ on S_2. Here the correlation kernels $K_{S_\ell}, \ell = 1, 2$ are given by*

$$K_{S_1}(x,x') = \int_\Gamma \psi_1(x,\gamma)\overline{\psi_1(x',\gamma)}\nu(d\gamma) = \langle \psi_1(x,\cdot), \psi_1(x',\cdot)\rangle_{L^2(\Gamma,\nu)},$$
$$K_{S_2}(y,y') = \int_\Gamma \psi_2(y,\gamma)\overline{\psi_2(y',\gamma)}\nu(d\gamma) = \langle \psi_2(y,\cdot), \psi_2(y',\cdot)\rangle_{L^2(\Gamma,\nu)}.$$

- **Now we consider a simplified version of the above setting of W.**
- **Let $\Gamma \subseteq S_2$ and $\nu = \lambda_2$. In the above setting of W, we put**

$$\psi_2(y,\gamma)\nu(d\gamma) = \delta(y-\gamma)d\gamma, \quad y \in S_2, \quad \gamma \in \Gamma,$$

and hence

$$W(y,x) = \overline{\psi_1(x,y)}\mathbf{1}_\Gamma(y).$$

- **Assumption 3 is replaced by the following.**

Assumption 3' **For $\Gamma \subseteq S_2$, there is a set of functions $\{\psi_1(\cdot,y) \in L^2(S_1,\lambda_1) : y \in \Gamma\}$ which satisfies the following.**
(i) The orthonormality relation holds,

$$\langle \psi_1(\cdot,y), \psi_1(\cdot,y')\rangle_{L^2(S_1,\lambda_1)}\lambda_2(dy) = \delta(y-y')dy, \quad y,y' \in \Gamma.$$

(ii) $\psi_1(x,\cdot) \in L^2(\Gamma,\lambda_2)$, λ_1-a.e. $x \in S_1$.

- **Corollary 2.3 is reduced to the following.**

Corollary 2.4 *Under Assumption 3', if we consider W in the simplified setting, then there exists unique DPP, (Ξ, K, λ_1) on S_1 with the correlation kernel*

$$K_{S_1}(x,x') = \int_\Gamma \psi_1(x,y)\overline{\psi_1(x',y)}\lambda_2(dy).$$

2.4 Simple examples
(i) DPP with sinc kernel

- **We set** $S_1 = \mathbb{R}$, $\lambda_1(dx) = dx$, $\Gamma = (-1, 1)$, $\nu(dy) = \lambda_2(dy) = dy$, **and put**

$$\psi_1(x, y) = \frac{1}{\sqrt{2\pi}} e^{ixy}.$$

- **The correlation kernel** K_{S_1} **is given by**

$$K_{\rm sinc}(x, x') = \frac{1}{2\pi} \int_{-1}^{1} e^{iy(x-x')} dy = \frac{\sin(x - x')}{\pi(x - x')}, \quad x, x' \in \mathbb{R}.$$

(ii) Three types of Ginibre ensembles

- **Let $S = \mathbb{C}$ with $\lambda(dx) = \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx)$, where $\lambda_{\mathrm{N}(m,\sigma^2);\mathbb{C})}(dx)$ denotes the complex normal distribution,**

$$\lambda_{\mathrm{N}(m,\sigma;\mathbb{C})}(dx) := \frac{1}{\pi\sigma^2} e^{-|x-m|^2/\sigma^2} dx$$
$$= \frac{1}{\pi\sigma^2} e^{-(x_{\mathrm{R}}-m_{\mathrm{R}})^2/\sigma^2-(x_{\mathrm{I}}-m_{\mathrm{I}})^2/\sigma^2} dx_{\mathrm{R}} dx_{\mathrm{I}},$$

$m \in \mathbb{C}, m_{\mathrm{R}} := \Re m, m_{\mathrm{I}} := \Im m, \sigma > 0$.

- **We put**

$$\psi^A(x,\gamma) = e^{-(x_{\mathrm{R}}^2-x_{\mathrm{I}}^2)/2+2x\gamma},$$
$$\psi^C(x,\gamma) = \sqrt{2}\sinh(2x\gamma)e^{-(x_{\mathrm{R}}^2-x_{\mathrm{I}}^2)/2},$$
$$\psi^D(x,\gamma) = \sqrt{2}\cosh(2x\gamma)e^{-(x_{\mathrm{R}}^2-x_{\mathrm{I}}^2)/2}.$$

- **It is easy to confirm that**

$$\frac{1}{\pi}\int_{\mathbb{R}} \psi^A(x,\gamma)\overline{\psi^A(x,\gamma')}e^{-x_{\rm I}^2}dx_{\rm I} = e^{-(x_{\rm R}^2-4x_{\rm R}\gamma)}\delta(\gamma-\gamma'),$$

$$\frac{1}{\pi}\int_{\mathbb{R}} \psi^R(x,\gamma)\overline{\psi^R(x,\gamma')}e^{-x_{\rm I}^2}dx_{\rm I} = e^{-x_{\rm R}^2}\cosh(4x_{\rm R}\gamma)\times\begin{cases}\delta(\gamma-\gamma')-\delta(\gamma+\gamma'), & R=C,\\ \delta(\gamma-\gamma')+\delta(\gamma+\gamma'), & R=D.\end{cases}$$

- **Therefore, we have**

$$\langle\psi^A(\cdot,\gamma),\psi^A(\cdot,\gamma')\rangle_{L^2(\mathbb{C},\lambda_{\rm N(0,1;\mathbb{C})})}\nu(d\gamma)=\delta(\gamma-\gamma')d\gamma, \quad \gamma,\gamma'\in\Gamma^A:=\mathbb{R},$$

$$\langle\psi^R(\cdot,\gamma),\psi^R(\cdot,\gamma')\rangle_{L^2(\mathbb{C},\lambda_{\rm N(0,1;\mathbb{C})})}\nu(d\gamma)=\delta(\gamma-\gamma')d\gamma, \quad \gamma,\gamma'\in\Gamma^R:=(0,\infty), \quad R=C,D,$$

with $\nu(d\gamma)=\lambda_{\rm N(0,1/4)}(d\gamma)$**, where** $\lambda_{\rm N(m,\sigma^2)}(dx)$ **denotes the normal distribution,**

$$\lambda_{\rm N(m,\sigma^2)}(dx)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-m)^2/(2\sigma^2)}dx, \quad m\in\mathbb{R}, \quad \sigma>0.$$

- **Then we can apply Corollaries 2.3 or 2.4.**
- **The obtained kernels are given as**

$$K^A(x,x') = \sqrt{\frac{2}{\pi}} e^{-\{(x_{\mathrm{R}}^2 - x_{\mathrm{I}}^2) + ({x'_{\mathrm{R}}}^2 - {x'_{\mathrm{I}}}^2)\}/2} \int_{-\infty}^{\infty} e^{-2\{\gamma^2 - (x + \overline{x'})\gamma\}} d\gamma,$$

$$K^C(x,x') = 2\sqrt{\frac{2}{\pi}} e^{-\{(x_{\mathrm{R}}^2 - x_{\mathrm{I}}^2) + ({x'_{\mathrm{R}}}^2 - {x'_{\mathrm{I}}}^2)\}/2} \int_{0}^{\infty} e^{-2\gamma^2} \sinh(2x\gamma) \sinh(2\overline{x'}\gamma) d\gamma,$$

$$K^C(x,x') = 2\sqrt{\frac{2}{\pi}} e^{-\{(x_{\mathrm{R}}^2 - x_{\mathrm{I}}^2) + ({x'_{\mathrm{R}}}^2 - {x'_{\mathrm{I}}}^2)\}/2} \int_{0}^{\infty} e^{-2\gamma^2} \cosh(2x\gamma) \cosh(2\overline{x'}\gamma) d\gamma.$$

- **The integrals are performed and we obtain**

$$K^R(x,x') = e^{ix_{\mathrm{R}}x_{\mathrm{I}}} K^R_{\mathrm{Ginibre}}(x,x') e^{-ix'_{\mathrm{R}}x'_{\mathrm{I}}}, \qquad R = A, C, D,$$

with

$$K^A_{\mathrm{Ginibre}}(x,x') = e^{x\overline{x'}},$$
$$K^C_{\mathrm{Ginibre}}(x,x') = \sinh(x\overline{x'}),$$
$$K^D_{\mathrm{Ginibre}}(x,x') = \cosh(x\overline{x'}), \qquad x, x' \in \mathbb{C}.$$

- **Due to the gauge invariance of DPP mentioned above, the obtained three types of infinite DPPs on $\mathbb{C}$ are written as $(\Xi, K^R_{\mathrm{Ginibre}}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx))$, $R = A, C, D$.**

- The DPP, $(\Xi, K^A_{\mathrm{Ginibre}}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx))$ describes the eigenvalue distribution of the Gaussian random complex matrix in the bulk scaling limit, which is called the complex Ginibre ensemble[Ginibre(1965)].

- This is uniform on $\mathbb{C}$ with the density

$$\rho_{\mathrm{Ginibre}}(x)dx = K^A_{\mathrm{Ginibre}}(x,x)\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx) = \frac{1}{\pi} dx_{\mathrm{R}} dx_{\mathrm{I}}, \quad x \in \mathbb{C}.$$

- On the other hands, the Ginibre DPPs of types C and D are rotationally symmetric around the origin, but non-uniform on $\mathbb{C}$.

- The density profiles are given by

$$\rho^C_{\mathrm{Ginibre}}(x)dx = K^C_{\mathrm{Ginibre}}(x,x)\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx) = \frac{1}{2\pi}(1 - e^{-2|x|^2}) dx_{\mathrm{R}} dx_{\mathrm{I}}, \quad x \in \mathbb{C},$$
$$\rho^D_{\mathrm{Ginibre}}(x)dx = K^D_{\mathrm{Ginibre}}(x,x)\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx) = \frac{1}{2\pi}(1 + e^{-2|x|^2}) dx_{\mathrm{R}} dx_{\mathrm{I}}, \quad x \in \mathbb{C}.$$

- They were first obtained in [K(2019)] by taking the limit $W \to \infty$ keeping the density of points of the DPPs in the strip on $\mathbb{C}$, $\{z \in \mathbb{C} : 0 \leq \Im z \leq W\}$.

3. DPPs on Sphere and Torus

3.1 Finite DPPs on sphere $\mathbb{S}^2$

- **Let $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : ||x||_{\mathbb{R}^3} = 1\}$ be the two-dimensional unit sphere centered at the origin in the three-dimensional Euclidean space $\mathbb{R}^3$, where $||\cdot||_{\mathbb{R}^3}$ denotes the Euclidean distance in $\mathbb{R}^3$.**

- **We will use the following coordinates for $x = (x^{(1)}, x^{(2)}, x^{(3)})$ on $\mathbb{S}^2$,**

$$x^{(1)} = \sin\theta\cos\varphi, \quad x^{(2)} = \sin\theta\sin\varphi, \quad x^{(3)} = \cos\theta, \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi).$$

- **We consider the case that $S_1 = \mathbb{N}_0$ and $S_2 = \mathbb{S}^2$, in which we assume that $\lambda_2(dx)$ is given by the Lebesgue surface area measure $d\sigma_2(x)$ on $\mathbb{S}^2$ such that**

$$\lambda_2(dx) = d\sigma_2(x) = d\sigma_2(\theta, \varphi) = \sin\theta d\theta d\varphi, \quad \lambda_2(\mathbb{S}^2) = \sigma_2(\mathbb{S}^2) = 4\pi.$$

- **For** $n \in \{0, 1, \ldots, N-1\}, N \in \mathbb{N}$**, put**

$$\varphi_n^{\mathbb{S}^2}(x) = \varphi_n^{\mathbb{S}^2}(\theta, \varphi) = \frac{1}{\sqrt{h_n}} e^{-in\varphi} \sin^n(\theta/2) \cos^{N-1-n}(\theta/2), \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi),$$

with

$$h_n = h_n^{(N)} = \frac{4\pi}{N} \binom{N-1}{n}^{-1}.$$

- **It is easy to confirm the following orthonormality relations on** $\mathbb{S}^2$**,**

$$\langle \varphi_n^{\mathbb{S}^2}(\cdot), \varphi_m^{\mathbb{S}^2}(\cdot) \rangle_{L^2(\mathbb{S}^2; d\sigma_2)} = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \, \varphi_n^{\mathbb{S}^2}(\theta, \varphi) \overline{\varphi_m^{\mathbb{S}^2}(\theta, \varphi)} d\sigma_2(\theta, \varphi) = \delta_{nm}, \quad n, m \in \mathbb{N}_0.$$

- Assumption 3' is satisfied and, if we set $L^2(\Gamma, \nu) = \ell^2(\{0, 1, \dots, N-1\})$, $N \in \mathbb{N}_0$, Corollary 2.4 gives the DPP with N points on $\mathbb{S}^2$, $(\Xi, K_{\mathbb{S}^2}^{(N)}, d\sigma_2(x))$, whose correlation kernel is given by

$$\begin{aligned} K_{\mathbb{S}^2}^{(N)}(x, x') &= K_{\mathbb{S}^2}^{(N)}((\theta, \varphi), (\theta', \varphi')) \\ &= \frac{N}{4\pi} \sum_{n=0}^{N-1} \binom{N-1}{n} \Big(e^{-i(\varphi-\varphi')} \sin(\theta/2) \sin(\theta'/2)\Big)^n \Big(\cos(\theta/2) \cos(\theta'/2)\Big)^{N-1-n} \\ &= \frac{N}{4\pi} \Big(e^{-i(\varphi-\varphi')} \sin(\theta/2) \sin(\theta'/2) + \cos(\theta/2) \cos(\theta'/2)\Big)^{N-1}. \end{aligned}$$

- The density of points with respect to $d\sigma_2(x)$ is given by

$$\rho(x) = K_{\mathbb{S}^2}^{(N)}(x, x) = \frac{N}{4\pi} = \textbf{constant}, \quad x \in \mathbb{S}^2.$$

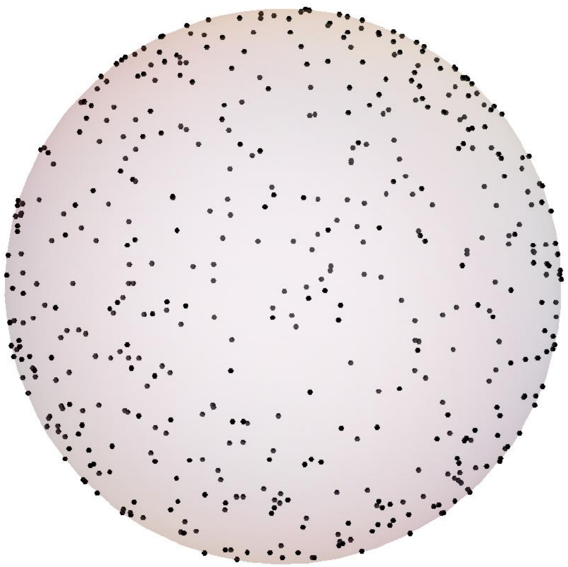


Figure made by T. Shirai

- **For two points** $x = (\theta, \varphi)$ **and** $x' = (\theta', \varphi')$ **on** $\mathbb{S}^2$**,**

$$\begin{aligned} ||x - x'||^2_{\mathbb{R}^3} &= (\sin\theta\cos\varphi - \sin\theta'\cos\varphi')^2 + (\sin\theta\sin\varphi - \sin\theta'\sin\varphi')^2 + (\cos\theta - \cos\theta')^2 \\ &= |\Phi(x - x')|^2, \end{aligned}$$

with

$$\Phi(x - x') = 2\cos\frac{\theta}{2}\cos\frac{\theta'}{2}e^{i(\varphi+\varphi')/2}\left[e^{-i\varphi}\tan\frac{\theta}{2} - e^{-i\varphi'}\tan\frac{\theta'}{2}\right].$$

- **Then we can show that the probability density of this DPP with respect to** $d\sigma_2(x) = \prod_{j=1}^N d\sigma_2(x_j)$ **is given as**

$$\mathbf{p}^{(N)}_{\mathbb{S}^2}(\boldsymbol{x}) = \frac{1}{Z^{(N)}_{\mathbb{S}^2}} \prod_{1 \le j < k \le N} ||x_k - x_j||^2_{\mathbb{R}^3},$$

with

$$Z^{(N)}_{\mathbb{S}^2} = \frac{2^{N(N+1)}\pi^N}{(N!)^{N-1}} \left(\prod_{j=1}^N (j-1)!\right)^2.$$

- **Since** $||x - x'||^2_{\mathbb{R}^3} = 2 - 2x \cdot x'$ **for** $x, x' \in \mathbb{S}^2$, **we have the equality**

$$\frac{1}{2}(1 + x \cdot x') = \left| e^{-i(\varphi - \varphi')} \sin(\theta/2) \sin(\theta'/2) + \cos(\theta/2) \cos(\theta'/2) \right|^2 .$$

- **Hence the absolute value of the correlation kernel is written as**

$$\left| K^{(N)}_{\mathbb{S}^2}(x, x') \right| = \frac{N}{4\pi} \left(\frac{1 + x \cdot x'}{2} \right)^{(N-1)/2},$$

and hence the two-point correlation function with respect to $d\sigma_2(x)$ **is given by**

$$\rho^2(x, x') = \left(\frac{N}{4\pi} \right)^2 \left[1 - \left(\frac{1 + x \cdot x'}{2} \right)^{N-1} \right], \quad x, x' \in \mathbb{S}^2.$$

- **The system** $(\Xi, K^{(N)}_{\mathbb{S}^2}, d\sigma_2(x))$ **is a uniform and isotropic DPP on** $\mathbb{S}^2$, **which is called the the spherical ensemble [Krishnapur (2009), Alishashi–Zamani (2015), Beltrán–Etayo(2018+)].**

- The equivalent system with the spherical ensemble of DPPs was studied by Caillol(1981) as a two-dimensional one-component plasma model in physics.

- It is interesting to see that he used the Cayley-Klein parameters defined by

$$\alpha = e^{i\varphi/2} \cos \frac{\theta}{2}, \quad \beta = -ie^{-i\varphi/2} \sin \frac{\theta}{2}, \quad \varphi \in [0, 2\pi), \quad \theta \in [0, \pi].$$

- The above orthonormal functions can be identified with the follows up to irrelevant factors,

$$\widetilde{\varphi}_n^{\mathbb{S}^2}(\alpha, \beta) = \frac{1}{\sqrt{h_n}} \alpha^{N-1-n} \beta^n, \quad n \in \{0, 1, \ldots, N-1\}.$$

- If we define

$$\langle (\alpha, \beta), (\alpha', \beta') \rangle_{\mathrm{CK}} := \alpha \overline{\alpha'} + \beta \overline{\beta'},$$

the correlation kernel is written as

$$K_{\mathbb{S}^2}^{(N)}(x, x') = K_{\mathbb{S}^2}^{(N)}((\alpha, \beta), (\alpha', \beta')) = \frac{N}{4\pi} (\langle (\alpha, \beta), (\alpha', \beta') \rangle_{\mathrm{CK}})^{N-1}.$$

- **Following the claim given in Caillol (1981), we consider the vicinity of the north pole, $x_{\rm np} = (0, 0, 1) \in \mathbb{R}^3$, that is $\theta \simeq 0$.**
- **We put**

$$\theta = \frac{2r}{\sqrt{N}}, \quad \theta' = \frac{2r'}{\sqrt{N}},$$

and take the limit $N \to \infty$ keeping r and r' be constants.

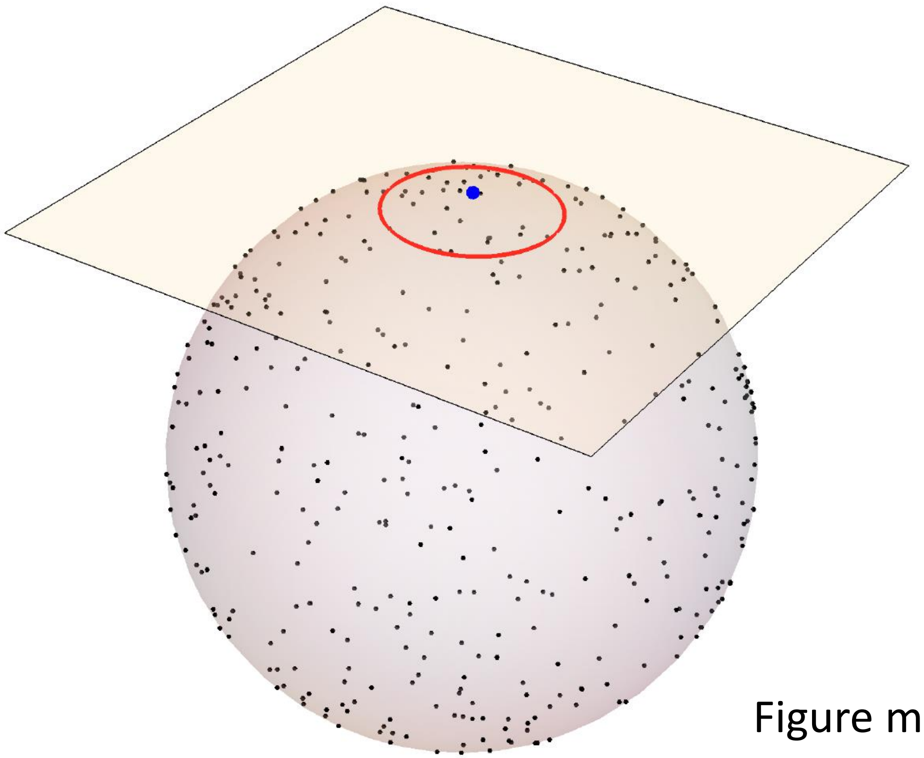


Figure made by T. Shirai

- **Then we see that**

$$\sin(\theta/2)\sin(\theta'/2) \simeq \frac{1}{4}\theta\theta' = \frac{rr'}{N},$$
$$\cos(\theta/2)\cos(\theta'/2) \simeq 1 - \frac{\theta^2 + \theta'^2}{8} = 1 - \frac{r^2 + r'^2}{2N}.$$

- **We set** $re^{i\varphi} = z, r'e^{i\varphi'} = z' \in \mathbb{C}$ **with** $rdrd\varphi = dz$. **Then the kernel**

$$\lim_{N\to\infty} K_{\mathbb{S}^2}^{(N)}((\theta,\varphi),(\theta',\varphi'))d\sigma_2(\theta,\varphi)\Big|_{\theta=2|z|/\sqrt{N},\theta'=2|z'|/\sqrt{N}}$$
$$= \lim_{N\to\infty} \frac{N}{4\pi}\left(1 + \frac{1}{N}\left\{z\overline{z'} - \frac{|z|^2 + |z'|^2}{2}\right\}\right)^N \frac{4}{N}dz$$
$$= \frac{1}{\pi}e^{z\overline{z'} - (|z|^2+|z'|^2)/2}dz.$$

- **This implies the following limit theorem.**

Proposition 3.1 *The following weak convergence is established,*

$$\frac{\sqrt{N}}{2} \circ \left(\Xi, K_{\mathbb{S}^2}^{(N)}, d\sigma_2(x)\right) \overset{N\to\infty}{\Longrightarrow} \left(\Xi, K_{\text{Ginibre}}^{A}, \lambda_{\text{N}(0,1;\mathbb{C})}(dx)\right),$$

where the limit point process is the Ginibre DPP of type A.

3.2 Finite DPPs on torus $\mathbb{T}^2$

- Let
$$z = e^{v\pi i}, \quad q = e^{\tau\pi i},$$
for $v \in \mathbb{C}$ and $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$. The Jacobi theta functions are defined as follows,
$$\vartheta_0(v;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} z^{2n}, \qquad \vartheta_1(v;\tau) = i \sum_{n\in\mathbb{Z}} (-1)^n q^{(n-1/2)^2} z^{2n-1},$$
$$\vartheta_2(v;\tau) = \sum_{n\in\mathbb{Z}} q^{(n-1/2)^2} z^{2n-1}, \qquad \vartheta_3(v;\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} z^{2n}.$$

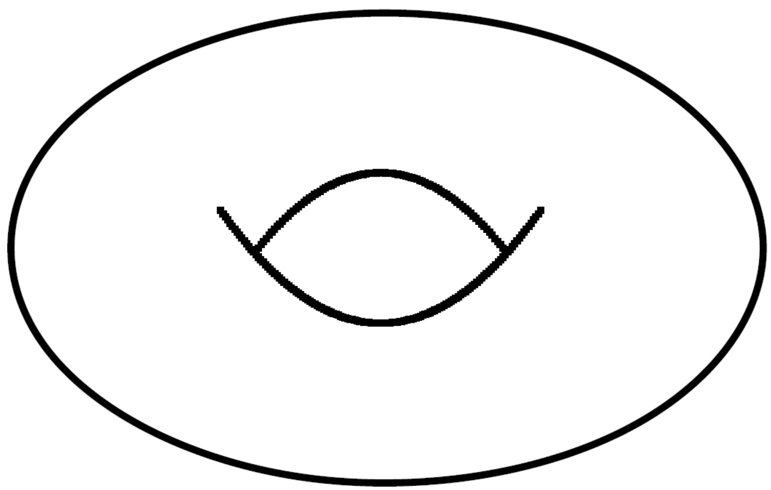
- We define the following four types of functions;
$$\Theta^A(\sigma, z, \tau) = e^{2\pi i\sigma z}\vartheta_2(\sigma\tau + z;\tau),$$
$$\Theta^B(\sigma, z, \tau) = e^{2\pi i\sigma z}\vartheta_1(\sigma\tau + z;\tau) - e^{-2\pi i\sigma z}\vartheta_1(\sigma\tau - z;\tau),$$
$$\Theta^C(\sigma, z, \tau) = e^{2\pi i\sigma z}\vartheta_2(\sigma\tau + z;\tau) - e^{-2\pi i\sigma z}\vartheta_2(\sigma\tau - z;\tau),$$
$$\Theta^D(\sigma, z, \tau) = e^{2\pi i\sigma z}\vartheta_2(\sigma\tau + z;\tau) + e^{-2\pi i\sigma z}\vartheta_2(\sigma\tau - z;\tau),$$
for $\sigma \in \mathbb{R}, z \in \mathbb{C}, \tau \in \mathbb{H}$.

- **We will consider the finite DPPs on a surface of torus with double periodicity** $\omega_1 = 2\pi$, $\omega_3 = 2\tau\pi$ **with** $\tau = i\Im\tau \in \mathbb{H}$.

- **The surface of such the torus** $\mathbb{T}^2 = \mathbb{T}^2(2\pi, 2\tau\pi) := \mathbb{S}^1(2\pi) \times \mathbb{S}^1(2\tau\pi)$ **can be identified with a rectangular domain in** $\mathbb{C}$**,**

$$D_{(2\pi,2\tau\pi)} = \{z \in \mathbb{C} : 0 \leq \Re z \leq 2\pi, 0 \leq \Im z \leq 2\pi\Im\tau\} \subset \mathbb{C} \quad \textbf{with double periodicity } (2\pi, 2\tau\pi).$$

So we first consider the systems on $D_{(2\pi,2\tau\pi)}$.



- **Let** $S = \mathbb{C}$ **with** $\lambda(dx) = \mathbf{1}_{D_{(2\pi, 2\tau\pi)}}(x) dx_{\rm R} dx_{\rm I}$**. For** $N \in \mathbb{N}$**, put**

$$\varphi_n^{R_N, (2\pi, 2\tau\pi)}(x) = \frac{e^{-\mathcal{N}^{R_N} i x_{\rm I}^2/(4\tau\pi)}}{\sqrt{h_n^{R_N}(\tau)}} \Theta^{\sharp(R_N)}\left(\frac{J^{R_N}(n)}{\mathcal{N}^{R_N}}, \mathcal{N}^{R_N} \frac{x}{2\pi}, \mathcal{N}^{R_N}\tau \right), \quad n \in \{1, 2, \dots, N\}$$

where

$$\sharp(R_N) = \begin{cases} A, & \textbf{if } R_N = A_{N-1}, \\ B, & \textbf{if } R_N = B_N, B_N^{\vee}, \\ C, & \textbf{if } R_N = C_N, C_N^{\vee}, BC_N, \\ D, & \textbf{if } R_N = D_N, \end{cases}$$

$$J^{R_N}(n) = \begin{cases} n - 1/2, & R_N = A_{N-1}, C_N^{\vee}, \\ n - 1, & R_N = B_N, B_N^{\vee}, D_N, \\ n, & R_N = C_N, BC_N, \end{cases}$$

$$\mathcal{N}^{R_N} = \begin{cases} N, & R_N = A_{N-1}, \\ 2N - 1, & R_N = B_N, \\ 2N, & R_N = B_N^{\vee}, C_N^{\vee}, \\ 2(N+1), & R_N = C_N, \\ 2N + 1, & R_N = BC_N, \\ 2(N-1), & R_N = D_N, \end{cases}$$

and $\{h_n^{R_N}\}$ **are proper normalization factors.**

- **The following orthonormal relations were proved in [K2019b],**

$$\langle \varphi_n^{R_N,(2\pi,2\tau\pi)}, \varphi_m^{R_N,(2\pi,2\tau\pi)} \rangle_{L^2(\mathbb{C},\mathbf{1}_{D_{(2\pi,2\tau\pi)}}(x)dx)} = \delta_{nm}, \quad n,m \in \Gamma := \{1,2,\dots,N\},$$

$R_N = A_{N-1}, B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N$.

- **Then Corollary 2.4 gives the seven types of DPPs with the correlation kernels,**

$$K^{R_N,(2\pi,2\tau\pi)}(x,x') = \sum_{n=1}^{N} \varphi_n^{R_N,(2\pi,2\tau\pi)}(x)\overline{\varphi_n^{R_N,(2\pi,2\tau\pi)}(x')},$$

with respect to the measure $\lambda(dx) = \mathbf{1}_{D_{(2\pi,2\tau\pi)}} dx$ **on** $\mathbb{C}$ **for** $R_N = A_{N-1}, B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N$.

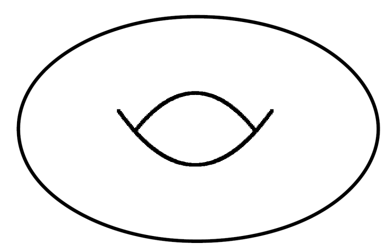
- The correlation kernels are quasi-double-periodic,

$$
\begin{aligned}
K^{R_N,(2\pi,2\tau\pi)}(x+2\pi,x') &= K^{R_N,(2\pi,2\tau\pi)}(x,x'+2\pi) \\
&= \begin{cases} (-1)^{\mathcal{N}^{A_{N-1}}} K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = A_{N-1}, \\ -K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = B_N, C_N^\vee, BC_N, \\ K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = B_N^\vee, C_N, D_N, \end{cases} \\
K^{R_N,(2\pi,2\tau\pi)}(x+2\tau\pi,x') &= \begin{cases} e^{-\mathcal{N}^{R_N} i x_{\mathrm{R}}} K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = A_{N-1}, C_N, C_N^\vee, BC_N, D_N, \\ -e^{-\mathcal{N}^{R_N} i x_{\mathrm{R}}} K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = B_N, B_N^\vee, \end{cases} \\
K^{R_N,(2\pi,2\tau\pi)}(x,x'+2\tau\pi) &= \begin{cases} e^{\mathcal{N}^{R_N} i x'_{\mathrm{R}}} K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = A_{N-1}, C_N, C_N^\vee, BC_N, D_N, \\ -e^{\mathcal{N}^{R_N} i x'_{\mathrm{R}}} K^{R_N,(2\pi,2\tau\pi)}(x,x'), & R_N = B_N, B_N^\vee. \end{cases}
\end{aligned}
$$

- The above implies that

$$
\begin{aligned}
\tau_{2\pi} K^{R_N,(2\pi,2\tau\pi)}(x,x') &= \frac{e^{\mathcal{N}^{R_N} i x_{\mathrm{R}}}}{e^{\mathcal{N}^{R_N} i x'_{\mathrm{R}}}} \tau_{2\tau\pi} K^{R_N,(2\pi,2\tau\pi)}(x,x') \\
&= K^{R_N,(2\pi,2\tau\pi)}(x,x'), \quad x, x' \in D_{(2\pi,2\tau\pi)}.
\end{aligned}
$$

- In other words, we have obtained the seven types of DPPs with a finite number of points N on a surface of torus $\mathbb{T}^2(2\pi, 2\tau\pi)$. Hence here we write them as $\left(\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx\right)$, $R_N = A_{N-1}$, B_N, $B_N^\vee$, C_N, $C_N^\vee$, BC_N, D_N.



- **We can prove the following limit theorem.**

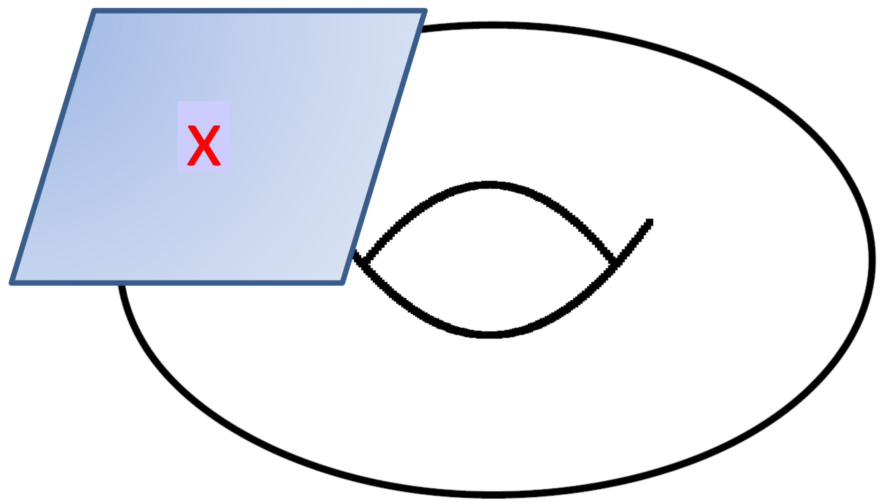
Proposition 3.2 *The following weak convergence is established,*

$$\frac{1}{2}\sqrt{\frac{N}{\pi \Im \tau}} \circ \left(\Xi, K_{\mathbb{T}^2(2\pi, 2\tau\pi)}^{A_{N-1}}, dx\right) \overset{N\to\infty}{\Longrightarrow} \left(\Xi, K_{\text{Ginibre}}^{A}, \lambda_{\text{N}(0,1;\mathbb{C})}(dx)\right),$$

$$\sqrt{\frac{N}{2\pi \Im \tau}} \circ \left(\Xi, K_{\mathbb{T}^2(2\pi, 2\tau\pi)}^{R_N}, dx\right) \overset{N\to\infty}{\Longrightarrow} \left(\Xi, K_{\text{Ginibre}}^{C}, 2\lambda_{\text{N}(0,1;\mathbb{C})}(dx)\right), \quad R_N = B_N, B_N^{\vee}, C_N, C_N^{\vee}, BC_N,$$

$$\sqrt{\frac{N}{2\pi \Im \tau}} \circ \left(\Xi, K_{\mathbb{T}^2(2\pi, 2\tau\pi)}^{D_N}, dx\right) \overset{N\to\infty}{\Longrightarrow} \left(\Xi, K_{\text{Ginibre}}^{D}, 2\lambda_{\text{N}(0,1;\mathbb{C})}(dx)\right),$$

where the limit point processes are the three types of Ginibre DPPs.



4. Two Families of Universal DPPs in Arbitrary Dimensions

4.1 Heisenberg Family of DPPs

- The Ginibre DPP of type A on $\mathbb{C}$ can be generalized to the DPPs on $\mathbb{C}^d$ for $d \geq 2$.

- This generalization was done by Abreu *et al.* (2017,2019) as the Weyl-Heisenberg ensemble of DPP, but here we derive the DPPs on $\mathbb{C}^d$, $d \in \mathbb{N}$, following Corollary 2.4.

- Let $S_1 = \mathbb{C}^d$, $S_2 = \Gamma = \mathbb{R}^d$,

$$\begin{aligned}\lambda_1(dx) &= \prod_{a=1}^{d} \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx^{(a)}) = \frac{1}{\pi^d} e^{-|x|^2} = \frac{1}{\pi^d} e^{-(|x_{\mathrm{R}}|^2+|x_{\mathrm{I}}|^2)} \\ &=: \lambda_{\mathrm{N}(0,1;\mathbb{C}^d)}(dx), \\ \lambda_2(dy) &= \prod_{a=1}^{d} \lambda_{\mathrm{N}(0,1/4)}(dy^{(a)}) = \left(\frac{2}{\pi}\right)^{d/2} e^{-2|\gamma|^2},\end{aligned}$$

and

$$\psi_1(x, \gamma) = e^{-(|x_{\mathrm{R}}|^2 - |x_{\mathrm{I}}|^2)/2 + 2(x_{\mathrm{R}} \cdot \gamma + i x_{\mathrm{I}} \cdot \gamma)}, \quad x = x_{\mathrm{R}} + i x_{rI} \in \mathbb{C}^d, \quad \gamma \in \mathbb{R}^d.$$

- **It is easy to verify Assumption 3' and then, by Corollary 2.4, we obtain the DPP on $\mathbb{C}^d$ with the correlation kernel,**

$$K^{(d)}(x, x') = \left(\frac{2}{\pi}\right)^{d/2} e^{\{(|x_{\rm R}|^2 - |x_{\rm I}|^2) + (|x'_{\rm R}|^2 - |x'_{\rm I}|^2)\}/2} \int_{\mathbb{R}^d} e^{-2[|\gamma|^2 - \{(x_{\rm R} + i x_{\rm I}) + (x'_{\rm R} - i x'_{\rm I})\} \cdot \gamma]} d\gamma$$
$$= \frac{e^{i x_{\rm R} \cdot x_{\rm I}}}{e^{i x'_{\rm R} \cdot x'_{\rm I}}} K^{(d)}_{\rm Heisenberg}(x, x')$$

with

$$K^{(d)}_{\rm Heisenberg}(x, x') = e^{x \cdot \overline{x'}}, \quad x, x' \in \mathbb{C}^d.$$

- **The kernels in this form on $\mathbb{C}^d, d \in \mathbb{N}$ have been studied by Zelditch and his coworkers [Zelditch (2000), Bleher–Shiffman–Zerditch (2000)], who identified them with the Szegö kernels for the reduced Heisenberg group.**

- **Here we call the DPPs associated with the correlation kernels in this form the Heisenberg family of DPPs on $\mathbb{C}^d, d \in \mathbb{N}$.**

- **This class includes the Ginibre DPP of type A as the lowest dimensional case with $d = 1$.**

Definition 4.1 *The Heisenberg family of DPP on $\mathbb{C}^d, d \in \mathbb{N}$ is defined by* $\Big(\Xi, K^{(d)}_{\text{Heisenberg}}, \lambda_{\text{N}(0,1;\mathbb{C}^d)}(dx)\Big)$ *with*

$$K^{(d)}_{\text{Heisenberg}}(x, x') = e^{x \cdot \overline{x'}}, \quad x, x \in \mathbb{C}^d.$$

- **Since**

$$K^{(d)}_{\text{Heisenberg}}(x, x) \lambda_{\text{N}(0,1;\mathbb{C}^d)}(dx) = \frac{1}{\pi^d} dx, \quad x \in \mathbb{C}^d,$$

every DPP in the Heisenberg family is uniform on $\mathbb{C}^d$ and the density with respect to the Lebesgue measure dx is given by $1/\pi^d$.

4.2 Finite DPPs on $\mathbb{S}^d$

- For $d \in \mathbb{N}$, let $\mathcal{P} = \mathcal{P}(\mathbb{R}^{d+1})$ be a vector space of all complex-valued polynomials on $\mathbb{R}^{d+1}$, and $\mathcal{P}_k, k \in \mathbb{N}_0$, be its subspaces consisting of homogeneous polynomials of degree k; $p(\boldsymbol{x}) = \sum_{|\alpha|=k} c_\alpha x^\alpha$, $c_\alpha \in \mathbb{C}, \boldsymbol{x} = (x^{(1)}, \dots, x^{(d+1)}) \in \mathbb{R}^{d+1}$, where we have used the notations $x^\alpha := \prod_{a=1}^{d+1} (x^{(a)})^{\alpha_a}$ with $\alpha := (\alpha_1, \dots, \alpha_{d+1}) \in \mathbb{N}_0^{d+1}$, $|\alpha| := \sum_{a=1}^{d+1} \alpha_a$.
- The vector space of all harmonic functions in $\mathcal{P}$ is denoted by $\mathcal{H} = \{p \in \mathcal{P} : \Delta p = 0\}$ and let $\mathcal{H}_k = \mathcal{H} \cap \mathcal{P}_k$, $k \in \mathbb{N}_0$.

- **Now we consider a unit sphere in** $\mathbb{R}^{d+1}$ **denoted by** $\mathbb{S}^d$**, in which we use the polar coordinates for** $x = (x^{(1)}, \ldots, x^{(d+1)}) \in \mathbb{S}^d$**,**

$$\begin{aligned} x^{(1)} &= \sin\theta_d \cdots \sin\theta_2 \sin\theta_1, \\ x^{(a)} &= \sin\theta_d \cdots \sin\theta_a \cos\theta_{a-1}, \qquad a = 2, \ldots, d, \\ x^{(d+1)} &= \cos\theta_d, \quad \textbf{with } \theta_1 \in [0, 2\pi), \quad \theta_a \in [0, \pi], \quad a = 2, \ldots, d. \end{aligned}$$

  Note that $||x||^2_{\mathbb{R}^{d+1}} := \sum_{a=1}^{d+1} {x^{(a)}}^2 = 1$**. The standard measure on** $\mathbb{S}^d$ **is given by the Lebesgue area measure expressed as**

$$d\sigma_d(x) = \sin^{d-1}\theta_d \sin^{d-2}\theta_{d-1} \cdots \sin\theta_2 d\theta_1 \cdots d\theta_d, \qquad x \in \mathbb{S}^d.$$

- **The total measure of** $\mathbb{S}^d$ **is calculated as**

$$\omega_d = \sigma_d(\mathbb{S}^d) = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}.$$

- **We write the restriction of harmonic polynomials in $\mathcal{H}_k$ on $\mathbb{S}^d$ as**

$$\mathcal{Y}_{(d,k)} = \left\{ h\Big|_{\mathbb{S}^d} : h \in \mathcal{H}_k \right\}, \quad k \in \mathbb{N}_0.$$

We can see that

$$D(d,k) = \dim \mathcal{Y}_{(d,k)} = \frac{(d+2k-1)(d+k-2)!}{(d-1)!k!}.$$

- **Consider an orthonormal basis $\{Y_j^{(d,k)}\}_{j=1}^{D(d,k)}$ of $\mathcal{Y}_{(d,k)}$ with respect to $d\sigma_d$;**

$$\langle Y_n^{(d,k)}, Y_m^{(d,k)} \rangle_{L^2(\mathbb{S}^d, d\sigma_d)} = \int_{\mathbb{S}^d} Y_n^{(d,k)}(x) \overline{Y_m^{(d,k)}(x)} d\sigma_d(x) = \delta_{nm}, \quad n, m \in \mathbb{N}_0.$$

Then, if we put

$$K^{\mathcal{Y}_{(d,k)}}(x, x') = \sum_{j=1}^{D(d,k)} Y_j^{(d,k)}(x) \overline{Y_j^{(d,k)}(x')}, \quad x' \in \mathbb{S}^d,$$

then $\{K^{\mathcal{Y}_{(d,k)}}(x,x')\}_{x,x' \in \mathbb{S}^d}$ give the reproducing kernels in $\mathcal{Y}^{(d,k)}$ in the sense that

$$Y(x') = \int_{\mathbb{S}^d} Y(x) \overline{K^{\mathcal{Y}_{(d,k)}}(x,x')} d\sigma_d(x), \quad \forall Y \in \mathcal{Y}_{(d,k)}.$$

- **For $\lambda > -1/2$, we define**

$$P_k^\lambda(x) = F\left(-k, k+2\lambda; \lambda + \frac{1}{2}; \frac{1-x}{2}\right),$$

where F denotes the Gauss hypergeometric function.

- **Then the following equality is established,**

$$K^{\mathcal{Y}_{(d,k)}}(x, x') = \frac{D(d,k)}{\omega_d} P_k^{(d-1)/2}(x \cdot x'), \quad x, x' \in \mathbb{S}^d,$$

where $x \cdot x' := \sum_{a=1}^{d+1} x^{(a)} x'^{(a)}$.

- **The function $P_k^\lambda(s)$ is called the ultraspherical polynomial. This is the zonal harmonics of degree k.**

- **Note that, when we set**

$$C_k^\lambda(x) = \binom{k+2\lambda-1}{k} P_k^\lambda(x),$$

we call $C_k^\lambda(s)$ the Gegenbauer polynomial of degree k.

- **Fix** $d \in \mathbb{N}$ **and** $k \in \mathbb{N}_0$**.**
- **Then, if we consider the case that** $S_1 = \mathbb{S}^d$**,** $S_2 = \mathbb{N}$ **with** $\lambda_1(dx) = d\sigma_d(x)$**,** $L^2(\Gamma, \nu) = \ell^2(\{1, 2, \dots, D(d,k)\}) \subset S_2$**, and** $\psi_1(x,n) = Y_n^{(d,k)}(x)$**, Assumption 3' is guaranteed.**
- **Hence Theorem 2.4 determines a unique DPP on** $\mathbb{S}^d$**, in which the correlation kernel is given by**

$$
\begin{aligned}
K^{\mathcal{Y}_{(d,k)}}(x,x') &= \frac{D(d,k)}{\omega_d} P_k^{(d-1)/2}(x \cdot x') \\
&= \frac{d-1+2k}{(d-1)\omega_d} C_k^{(d-1)/2}(x \cdot x').
\end{aligned}
$$

- **The density of points is uniform on** $\mathbb{S}^d$ **and is given with respect to** $\sigma_d(dx)$ **by**

$$
\begin{aligned}
\rho^{\mathcal{Y}_{(d,k)}} = K^{\mathcal{Y}_{(d,k)}}(x,x) &= \frac{D(d,k)}{\omega_d} P_k^{(d-1)/2}(1) \\
&= \frac{D(d,k)}{\omega_d},
\end{aligned}
$$

where we have used the fact that $P_k^{\lambda}(1) = F(-k, k+2\lambda, \lambda+1/2; 0) = 1, \lambda > -1/2$**.**

- **Next we consider the DPP on $\mathbb{S}^d$ for fixed $d \in \mathbb{N}$ and $N \in \mathbb{N}$ such that the correlation kernel is given by the following finite sum,**

$$K^{(N(d,L))}_{\mathrm{harmonic}(\mathbb{S}^d)}(x,x') := \sum_{k=0}^{L-1} K^{\mathcal{Y}_{(d,k)}}(x,x') = \frac{1}{\omega_d}\sum_{k=0}^{L-1} D(d,k) P_k^{(d-1)/2}(x\cdot x')$$
$$= \frac{1}{\omega_d}\sum_{k=0}^{L-1} \frac{d-1+2k}{d-1} C_k^{(d-1)/2}(x\cdot x'),$$

where the total number of points on $\mathbb{S}^d$ is given by

$$N(d,L) = \sum_{k=0}^{L-1} D(d,k) = \frac{2L+d-2}{d}\binom{d+L-2}{L-1} = \frac{2}{d!}L^d + \mathrm{o}(L^d).$$

- **The DPP $(\Xi, K^{(N)}_{\mathrm{harmonic}(\mathbb{S}^d)}, d\sigma_d(x))$ was called the harmonic ensemble in $\mathbb{S}^d$ with N points by Beltrán *et al.* (2016).**

- **We note the recurrence relation of the Gegenbauer polynomials,**

$$(n+\lambda)C_n^{\lambda}(x) = \lambda(C_n^{\lambda+1}(x) - C_{n-2}^{\lambda+1}(x)).$$

This implies that

$$\frac{d-1+2k}{d-1} C_k^{(d-1)/2}(x) = C_k^{(d+1)/2}(x) - C_{k-2}^{(d+1)/2}(x), \qquad k \geq 2.$$

Since $C_0^{\lambda}(x) = 1, C_1^{\lambda}(x) = 2\lambda x$, **we obtain the following expression for the correlation kernel,**

$$K_{\mathrm{harmonic}(\mathbb{S}^d)}^{(N(d,L))}(x,x') = \frac{1}{\omega_d}\left[C_{L-1}^{(d+1)/2}(x \cdot x') + C_{L-2}^{(d+1)/2}(x \cdot x')\right].$$

- **If we introduce the Jacobi polynomials defined as**

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} F\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right),$$

and use the contiguous relation, $(b-a)F(a,b;c;z) + aF(a+1,b;c;z) - bF(a,b+1;c;z) = 0$, **the above is written as follows,**

$$K_{\mathrm{harmonic}(\mathbb{S}^d)}^{(N(d,L))}(x,x') = \frac{1}{\omega_d} \frac{N(d,L)}{\binom{L+d/2-1}{L-1}} P_{L-1}^{(d/2,(d-2)/2)}(x \cdot x'),$$

where $\binom{L+d/2-1}{L-1} := \Gamma(L+d/2)/\{(L-1)!\Gamma(d/2+1)\} = P_{L-1}^{(d/2,(d-2)/2)}(1)$.

- **In particular, when** $d = 1$, **for** $x = (x^{(1)}, x^{(2)}) = (\sin\theta, \cos\theta)$, $x' = (x'^{(1)}, x'^{(2)}) = (\sin\theta', \cos\theta') \in \mathbb{S}^1 \subset \mathbb{R}^2$, $\theta, \theta' \in [0, 2\pi)$, **we have** $x \cdot x' = \cos(\theta - \theta')$ **and**

$$
\begin{aligned}
K^{(N(1,L))}_{\mathrm{harmonic}(\mathbb{S}^1)}(x, x') d\sigma_1(x) &= \frac{1}{2\pi} F\left(\frac{1-(2L-1)}{2}, \frac{1+(2L-1)}{2}; \frac{3}{2}; \sin^2 \frac{\theta-\theta'}{2} \right) d\theta \\
&= \frac{\sin\{(2L-1)(\theta-\theta')/2\}}{\sin\{(\theta-\theta')/2\}} \frac{d\theta}{2\pi} \\
&= \frac{\sin\{N(\theta-\theta')/2\}}{\sin\{(\theta-\theta')/2\}} \frac{d\theta}{2\pi},
\end{aligned}
$$

where we have used the fact that $N(1, L) = 2L - 1$.

- **This verifies the identification of the 1-sphere case of the present DPP with the Curcular Unitary Ensemble studied in random matrix theory.**

- **On the other hand, when** $d = 2$, **we have** $N(2, L) = L^2$ **and**

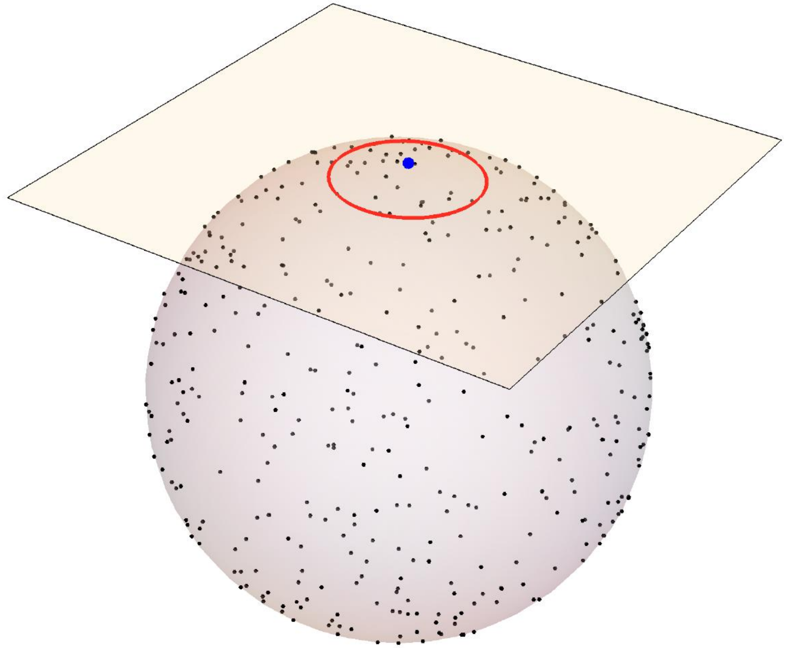
$$
\begin{aligned}
K^{(N(2,L))}_{\mathrm{harmonic}(\mathbb{S}^2)}(x, x') &= \frac{L^2}{4\pi} F\left(-L+1, L+1; 2; \frac{1 - x \cdot x'}{2} \right) \\
&= \frac{N}{4\pi} F\left(-\sqrt{N}+1, \sqrt{N}+1; 2; \frac{||x - x'||^2_{\mathbb{R}^3}}{4} \right),
\end{aligned}
$$

which is different from $K^{(N)}_{\mathbb{S}^2}(x, x')$.

4.3 Euclidean Family of DPPs

- **We consider the vicinity of the north pole e_{d+1} on $\mathbb{S}^d$ and put $\theta_d = r/L$, $r \in [0, \infty)$. Then the polar coordinates behave as**

$$
\begin{aligned}
x^{(1)} &\simeq \frac{r}{L} \sin\theta_{d-1} \cdots \sin\theta_2 \sin\theta_1 =: \frac{1}{L} x^{(1)}, \\
x^{(a)} &\simeq \frac{r}{L} \sin\theta_{d-1} \cdots \sin\theta_k \cos\theta_{a-1} =: \frac{1}{L} x^{(a)}, \qquad a = 2, \dots, d, \\
x^{(d+1)} &\simeq 1 - \frac{1}{2}\left(\frac{r}{L}\right)^2.
\end{aligned}
$$



- **In this case, for** $x, x' \in \mathbb{S}^d$**,**

$$x \cdot x' = \sum_{a=1}^{d+1} x^{(a)} x'^{(a)} = 1 - \frac{1}{2L^2} ||x - x'||_{\mathbb{R}^d}^2 + o(1/L^2), \quad \textbf{as } L \to \infty,$$

where $x, x' \in \mathbb{R}^d$ **and** $||\cdot||_{\mathbb{R}^d}$ **denotes the Euclidean norm in** $\mathbb{R}^d$**.**

- **Hence we can conclude that**

$$x \cdot x' = \cos\left(\frac{r}{L}\right) + \mathrm{o}\left(\frac{1}{L^2}\right), \quad \textbf{with } r := ||x - x'||_{\mathbb{R}^d}, \quad \textbf{as } L \to \infty.$$

- **In this limit, the measure on** $\mathbb{S}^d$ **behaves as**

$$\begin{aligned} d\sigma_d(x) &= \frac{1}{L^d} r^{d-1} \sin^{d-3}\theta_{d-2} \cdots \sin\theta_2 \, dr d\theta_1 \cdots d\theta_{d-1} \\ &= \frac{1}{L^d} dx, \quad x \in \mathbb{S}^d, \quad x \in \mathbb{R}^d. \end{aligned}$$

- **The following limit is proved for the correlation kernel $K^{(N(d,L))}_{\mathrm{harmonic}(\mathbb{S}^d)}$.**

Lemma 4.2 *When*

$$x \cdot x' = \cos\left(\frac{r}{L}\right) + \mathrm{o}\left(\frac{1}{L^2}\right), \quad \text{with } r := ||x - x'||_{\mathbb{R}^d}, \quad \text{as } L \to \infty.$$

holds, the limit

$$k^{(d)}(r) = \lim_{L\to\infty} \frac{1}{L^d} K^{(N(d,L))}_{\mathrm{harmonic}(\mathbb{S}^d)}(x, x')$$

exists and have the following expressions,

$$\begin{aligned} k^{(d)}(r) &= \frac{J_{d/2}(r)}{(2\pi r)^{d/2}}, \\ &= \frac{1}{(2\pi)^{d/2} r^{(d-2)/2}} \int_0^1 s^{d/2} J_{(d-2)/2}(rs)\,ds, \end{aligned}$$

where $J_\nu(z)$ is the Bessel function of the first kind with index ν.

- **We can give the following alternative expression for $K^{(d)}$.**

Lemma 4.3 *For $d \in \mathbb{N}$, the correlation kernel $K^{(d)}$ given above is written as*

$$K^{(d)}(x, x') = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbf{1}_{\mathbb{B}^d}(y) e^{i(x-x')\cdot y} dy,$$

where $\mathbb{B}^d$ denotes the unit ball centered at the origin, $\mathbb{B}^d := \{y \in \mathbb{R}^d : |y| \leq 1\}$.

- **The above kernel is obtained as the correlation kernel K_{S_1} in Corollary 2.4, if we consider the case such that $S_1 = S_2 = \mathbb{R}^d$, $\lambda_1(dx) = dx$, $\lambda_2(dy) = \nu(dy) = dy$, $\psi_1(x, y) = e^{ix\cdot y}$, and $\Gamma = \mathbb{B}^d \subsetneq \mathbb{R}^d$.**

- **This kernel $K^{(d)}$ on $\mathbb{R}^d, d \geq 1$ have been studied by Zelditch (2000), who regarded them as the Szegö kernels for the reduced Euclidean motion group. Here we call the DPPs associated with the kernels in this form as correlation kernels the Euclidean family of DPPs on $\mathbb{R}^d, d \in \mathbb{N}$. See also [Zelditch (2000), Sogge–Zelditch (2002), Zelditch (2009), Canzani–Hamin (2015)].**

Definition 4.4 *The Euclidean family of DPP on $\mathbb{R}^d, d \in \mathbb{N}$ is defined by $\left(\Xi, K^{(d)}_{\text{Euclidean}}, dx\right)$ with the correlation kernel*

$$
\begin{aligned}
K^{(d)}_{\text{Euclid}}(x,x') &= \frac{1}{(2\pi)^{d/2}} \frac{J_{d/2}(||x-x'||_{\mathbb{R}^d})}{||x-x'||^{d/2}_{\mathbb{R}^d}} \\
&= \frac{1}{(2\pi)^{d/2}} \frac{1}{||x-x'||^{(d-2)/2}_{\mathbb{R}^d}} \int_0^1 s^{d/2} J_{(d-2)/2}(||x-x'||_{\mathbb{R}^d} s) \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbf{1}_{\mathbb{B}^d}(y) e^{i(x-x')\cdot y} dy = \frac{1}{(2\pi)^d} \int_{\mathbb{B}^d} e^{i(x-x')\cdot y} dy, \quad x, x' \in \mathbb{R}^d.
\end{aligned}
$$

- **The above result is summarized as follows.**

Proposition 4.5 *The following is established for $d \in \mathbb{N}$,*

$$\left(\frac{d!}{2}\right)^{1/d} N^{1/d} \circ \left(\Xi, K^{(N)}_{\mathrm{harmonic}(\mathbb{S}^d)}, d\sigma_d(x)\right) \stackrel{N\to\infty}{\Longrightarrow} \left(\Xi, K^{(d)}_{\mathrm{Euclid}}, dx\right).$$

- **We see that**

$$K^{(d)}_{\mathrm{Euclid}}(x,x) = \lim_{r\to 0} \frac{1}{(2\pi)^{d/2}} \frac{J_{d/2}(r)}{r^{d/2}} = \frac{1}{2^d \pi^{d/2} \Gamma((d+2)/2)}.$$

Then the Euclidean family of DPP is uniform on $\mathbb{R}^d$ with the density with respect to the Lebesgue measure dx is given by

$$\rho^{(d)}_{\mathrm{Euclid}} = \frac{1}{2^d \pi^{d/2} \Gamma((d+2)/2)}.$$

- **For lower dimensions, the correlation kernels and the densities are given as follows,**

$$K^{(1)}_{\text{Euclid}}(x,x') = \frac{\sin(x-x')}{\pi(x-x')} = K_{\text{sinc}}(x,x') \quad \textbf{with} \quad \rho^{(1)}_{\text{Euclid}} = \frac{1}{\pi},$$
$$K^{(2)}_{\text{Euclid}}(x,x') = \frac{J_1(||x-x'||_{\mathbb{R}^2})}{2\pi||x-x'||_{\mathbb{R}^2}} \quad \textbf{with} \quad \rho^{(2)}_{\text{Euclid}} = \frac{1}{4\pi},$$
$$K^{(3)}_{\text{Euclid}}(x,x') = \frac{1}{2\pi^2||x-x'||^2_{\mathbb{R}^3}} \left(\frac{\sin||x-x'||_{\mathbb{R}^3}}{||x-x||_{\mathbb{R}^3}} - \cos||x-x'||_{\mathbb{R}^3} \right) \quad \textbf{with} \quad \rho^{(3)}_{\text{Euclid}} = \frac{1}{6\pi^2}.$$

- **This class of DPPs includes the DPP with the sinc kernel K_{sinc} as the lowest dimensional case with $d=1$.**

- **Note that, if d is odd,**

$$K^{(d)}_{\text{Euclid}}(x,x') = k^{(d)}(||x-x'||_{\mathbb{R}^d}) \quad \textbf{with} \quad k^{(d)}(r) = \left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-1)/2} \frac{\sin r}{\pi r}.$$

Thank you very much for your attention.

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