四元数CR多様体のツイスター空間について
Twistor space for a quaternionic CR manifold *

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Question: What is is a natural geometric structure modeled on any real hypersurface M^{4n+3} in a quaternionic manifold $\mathcal{N}^{4n+4}$ (e.g., $\mathbb{H}^{n+1}$)?

Fact: A CR structure is a natural geometric structure on every real hypersurface in a complex manifold (e.g., $\mathbb{C}^{n+1}$).

Quaternionic analogues of CR structures:

- Kuo (1970). Almost contact 3-structure. Sasakian 3-structure
- Boyer-Galicki-Mann (1993): 3-Sasakian manifolds
- Hernandez (1996): Hyper f-structure. Hyper PS-structure
- Biquard (2000): Quaternionic contact structure (QC)
  (Duchemin (2006): QC hypersurface)
- 待田芳徳 (2004): インスタントン分布の理論と3-接触構造への一般化
- Alekseevsky-Kamishima (2004): Pseudo-conformal quaternionic CR structure.
- K.-Nayatani(2001, 2013)*: Quaternionic CR structure (QCR).
- Marchiafava-Ornea-Pantilie (2012): CR quaternionic structure

Question: What are different?

*The notion of QCR str. of the 2001 version was defined for a real hypersurface in a certain (but not general) quaternionic manifold (e.g., $\mathbb{H}^{n+1}$); So we change the definition of QCR str. so that any real hypersurface in a quaternionic manifold naturally admits such a structure.

Extracted from Machida (2004):

In $2n$-dimensions:

$$\text{シンプレクティック構造} \leftarrow \text{Kähler 計量} \;\; \rightarrow \text{複素構造}.$$

In $(2n+1)$-dimensions:

$$\text{接触構造} \leftarrow \text{Sasaki 計量} \;\; \rightarrow \text{CR 構造}.$$

In $4n$-dimensions:

超シンプレクティック構造

In $(4n+3)$-dimensions:

超接触構造* $\leftarrow$ 超 Sasaki 計量† $\rightarrow$ 超 CR 構造‡.

*This is also called tri-contact structure (鼎接触構造) of quaternionic type (4元型3-接触構造).

†This is usually called a 3-Sasakian metric.

‡This may be equivalent to our HCR str. in the 2001 version.

Twistor theory

• Penrose (1976)

• Atiyah-Hitchin-Singer (1978)

• Salamon (1982, 1986): quaternionic-Kähler / quaternionic manifolds

• LeBrun (1984): Twistor CR manifolds and three-dimensional conformal geometry

Cf. Sato-Yamaguchi (1989): Lie contact manifolds

• Machida-Sato (2000)

• LeBrun-Mason (2007)

Quaternionic contact case (Biquard):

- Typical examples: S^{4n+3}(sphere), $\mathcal{H}^{4n+3}$(quaternionic Heisenberg group).

- Analogue of the Tanaka-Webster connection:

  Biquard connection D^{biq}.

- The twistor CR spaces for QC manifolds are constructed.

- QC structures sufficiently close to the standard one on S^{4n+3} exist, and are realized as the conformal infinity of $B_{\mathbb{H}}^{n+1}$ with a certain complete Einstein metric.

Quaternionic CR case (Nayatani-K.):

- Typical examples: S^{4n+3}, $\mathcal{H}^{4n+3}$, and <u>any</u> real hyersurface in a quaternionic manifold
- Analogue of the Tanaka-Webster connection: canonical connection D^{can}, under "ultra-pseudoconvexity"

Twistor CR spaces should be constructed.

Main Result The twistor space for a QCR manifold has a natural partially integrable almost CR structure.

In the reminder of this talk, unless otherwise stated, we assume $\dim_{\mathbb{R}} M > 7$.

§1. Review of CR structure

Definition: Let M be an odd-dim. (orientable) manifold. An almost CR str. on M is a pair (Q, J) of a corank 1 subbundle $Q \subset TM$ and a complex str. $J : Q \to Q$ (that is, $J^2 = -\mathrm{id}_Q$).

- (Q, J) is partially integrable if it satisfies
  $[JX, Y] + [X, JY] \in \Gamma(Q), \ X, Y \in \Gamma(Q)$
- (Q, J) is integrable if it further satisfies
  $[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0,$
  $X, Y \in \Gamma(Q)$.
  Then (Q, J) is called a CR structure.

Let (Q, J) be an almost CR str. on M with 1-form θ on M such that $\ker\theta = Q$. Define a bilinear form L_θ on Q by

$$L_\theta(X, Y) := d\theta(X, JY), \ \ X, Y \in Q.$$

- L_θ is symmetric and J-invariant on Q
  (i.e., $d\theta(X, JY) = d\theta(Y, JX), \ \ X, Y \in Q$)
  iff (Q, J) is partially integrable.
  (θ is unique up to $\theta \mapsto f\theta$ for $f \neq 0$, and
  satisfies $L_{f\theta} = fL_\theta$, because $d(f\theta) = df \wedge \theta + fd\theta$.)
  L_θ is called the Levi form for (Q, J)
- (Q, J) is strongly pseudoconvex if $L_\theta > 0$ for some θ,
  and such a θ is called a pseudohermitian structure.
  Then $\exists$a vector field T on M satisfying
  $\theta(T) = 1$ and $d\theta(T, \cdot) = 0$ (Reeb vector field).

§2. Quaternionic CR structures:

Unless otherwise stated, (a, b, c) denotes a cyclic permutation of $(1, 2, 3)$.

Definition A hyper CR str. on M^{4n+3} is a triple of almost CR str.s $(Q_a, I_a), a = 1, 2, 3$, satisfying

(i) Q_a and Q_b are transverse to each other;

(ii) $I_a(Q_a \cap Q_b) = Q_a \cap Q_c$;

(iii) $I_a I_b = I_c$ on $Q_b \cap Q_c$ and $I_b I_a = -I_c$ on $Q_a \cap Q_c$;

(iv) $[I_a X, Y] + [X, I_a Y] \in \Gamma(Q_a)$, $X, Y \in \Gamma(Q)$;

(v) $[I_a X, I_a Y] - [X, Y] - I_a([I_a X, Y] + [X, I_a Y]) \in \Gamma(Q)$, $X, Y \in \Gamma(Q)$.

Here, $Q := \bigcap_{a=1}^{3} Q_a$, and $I_a I_b = -I_b I_a = I_c$ hold on Q.

Definition (Strong integrability) A hyper CR structure $\{(Q_a, I_a)\}$ on M is said to be strongly integrable if $\exists$an $so(3)$-valued 1-form (γ_{ab}) on M such that

$$\begin{aligned} I_a([X,Y]-[I_aX,I_aY]) &= [X,I_aY]+[I_aX,Y] \\ &\quad -((\gamma_{ab}-\gamma_{ac}\circ I_a)\wedge I_b)(X,Y) \\ &\quad +((\gamma_{ab}-\gamma_{ac}\circ I_a)\wedge I_b)(I_aX,I_aY), \\ &\quad X,Y\in \Gamma(Q). \end{aligned}$$

Definition A quaternionic CR (QCR) str. is a covering by local hyper CR str.s satisfying an appropriate gluing condition.

Note The local str.s $\{(Q_a, I_a)\}$ of any real hypersurface in a quaternionic manifold satisfy the strong integrability.

Proposition For a hyper CR str. $(Q_a, I_a), a = 1, 2, 3$, there exists an $\mathbb{R}^3$-valued 1-form $\theta = (\theta_1, \theta_2, \theta_3)$, called a compatible 1-form, that satisfies $Q_a = \ker \theta_a$, and

$$\theta_c = \theta_a \circ I_b \text{ on } Q_b, \qquad \theta_c = -\theta_b \circ I_a \text{ on } Q_a.$$

(θ is also unique up to conformal change $\theta \mapsto f\theta$.)

Definition (Levi form) The Levi form L_θ is a symmetric bilinear form on Q invariant by all I_a's , defined by

$$L_\theta(X, Y) = \frac{1}{2}\left\{d\theta_a(X, I_a Y) + d\theta_a(I_b X, I_c Y)\right\}, \ X, Y \in Q.$$

(The RHS is independent of the choice of (a, b, c).)

Note The Levi form, the strong pseudoconvexity, and the notions of pseudohermitian str. and strong integrability can also be defined for a QCR str.. A strongly pseudoconvex QCR str. has a natural $CSp(n){\cdot}Sp(1)$-str. $(Q, [L_\theta], \mathbb{I})$.

Definition ($CSp(n){\cdot}Sp(1)$-str.) Let $Q \to M$ be a real vector bundle of rank $4n$. A $CSp(n){\cdot}Sp(1)$-str. on Q is a pair $([\gamma], \mathbb{I})$ of a conformal class $[\gamma]$ of metrics on Q and an S^2-bundle $\mathbb{I}$ over M consisting of complex str.s on Q satisfying

(i) $\mathbb{I}$ locally admits sections $I_a, a = 1, 2, 3$ that satisfy the quaternion relation $I_1 I_2 = -I_2 I_1 = I_3$ and

$$\mathbb{I} = \left\{ v_1 I_1 + v_2 I_2 + v_3 I_3 \mid {v_1}^2 + {v_2}^2 + {v_3}^2 = 1 \right\}.$$

(ii) $\gamma(IX, IY) = \gamma(X, Y),\ X, Y \in Q$, for any $I \in \mathbb{I}$.

§3. Real hypersurface

Let $\mathcal{N}$ be a real $(4n+4)$-dim. quaternionic manifold, that is, a $GL(n+1,\mathbb{H})\cdot Sp(1)$-manifold admitting a compatible torsion-free connection. Here,

- A $GL(n+1,\mathbb{H})\cdot Sp(1)$-str. on $T\mathcal{N}$ is given by

$$\mathcal{J} = \left\{ v_1\mathcal{J}_1 + v_2\mathcal{J}_2 + v_3\mathcal{J}_3 \mid {v_1}^2 + {v_2}^2 + {v_3}^2 = 1 \right\}$$

for some (local) triple of almost complex str. $\mathcal{J}_a$ that satisfies $\mathcal{J}_a\mathcal{J}_b = \mathcal{J}_c = -\mathcal{J}_b\mathcal{J}_a$.

- A torsion-free affine connection $\mathcal{D}$ on $\mathcal{N}$ is compatible with $\mathcal{J}$, if $\exists$a local $so(3)$-valued 1-form (γ_{ab}) such that $\mathcal{D}\mathcal{J}_a = \gamma_{ab} \otimes \mathcal{J}_b + \gamma_{ac} \otimes \mathcal{J}_c$.

Let $M = \rho^{-1}(0) \subset \mathcal{N}$ be a real hypersurface and set

$$Q_a = TM \cap \mathcal{J}_a TM, \qquad I_a = \mathcal{J}_a|_{Q_a}.$$

Then $\{(Q_a, I_a)\}$ is a local hyper CR structure on M, which defines a QCR structure on M equipped with compatible 1-form $\theta = (\theta_1, \theta_2, \theta_3)$,

$$\theta_a = -\frac{1}{2}\mathcal{J}_a d\rho|_{TM}, \ \ a = 1, 2, 3.$$

The $GL(n, \mathbb{H}) \cdot Sp(1)$-structure $(Q, \mathbb{I})$ of this QCR structure is called the standard underlying structure.

§4. Almost CR structure on the twistor space

Let M be a strongly integrable QCR manifold and $\pi\colon \mathcal{Z} = \mathbb{I} \to M$ the associated S^2-bundle

$$\mathcal{Z} = \{I_{\mathbf{v}} = v_1 I_1 + v_2 I_2 + v_3 I_3 \mid \mathbf{v} = (v_1, v_2, v_3) \in S^2\}.$$

Then one can define a hyperplane $Q_{\mathbf{v}} \subset T_q M,\ q \in M$ on which $I_{\mathbf{v}}$ makes sense. In fact, $Q_{\mathbf{v}} = \mathrm{Ker}\, \theta_{\mathbf{v}}$, where $\theta_{\mathbf{v}} = v_1\theta_1 + v_2\theta_2 + v_3\theta_3$.

Assume that M is ultra pseudoconvex and fix a pseudohermitian str..

<u>Definition</u> A quaternionic CR str. is ultra pseudoconvex if

$$h_\theta = (2n+4)L_\theta - \sum_{a=1}^{3} d\theta_a(\cdot, I_a\cdot)|_{Q\times Q} > 0.$$

Set $I = I_{\mathbf{v}} \in \mathcal{Z}$ and $q = \pi(I)$.

Using the canonical connection D^{can}, we decompose $T_I\mathcal{Z} = \mathcal{V}_I \oplus \mathcal{H}_I$, where $\mathcal{V}_I = T_I\pi^{-1}(q)$ and $\mathcal{H}_I$ is the horizontal space with respect to D^{can}.

Define a complex str. $\mathcal{J} = \mathcal{J}_I$ on $\mathcal{P}_I = \mathcal{V}_I \oplus \widetilde{Q_{\mathbf{v}}}$, where $\widetilde{Q_{\mathbf{v}}} = (\pi_*|_{\mathcal{H}_I})^{-1}(Q_{\mathbf{v}})$, so that

(i) $\mathcal{J}|_{\mathcal{V}_I}$ is the natural complex str. of $\pi^{-1}(q) = S^2$.

(ii) $\mathcal{J}|_{\widetilde{Q_{\mathbf{v}}}}$ is the lift of $I_{\mathbf{v}} : Q_{\mathbf{v}} \to Q_{\mathbf{v}}$.

Theorem $\mathcal{J} : \mathcal{P}_I \to \mathcal{P}_I$ is independent of the choice of θ. Then we have a well-defined almost CR str. $(\mathcal{P}, \mathcal{J})$ on $\mathcal{Z}$. Moreover, $(\mathcal{P}, \mathcal{J})$ is also partially integrable.

It remains to see whether $(\mathcal{P}, \mathcal{J})$ is integrable.

Note

• It is known that for an HCR str. with integrable CR structures $\{(Q_a, I_a)\}$ on M, the product space $\mathcal{Z} = M \times S^2$ has a natural CR str..

• Recently, Marchiafava et al. introduce "CR quaternonic structure" and "co-CR quaternionic structure".

§5. Comparison of QCR with QC

Definition (Quaternionic contact str.) A quaternionic contact (QC) str. on a $(4n+3)$-dim. manifold M is a corank 3 subbundle $Q \subset TM$ equipped with a $CSp(n) \cdot Sp(1)$-str. $([\gamma], \mathbb{I})$ such that Q is locally the kernel of an $\mathbb{R}^3$-valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ that satisfies

$$d\eta_a(X, I_a Y) = \gamma(X, Y), \ X, Y \in Q \ (a = 1, 2, 3).$$

Note $d\eta_a(\cdot, I_a \cdot)|_{Q \times Q} > 0$ and is invariant by all I_a's.

Proposition Let $(Q, [\gamma], \mathbb{I})$ be any QC str.. Take any 3-plane field $Q^\perp$ satisfying $TM = Q \oplus Q^\perp$. Then we can define a QCR str. $\{(Q_a, I_a)\}$ compatible with $(Q, \mathbb{I})$. If one takes $Q^\perp = \langle R_a \rangle$, the bundle generated by the "Reeb vector fields" R_a, $a = 1, 2, 3$, then the corresponding QCR str. is strongly integrable.

Note In this sense, the geometry of QCR str.s contains that of QC ones. The latter statement of this proposition can be verified by using the Biquard connection D^{biq}.

Theorem (K.-Nayatani) Let $(Q, \mathbb{I})$ be a $GL(n, \mathbb{H}) \cdot Sp(1)$-str. compatible with both a QC structure $(Q, [\gamma], \mathbb{I})$ and a QCR structure $\{(Q_a, I_a)\}$. Then $[\gamma] = [L_\theta]$ and

$$L_\theta(X, Y) = d\theta_a(X, I_a Y), \;\; X, Y \in Q \;\; (a = 1, 2, 3).$$

Note Such a QCR str. $\{(Q_a, I_a)\}$ is ultra pseudoconvex. Thus, for each choice of a pseudohermitian st. $\theta = (\theta_a)$ together with $Q^\perp = \langle R_a \rangle$, the Biquard connection D^{biq} for QC and the canonical connection D^{can} for QCR restrict the same Q-partial connection on Q, that is, $D^{\mathrm{biq}}_X Y = D^{\mathrm{can}}_X Y$ for $X, Y \in \Gamma(Q)$.

Recall: a quaternionic CR str. is ultra pseudoconvex $\Leftrightarrow$

$$h_\theta = (2n+4) L_\theta - \sum_{a=1}^{3} d\theta_a(\cdot, I_a \cdot)|_{Q \times Q} > 0.$$

Corollary There exists a $GL(n,\mathbb{H})\cdot Sp(1)$-str. compatible with a QCR str., but never compatible with a QC str..

Example A deformation of the quaternionic Heisenberg group has a $GL(n,\mathbb{H})\cdot Sp(1)$-str. compatible with a QCR, but never compatible with QC str., when some of $d\theta_a(\cdot, I_a\cdot)|_{Q\times Q}$ is indefinite or degenerate.

Example A non-quaternionic ellipsoid $\mathcal{E}$ admits no QC str. having the standard underlying $GL(n,\mathbb{H})\cdot Sp(1)$-str. defined by the defining function of $\mathcal{E}$.

Real ellipsoid Let $\mathcal{E} = \rho^{-1}(0) \subset \mathbb{H}^{n+1} = \mathbb{C}^{2n+2}$ with

$$\rho(z,w) = \sum_{i=1}^{n+1} \{A_i({z_i}^2 + {\overline{z_i}}^2) + B_i z_i \overline{z_i} + C_i(w_i^2 + {\overline{w_i}}^2) + D_i \overline{w_i} w_i\} - 1$$

($A_i, C_i \in \mathbb{R}, B_i, D_i \in \mathbb{R}_{>0}$: constants). Then

$$\begin{aligned} L_{\theta_1} &= 2\sum_{i=1}^{n+1}(B_i dz_i d\overline{z_i} + D_i d\overline{w_i} dw_i) \\ L_{\theta_2} &= \mathrm{Re}\sum_{i=1}^{n+1}\{(B_i + D_i)(dz_i d\overline{z_i} + d\overline{w_i} dw_i) \\ &\quad +2(A_i + C_i)(dz_i^2 + d{\overline{w_i}}^2)\} \\ L_{\theta_3} &= \mathrm{Re}\sum_{i=1}^{n+1}\{(B_i + D_i)(dz_i \overline{dz_i} + d\overline{w_i} dw_i) \\ &\quad +2(A_i - C_i)(dz_i^2 - d{\overline{w_i}}^2)\}. \end{aligned}$$

In particular, all $L_{\theta_a}, a = 1,2,3,$ coincide if and only if $B_i = D_i$ and $A_i = C_i = 0$, i.e., $\mathcal{E}$ is a quaternionic ellipsoid.

Example (Deformation of $\mathcal{H}^{4n+3}$) Take the following vector fields X^a_α on $\mathbb{R}^{4n}\times\mathbb{R}^3$:

$$\begin{aligned}
X^0_\alpha &= \frac{\partial}{\partial x^0_\alpha} + A^1_\alpha x^1_\alpha\frac{\partial}{\partial t_1} + A^2_\alpha x^2_\alpha\frac{\partial}{\partial t_2} + A^3_\alpha x^3_\alpha\frac{\partial}{\partial t_3},\\
X^1_\alpha &= \frac{\partial}{\partial x^1_\alpha} - B^1_\alpha x^0_\alpha\frac{\partial}{\partial t_1} - B^2_\alpha x^3_\alpha\frac{\partial}{\partial t_2} + B^3_\alpha x^2_\alpha\frac{\partial}{\partial t_3},\\
X^2_\alpha &= \frac{\partial}{\partial x^2_\alpha} + C^1_\alpha x^3_\alpha\frac{\partial}{\partial t_1} - C^2_\alpha x^0_\alpha\frac{\partial}{\partial t_2} - C^3_\alpha x^1_\alpha\frac{\partial}{\partial t_3},\\
X^3_\alpha &= \frac{\partial}{\partial x^3_\alpha} - D^1_\alpha x^2_\alpha\frac{\partial}{\partial t_1} + D^2_\alpha x^1_\alpha\frac{\partial}{\partial t_2} - D^3_\alpha x^0_\alpha\frac{\partial}{\partial t_3},
\end{aligned}$$

where $A^a_\alpha, B^a_\alpha, C^a_\alpha, D^a_\alpha \in \mathbb{R}$ (constants).

Set $T_a = 2\partial/\partial t_a$ and define Q, (Q_a, I_a) by

$$Q = \mathrm{span}\{X^a_\alpha\}_{1\le\alpha\le n, 0\le a\le 3}, \qquad Q_a = Q\oplus\mathbb{R}T_b\oplus\mathbb{R}T_c,$$

$$I_aX^0_\alpha = X^a_\alpha,\ I_aX^b_\alpha = X^c_\alpha,\ I_aT_b = T_c.$$

Then almost CR structures $(Q_a, I_a), a = 1, 2, 3$, on $\mathbb{R}^{4n} \times \mathbb{R}^3$ always satisfy (iv).

If $\{(Q_a, I_a)\}$ is strongly integrable (or all (Q_a, I_a) are CR structures), then $A^a_\alpha + B^a_\alpha + C^a_\alpha + D^a_\alpha$ does not depend on a (though may depend on α).

Then an $\mathbb{R}^3$-valued 1-form $\theta = (\theta_1, \theta_2, \theta_3)$ given by

$$\begin{aligned}
\theta_1 &= \frac{1}{2}\left[dt_1 + \sum_\alpha \left(-A^1_\alpha x^1_\alpha dx^0_\alpha + B^1_\alpha x^0_\alpha dx^1_\alpha - C^1_\alpha x^3_\alpha dx^2_\alpha + D^1_\alpha x^2_\alpha dx^3_\alpha\right)\right], \\
\theta_2 &= \frac{1}{2}\left[dt_2 + \sum_\alpha \left(-A^2_\alpha x^2_\alpha dx^0_\alpha + B^2_\alpha x^3_\alpha dx^1_\alpha + C^2_\alpha x^0_\alpha dx^2_\alpha - D^2_\alpha x^1_\alpha dx^3_\alpha\right)\right], \\
\theta_3 &= \frac{1}{2}\left[dt_2 + \sum_\alpha \left(-A^3_\alpha x^3_\alpha dx^0_\alpha - B^3_\alpha x^2_\alpha dx^1_\alpha + C^3_\alpha x^1_\alpha dx^2_\alpha + D^3_\alpha x^0_\alpha dx^3_\alpha\right)\right]
\end{aligned}$$

is compatible with the hyper CR structure and satisfies $\theta_a(T_b) = \delta_{ab}$.

The Levi forms (restricted to Q) for CR structures (Q_a, I_a) are

$$
\begin{aligned}
L_{\theta_1} &= \frac{1}{2}\sum_\alpha \Big\{ \left(A^1_\alpha + B^1_\alpha\right)\left(\left(dx^0_\alpha\right)^2 + \left(dx^1_\alpha\right)^2\right) \\
&\qquad + \left(C^1_\alpha + D^1_\alpha\right)\left(\left(dx^2_\alpha\right)^2 + \left(dx^3_\alpha\right)^2\right)\Big\}, \\
L_{\theta_2} &= \frac{1}{2}\sum_\alpha \Big\{ \left(A^2_\alpha + C^2_\alpha\right)\left(\left(dx^0_\alpha\right)^2 + \left(dx^2_\alpha\right)^2\right) \\
&\qquad + \left(B^2_\alpha + D^2_\alpha\right)\left(\left(dx^3_\alpha\right)^2 + \left(dx^1_\alpha\right)^2\right)\Big\}, \\
L_{\theta_3} &= \frac{1}{2}\sum_\alpha \Big\{ \left(A^3_\alpha + D^3_\alpha\right)\left(\left(dx^0_\alpha\right)^2 + \left(dx^3_\alpha\right)^2\right) \\
&\qquad + \left(B^3_\alpha + C^3_\alpha\right)\left(\left(dx^1_\alpha\right)^2 + \left(dx^2_\alpha\right)^2\right)\Big\}.
\end{aligned}
$$

Then the Levi form L_θ is

$$L_\theta = \frac{1}{4}\sum_\alpha \Lambda_\alpha \left(\left(dx_\alpha^0\right)^2 + \left(dx_\alpha^1\right)^2 + \left(dx_\alpha^2\right)^2 + \left(dx_\alpha^3\right)^2\right),$$

where $\Lambda_\alpha := A_\alpha^a + B_\alpha^a + C_\alpha^a + D_\alpha^a$.

If $\Lambda_\alpha > 0$ for all α, then the hyper CR structure is strongly pseudoconvex.

Well-definedness of $(\mathcal{P},\mathcal{J})$: Need to show the vanishing of

$$
\begin{aligned}
&g((\nabla' - \nabla)_{T_3} I, \Phi) + g((\nabla' - \nabla)_{T_2} I, I\Phi) = \\
&\quad 4u\Big[\sum_{i=1}^{4n} g((D_{J\varepsilon_i} I)(JW), I\varepsilon_i) - \Big\{(d\theta_1(T_2, KW) + d\theta_1(T_3, JW)) \\
&\qquad +(d\theta_2(T_2, W) - d\theta_3(T_3, W) - d\theta_2(T_3, IW) - d\theta_3(T_2, IW))\Big\}\Big] \\
&\quad +4v\Big[\sum_{i=1}^{4n} g((D_{K\varepsilon_i} I)(JW), I\varepsilon_i) - \Big\{(-d\theta_1(T_2, JW) + d\theta_1(T_3, KW)) \\
&\qquad +(d\theta_2(T_2, IW) - d\theta_3(T_3, IW) + d\theta_2(T_3, W) + d\theta_3(T_2, W))\Big\}\Big].
\end{aligned}
$$

for $\Phi = uJ + vK$.

Lemma

$$(d\theta_2 - \sqrt{-1}d\theta_3)(T_2 - \sqrt{-1}T_3, Z) = 0, \tag{1}$$

$$\sum_{k=1}^{n} g([e_{2k-1}, e_{2k}]_Q, Z) = -\frac{\sqrt{-1}}{2} d\theta_1(T_2 - \sqrt{-1}T_3, Z), \tag{2}$$

$$g(D_Z e_{2k-1}, e_{2k}) = \frac{\sqrt{-1}}{2} d\theta_1(T_2 - \sqrt{-1}T_3, Z), \ k = 1, 2, \ldots, n \tag{3}$$

for any $Z \in Q^{1,0} = \{Z \in Q \otimes \mathbb{C} \mid I_1 Z = \sqrt{-1}Z\}$ Here, $\{e_1, \ldots, e_{2n}\}$ is a local unitary frame field on $Q^{1,0}$ with respect to the Levi form $g = L_\theta$ satisfying $I_2 e_{2k-1} = \overline{e_{2k}}$, $I_2 e_{2k} = -\overline{e_{2k-1}}$, $k = 1, 2, \ldots, n$.