la corrispondenza Γ è riducibile può considerarsi come preliminare relativamente alla ricerca delle curve γ di cui ora si è detto, giacchè queste rientrano evidentemente fra quelle (anzi vi rientrano già le proiezioni generiche delle curve γ eseguite su uno S_{h+1}).

Courmayeur, 31 agosto 1931.

ALESSANDRO TERRACINI.

---

[27]) Non so se siano stati dati esempi di curve algebriche relative al caso $r = 3$, $h = 1$, vale a dire di curve *algebriche* sghembe dello spazio ordinario le cui rette tangenti siano tutte ulteriormente secanti. Invece già entro classi molto semplici di curve se ne trovano di analitiche; per es. le

$$x_0 : x_1 : x_2 : x_3 = 1 : e^{\alpha t} : e^{\beta t} : e^{\gamma t}$$

dove α, β, γ sono costanti non nulle e diverse fra loro, legate a un'altra costante $k \neq 0$ dalle relazioni

$$\frac{e^{\alpha k} - 1}{\alpha} = \frac{e^{\beta k} - 1}{\beta} = \frac{e^{\gamma k} - 1}{\gamma}.$$

Si può certo soddisfare a queste condizioni, con k prefissato, prefissando anche il valore K comune a queste tre frazioni: dove basta prendere come K un valore (non nullo), non eccezionale secondo il teorema di PICARD per la funzione intiera della variabile complessa z

$$\frac{e^{kz} - 1}{z},$$

assumendo poi per α, β, γ tre valori di z per i quali questa funzione intiera diventa uguale a K: si vede subito che la retta tangente nel punto corrispondente al valore t del parametro si appoggia nuovamente alla curva nel punto ove il parametro vale $t + k$.

三点接割線の補題[1]
A tangential trisecant lemma

楫　元
横浜

第23回沼津研究会
———幾何，数理物理，そして量子論———
沼津工業高等専門学校
2016年3月8日

[1]待田芳徳先生から演題邦訳をいただきました．ありがとうございました．

万葉集

第六巻:0934: 朝なぎに楫の音聞こゆ御食つ国 ···

原文: 朝名寸二 梶音所聞 三食津國 野嶋乃海子乃 船二四有良信

作者: 山部赤人 (やまべのあかひと)

よみ: 朝なぎに、楫 (かぢ) の音 (おと) 聞こゆ、御食 (みけ) つ国、野島 (のしま) の海人 (あま) の、舟にしあるらし

意味: 朝凪 (あさなぎ) に舵 (かじ) の音が聞こえます。御食 (みけ) つ国の野島 (のしま) の海人 (あま) の舟なのでしょう。

第十九巻:4240: 大船に真楫しじ貫きこの我子を ···

原文: 大船尓 真梶繁貫 此吾子乎 韓國邊遣 伊波敝神多智

作者: 光明皇后 (こうみょうこうごう)

よみ: 大船 (おほぶね) に、楫 (まかぢ) しじ貫 (ぬ) き、この我子 (あこ) を、唐国 (からくに) へ遣 (や) る、斎 (いは) へ神たち

意味: 大船に櫂 (かい) をたくさん取りつけて、この我が子を唐の国へ遣 (つか) わします。どうかお守りください、神々よ。

歌風と万葉仮名編集 (https://ja.wikipedia.org/wiki/万葉集)

全文が漢字で書かれており、漢文の体裁をなしている。しかし、歌は、日本語の語順で書かれている。歌は、表意的に漢字で表したもの、表音的に漢字で表したもの、表意と表音とを併せたもの、文字を使っていないものなどがあり多種多様である。編纂された頃にはまだ仮名文字は作られていなかったので、万葉仮名とよばれる独特の表記法を用いた。つまり、漢字の意味とは関係なく、漢字の音訓だけを借用して日本語を表記しようとしたのである。その意味では、万葉仮名は、漢字を用いながらも、日本人による日本人のための最初の文字であったと言えよう。

Plan

1. **Introduction**
2. **Tangential Trisecant Lemma**
3. **Recent Result**
4. **Sketch of Proof**
5. **Conjectures**

We work over an algebarically closed field k of arbitrary characteristic $p \geq 0$.

1 Introduction

As a celebrated result in classical projective geometry, we have

Theorem (trisecant lemma)

Let $X \subseteq \mathbb{P}^N$ be a smooth projective curve.
If a general secant line of X is trisecant, then
X is planar, i.e., contained in a 2-plane.

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By virtue of the trisecant lemma, using generic projection, one can prove

Corollary (existence of a good plane-curve model)

A smooth projective curve is birationally equivalent to a plane curve with at most nodes for singularities.

Definition A line $L \subseteq \mathbb{P}^N$ is called

- a secant line of $X \overset{\text{def}}{\Leftrightarrow} \#(L \cap X) \geq 2$.
- a trisecant line of $X \overset{\text{def}}{\Leftrightarrow} \#(L \cap X) \geq 3$.

Question

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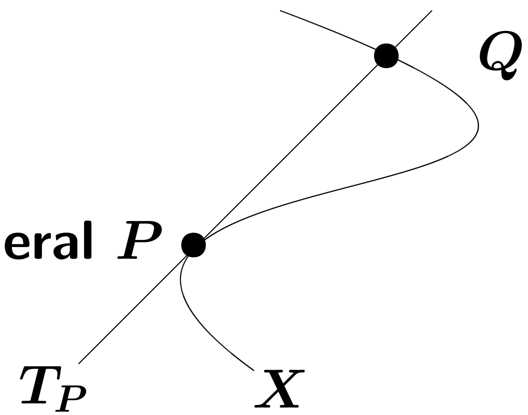

tangential trisecant line

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Definition A line $L \subseteq \mathbb{P}^N$ is called

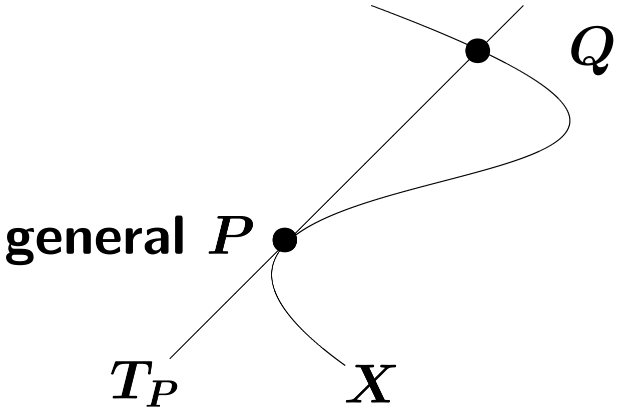
- a tangential trisecant line of $X \overset{\text{def}}{\Leftrightarrow} L$ tang to X & $\#(L \cap X) \geq 2$.

Definition A projective curve $X \subseteq \mathbb{P}^N$ is said to be **tangentially degenerate**

$\overset{\text{def}}{\Leftrightarrow}$ a general tangent line is tangential trisecant.

Question (naïve)

Is a projective curve $X \subseteq \mathbb{P}^N$ planar if it is tangentially degenerate?



tangential trisecant line

According to C.Ciliberto [MR0850959 (87i:14027)],
such a question was explicitly posed for the first time by A.Terracini:
In fact, in the footnote 27 on p.143 of his paper,

Alessandro TERRACINI:
"Sulla riducibilitá di alcune particolari corrispondenze algebriche," Rend.Circ.Mat.Palermo 56 (1932), 112–143.

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$\downarrow$ http://translate.google.com/

[27]) I don't know if have been given examples of algebraic curves related to the case $r = 3$, $h = 1$, that is to say of skew algebraic curves of the ordinary space whose tangent lines are further all secant.

r = dim of ambnt space, h = dim of linear spaces in question.

from "On the reducibility of some special algebraic correspondences"

In fact, he gave a counter-example of analytic curve in $\mathbb{A}^3_{\mathbb{C}}$, as follows:

Example (Terracini (1932), tang deg but non-planar affine analytic curve)

- Let $X = \varphi(\mathbb{C})$ be an analytic curve parametrized by
$$\varphi : \mathbb{C} \to \mathbb{C}^3; t \mapsto (e^{\alpha t}, e^{\beta t}, e^{\gamma t}), \quad (\alpha, \beta, \gamma \in \mathbb{C} \setminus \{0\}).$$
- Then, for $k \in \mathbb{C} \setminus \{0\}$, the tangent line to X at $\varphi(t)$ meets X again at $\varphi(t+k)$ iff $\boxed{\varphi(t+k) - \varphi(t) \parallel \dot{\varphi}(t)}$ as vectors in $\mathbb{C}^3$, where

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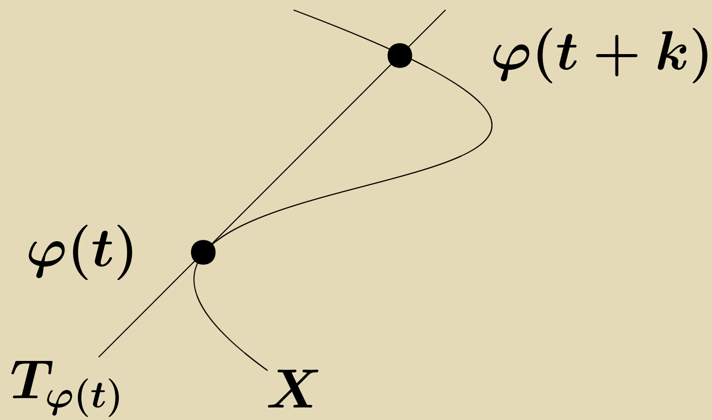

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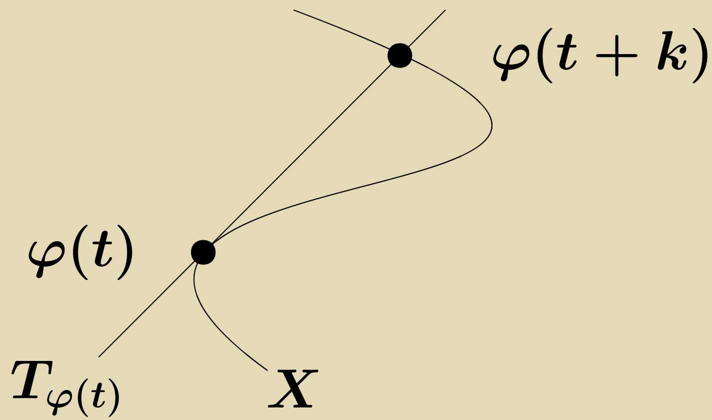

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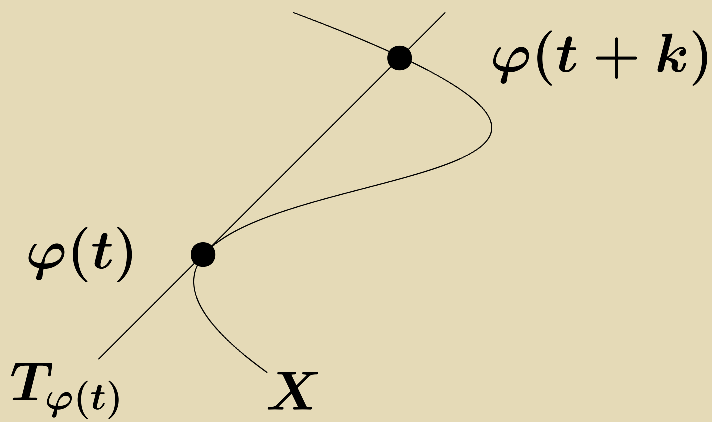

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  X is not planar in $\mathbb{C}^3$ if α, β, γ distinct.
- To show $\exists$ distinct $\alpha, \beta, \gamma \in \mathbb{C} \setminus \{0\}$ satisfying the relation above,

  consider a function on $\mathbb{C}$ as follows: $f(z) = \frac{e^{kz} - 1}{z}$.
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- According to Picard theorem (in complex analysis), for a general $K \in \mathbb{C}$, there exist infinitely many $z \in \mathbb{C}$ s.t. $f(z) = K$.

  $\rightsquigarrow$ just choose distinct $\alpha, \beta, \gamma \in f^{-1}(K)$. $\square$

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What's going on at
the infinity $\overline{X} \setminus X \subseteq \mathbb{CP}^3$?
e.g., $\dim_{\mathbb{R}}(\overline{X} \setminus X) = 0$ or 1?

Theorem (trisecant lemma, slightly generalized version)
For $X \subseteq \mathbb{P}^N$ a projective curve with normalization C, assume that

- the characteristic $p = 0$, or
- $b_1(P) = 1$ ($\forall P \in C$), i.e., $\iota : C \to \mathbb{P}^N$ unramified (e.g., X smooth).

If a general secant line of X is trisecant, then X is planar.

Proof May assume $N = 3$, by induction on N with generic projection.

- Suppose X were not planar (i.e., non-degenerate in $\mathbb{P}^3$).

$\rightsquigarrow \pi_z|_X : X \twoheadrightarrow \overline{X} := \pi_z(X) \subseteq \mathbb{P}^2$ is finite morph of deg ≥ 2,
$\because$ gen secant is trisec, where $\pi_z : \mathbb{P}^3 \setminus X \to \mathbb{P}^2 (\subseteq \mathbb{P}^3)$ proj from gen $z \in X$.
$\rightsquigarrow$ if $\pi_z|_X$ inseparable, then $z \in T_x$ for any smooth $x \in X$,
namely, X is strange with center z, and
if $\pi_z|_X$ sep, then for gen $P \in \overline{X}$, $\#(\pi_z|_X^{-1}(P)) \geq 2$.
$\rightsquigarrow \exists x \neq y \in X, \pi_z(x) = \pi_z(y) = P$.
$\rightsquigarrow T_x, T_y \subseteq \langle z, T_P \rangle \simeq \mathbb{P}^2$. $\rightsquigarrow T_x \cap T_y \neq \emptyset$.
$\rightsquigarrow$ for gen $z \in X$ and for gen $x, y \in X$ s.t. $z \in \langle x, y \rangle$, $T_x \cap T_y \neq \emptyset$.
$\rightsquigarrow$ for gen $x, y \in X$, $T_x \cap T_y \neq \emptyset$ (by dimension counting).
$\rightsquigarrow A := T_x \cap T_y$ (gen $x, y \in X$) $\Rightarrow T_w \ni A$ for gen $w \in X \setminus \langle T_x, T_y \rangle$.
$\rightsquigarrow X$ is strange with center A.
$\rightsquigarrow$ Whether $\pi_z|_X$ is separable or not, X would be strange.

- A strange curve X with unramified ι is classified either a line or a conic in $p = 2$. In particular, X is planar.

$\therefore$ This is a contradiction. (Note: only strange curve in $p = 0$ is a line.) $\square$

The unramifiedness of ι in $p > 0$ is essential. In fact, we have

Example (J.Roberts (1980)) Assume $p > 0$.

Let $X := \overline{\varphi(\mathbb{A}^1)} \subseteq \mathbb{P}^N$ the projective closure of $\varphi(\mathbb{A}^1)$, where

$$\varphi : \mathbb{A}^1 \to \mathbb{A}^N; t \mapsto (t, t^p, t^{p^2}, \ldots, t^{p^{N-2}}, t^{p^{N-1}}).$$

Then

- X is non-degenerate in $\mathbb{P}^N$, hence not planar if $N \geq 3$.
- A general secant line of X is trisecant: In fact,

  $\boxed{\varphi(t + t') \in \langle \varphi(t), \varphi(t') \rangle}$ for any $t \neq t' \in \mathbb{A}^1$.

  $\because$ $\varphi(t) + \varphi(t') = \varphi(t + t')$ as vectors in $\mathbb{A}^N$ by $p > 0$.
- X is strange: In fact, $T_{\varphi(t)} \ni \dot{\varphi}(t) = (1, 0, \ldots, 0)$ for any $t \in \mathbb{A}^1$.
- The induced morphism $\iota : \mathbb{P}^1 = C \to X \hookrightarrow \mathbb{P}^N$ is

  ramified at ∞ $\in \mathbb{P}^1$ unless $N = p = 2$ ($\Leftrightarrow$ X is smooth).

  $\because$ the order at ∞: $b_1(\infty) = p^{N-1} - p^{N-2} = p^{N-2}(p-1)$.

  $\rightsquigarrow b_1(\infty) = 1 \Leftrightarrow N = p = 2$. $\square$

Remark

Roberts' example above is introduced in F.Zak's textbook "Tangents and Secants of Algebraic Varieties" (p.41, Remark 1.12), as in origin a counter-example in $p > 0$ for "Terracini's Lemma," which asserts that $T_z \operatorname{Sec} X = \langle T_x X, T_y X \rangle$ for general $z \in \langle x, y \rangle$.

Corollary (existence of a good plane-curve model)

A smooth projective curve $X \subseteq \mathbb{P}^N$ is birationally equivalent to a plane curve with at most nodes for singularities.

Proof May assume $X \subseteq \mathbb{P}^3$ non-planar, and smooth (by gen projection).

Claim 1: $T_x \cap T_y = \emptyset$ for gen $x, y \in X$.

$\because$ shown in the proof of Trisecant Lemma.

Claim 2: gen pt $z \in \mathbb{P}^3$ is not on any trisecant line of X. $\square$

$\because$ The closure of the image,

$$p_{12} : \{(P, Q, R) | R \in \langle P, Q \rangle\} \subseteq X \times X \times X \to X \times X,$$

is proper closed in $X \times X$ by Trisecant Lemma, hence of dim ≤ 1.

$\rightsquigarrow \dim(\bigcup \text{trisecant lines}) \leq 2$, hence a proper subset of $\mathbb{P}^3$. $\square$

- Set $\overline{X} := \pi_z(X) \subseteq \mathbb{P}^2$, where $\pi_z : \mathbb{P}^3 \setminus \{z\} \to \mathbb{P}^2 (\subseteq \mathbb{P}^3)$ proj from $z \in \mathbb{P}^3$.
- for gen $z \in \mathbb{P}^3$,

  Claim 1 $\rightsquigarrow$ $\overline{X}$ has at most ordinary pts for sing.

  Claim 2 $\rightsquigarrow$ $\overline{X}$ has at most double pts for sing.

  Therefore $\overline{X}$ has at most ordinary double pts (i.e., nodes) for sing. $\square$

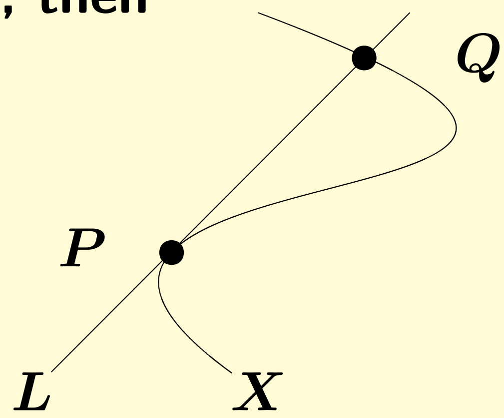
2 Tangential Trisecant Lemma

Theorem ('tangential trisecant lemma,' K (1986))

For a projective curve $X \subseteq \mathbb{P}^N$ with normalization C, assume that

- the characteristic $p = 0$, and
- the induced morphism $\iota : C \to \mathbb{P}^N$ is unramified.

If a general tangent line of X is a tangential trisecant line, then X is planar, that is, contained in a 2-plane.



Definition A line $L \subseteq \mathbb{P}^N$ is called a tangential trisecant line of X

$\overset{\text{def}}{\Leftrightarrow}$ L is tangent to X and $\#(L \cap X) \geq 2$ as a set.

Remark

- Some attempts to weaken the condition on singularities of X have been given, as I explain below.
- I believe that any condition on singularities is not necessary. Namely,

My Belief

The conclusion of Theorem above holds for any (possibly singular) projective curve $X \subseteq \mathbb{P}^N$ if $p = 0$.

On the other hand, ...

Counter examples in $p > 0$ (smooth, tang degen but non-planar curves):

Example 1 (K(1986), Rathmann(1987), Levcovitz(1991); graph of insep morph)

For $f : \mathbb{P}^1 \twoheadrightarrow \mathbb{P}^1$ of sep deg $s > 1$, insep deg $q = p^e$ with $e > 0$, set

$X :=$ (the image of $\Gamma_f \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$), Γ_f is the graph of f.

$\rightsquigarrow$ for gen $P \in X$, $\boxed{T_P \cap X = q \cdot P_1 + \cdots + q \cdot P_s}$ ($\exists P_1, \ldots, P_s = P$).

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Example 2 (K(1989); ordinary elliptic curves without inflection pt)

For ordinary elliptic curve X and for $s > 0$ s.t. $p \not| \, s$,

$\exists$ embedding $\varphi : X \hookrightarrow \mathbb{P}^N$ with $N \geq 3$ s.t.

for all $P \in X$, $\boxed{T_P \cap X = q \cdot P_1 + \cdots + q \cdot P_s}$, and

$\{P_1, \ldots, P_s = P\}$ form a cyclic subgroup of X with order s.

- If $q = 2$, then X has no inflection point.

An elliptic curve C in char $p > 0$ is said to be supersingular if $C_p = \{0\}$.
Otherwise C is said to be ordinary, and in that case $C_p \simeq \mathbb{Z}/p\mathbb{Z}$.

A point P of X is called an inflection point $\overset{\text{def}}{\Leftrightarrow}$ $i(X, T_P; P) \geq 3$.

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Example 3 (Garcia-Voloch(1991); Frobenius non-classical complete int)

Consider $X \subseteq \mathbb{P}^3 : x^{q+1} + y^{q+1} = 1, x^{q+1} + z^{q+1} = \lambda,$ $(1 \neq \lambda \in \mathbb{F}_q, p > 2)$.

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Counter examples in $p > 0$ (smooth, tang degen but non-planar curves):

Example 4 (Esteves-Homma (1994))

Assume $p > 3$ and set $X := \overline{\varphi(\mathbb{A}^1)} \subseteq \mathbb{P}^3$, where

$$\varphi : \mathbb{A}^1 \to \mathbb{A}^3, \quad \varphi(t) = (t, t^2 - t^p, t^3 + 2t^p - 3t^{p+1}).$$

$\rightsquigarrow$ for all $t \in \mathbb{A}^1$, $\boxed{T_{\varphi(t)} \cap X = 2 \cdot \varphi(t) + \varphi(t+1)}$.

In fact, $\varphi(t+1) - \varphi(t) = \dot{\varphi}(t)$ as vectors for all $t \in \mathbb{A}^1$.

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- The orders of X are $\{0, 1, 2, 3\}$.
- Surprisingly, it's reflexive!

(I will return to this example later)

Generalizations of the weak Tangential Trisecant Lemma in char $p = 0$:

- S.González, R.Mallavibarrena: "Osculating Degeneration of Curves," Comm.Alg. 31 (2003), 3829-3845.
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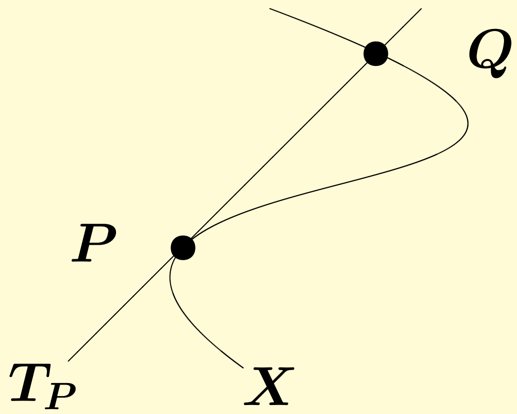
- M.Bolognesi, G.Pirola: "Osculating spaces and diophantine equations," Math.Nachr. 284 (2011), 960–972.
  They weaken the condition on singularities, treating locally toric curves, i.e., curves locally isomorphic to a monomial curve given by an analytical parameterization with relatively prime exponents:
  $t \mapsto (t^{a_1}, \ldots, t^{a_N})$ with $0 < a_1 < \cdots < a_N$ and $(a_1, \ldots, a_N) = 1$.

Theorem (Bolognesi-Pirola (2011), locally toric curves)

Let $X \subseteq \mathbb{P}^3$ a complex projective curve.
Assume that X is locally toric.
If X is tangentially degenerate, then X is planar.

3 Recent Result



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$\overset{\text{def}}{\Leftrightarrow}$ a gen tang line is tangential trisecant.

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If a proj curve $X \subseteq \mathbb{P}^N$ is tangentially degenerate, then is X planar?

Theorem ...

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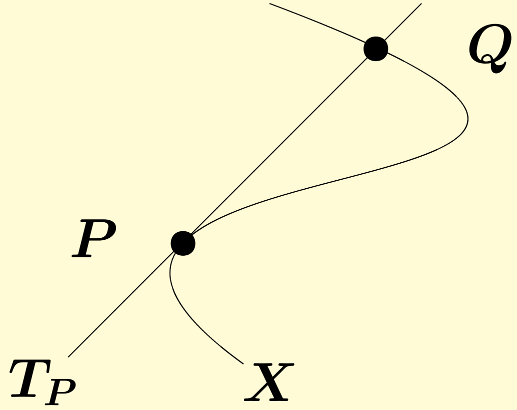

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$X \subseteq \mathbb{P}^N$ a non-deg proj curve with $N \geq 3$ in $p = 0$.

Assume that $\forall P \in C$ (normalization of X), $\exists$ distinct $i, j, k > 0$ s.t.
the orders, $b_i(P)$, $b_j(P)$ and $b_k(P)$ are relatively prime.

Then X is not tangentially degenarate.

...

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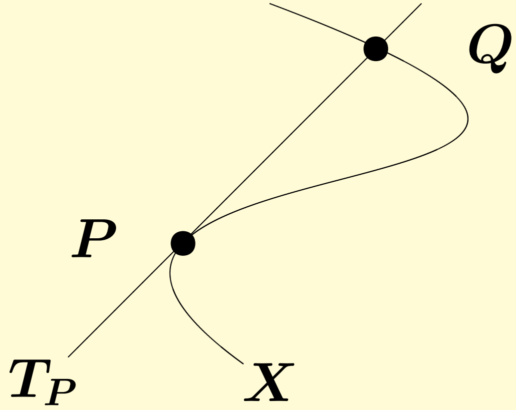

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What is "order"?

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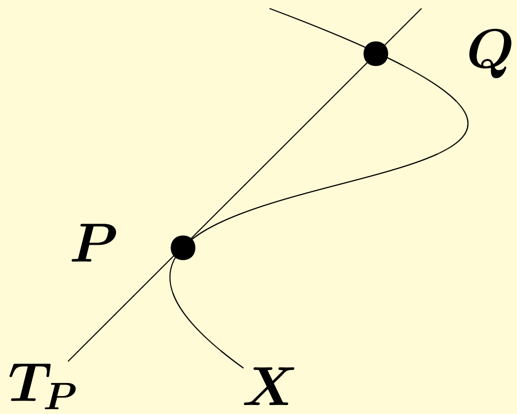

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Definition (orders)

The orders at $P \in C$ are a sequence of non-neg integers defined by

$$\{b_0(P) < b_1(P) < b_2(P) < \cdots < b_N(P)\} := \{v_P(f) | 0 \neq f \in \Lambda\},$$

where

$\Lambda \subseteq K(C)$ the linear system defining induced morph $\iota : C \to X \subseteq \mathbb{P}^N$

v_P a valuation of the local ring $\mathcal{O}_{C,P} \simeq \mathcal{O}_C(\Lambda)_P = \iota^* \mathcal{O}_{\mathbb{P}^N}(1)_P$.

The orders of X are defined to be the orders at a general pt of C.

Note: the function $b_i : P \mapsto b_i(P)$ is upper semi-continuous.

- The orders at $P \in C$ (normalization of $X \subseteq \mathbb{P}^N$):

$$\{b_0(P) < b_1(P) < b_2(P) < \cdots < b_N(P)\} := \{v_P(f) | 0 \neq f \in \Lambda\}.$$

- The orders of X: $\{b_i := b_i(P) \text{ for general } P \in C\}_{0 \leq i \leq N}$.

Remark Let $\iota : C \to \mathbb{P}^N$ induced morph from normalization $C \to X$.

- $b_0 = 0$ ($\because \mathrm{Bs}(\iota) = \emptyset \Leftrightarrow \forall P \in C, \exists f \in \Lambda, f(P) \neq 0$, i.e., $b_0(P) = 0$).
- $b_1 = 1$ ($\because \iota$ is bir & [ι unram at $P \Leftrightarrow \exists f \in \Lambda, \frac{df}{dt}(P) \neq 0 \Leftrightarrow b_1(P) = 1$]).

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What is "reflexive"?

- The **orders** at $P \in C$ (normalization of $X \subseteq \mathbb{P}^N$):
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Definition (reflexivity) A projective variety $X \subseteq \mathbb{P}^N$ said to be **reflexive** if
$$C(X) = C(X^*) \text{ via } \mathbb{P}^N \times \check{\mathbb{P}}^N \simeq \check{\check{\mathbb{P}}}^N \times \check{\mathbb{P}}^N,$$
where X^* the dual variety of X, and $C(X)$ the conormal variety of X.

- If X is reflexive, then one can expect $\{T_P\}_{P \in X}$ 'behaves' as in char $p = 0$.

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Fact (Hefez-Kakuta(1992), Homma-K(1992), K(1992)) [not used below]

Let b_i' be the highest power of p dividing b_i,

$\iota^{(i)} : X \dashrightarrow \mathbb{G}(i, \mathbb{P}^N)$ the i-th Gauss map, and

$\pi^{(i)} : C^{(i)}X \twoheadrightarrow X^{*(i)}$ the i-th conormal map of X

defined by osculating i-planes of X. Then for each $i \geq 1$, we have

$b_{i+1}' =$ insep-deg$(\iota^{(i)}) =$ insep-deg$(\pi^{(i)})$.

In particular, $\boxed{b_{i+1} \equiv 0 \bmod p} \Leftrightarrow \iota^{(i)}$ insep $\Leftrightarrow \pi^{(i)}$ insep. [End of §2: Main Result]

4 Sketch of Proof

Theorem **(K (2014), tangential trisecant lemma)**

$X \subseteq \mathbb{P}^N$ a non-deg proj curve with $N \geq 3$ in $p = 0$.
Assume that $\forall P \in C$ (normalization of X),
$\exists$ distinct $i, j, k > 0$ s.t. $(b_i(P), b_j(P), b_k(P)) = 1$.
Then X is not tangentially degenarate.

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To prove the above,
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we deduce contradiction.

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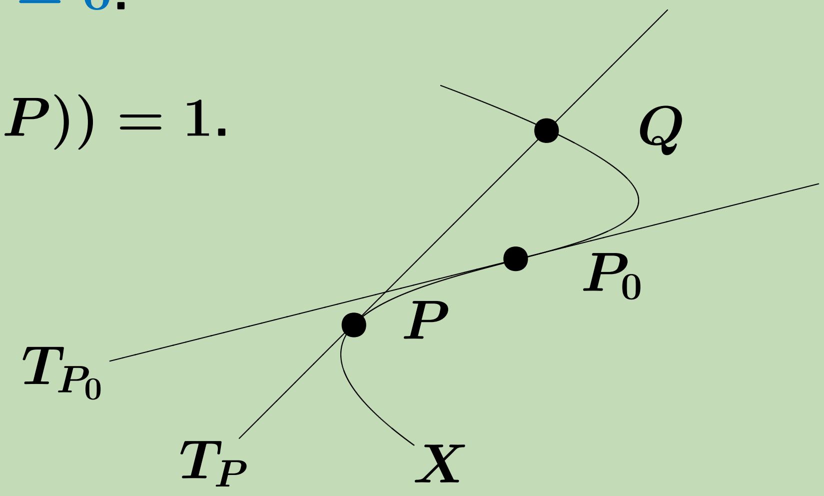

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Step 1: rephrase "tangential degeneration"
Step 2: parametrize the pts of contact P and of intersection Q on X
Step 3: find an inflection point P_0 of X where
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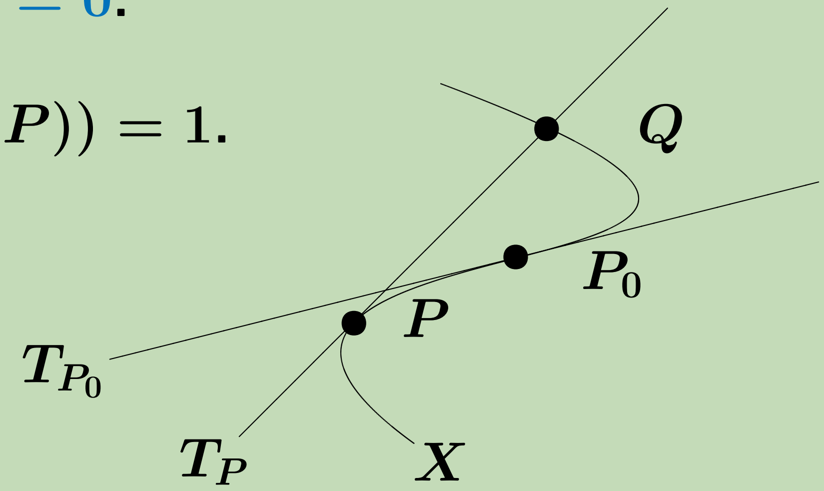

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Remark

- The proof here is different from the one for "trisecant lemma".
- The arguments here are similar to the ones of the weak version,
  except for Steps 3 and 5 in the plan above.

Step 1: rephrase "tangential degeneration".

- Let $C \to X$ the normal of $X \subseteq \mathbb{P}^N$, $\iota : C \to \mathbb{P}^N$ the induced morph.
- One can assign any $P \in C$ to a 'tangent line' T_P to X at $\iota(P)$.
  (just extend a rational map $C \dashrightarrow \mathbb{G}(1, \mathbb{P}^N); P \mapsto T_P$, to a morphism)
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- Set

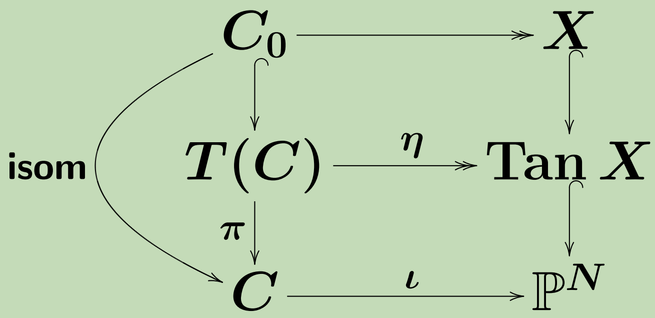
$$T(C) := \text{(projective tangent bundle)} = \coprod_{P \in C} T_P \subseteq C \times \mathbb{P}^N$$

with $\pi : T(C) \to C$ canonical projection,

$$\operatorname{Tan} X := \text{(tangential surface)} = \bigcup_{P \in C} T_P \subseteq \mathbb{P}^N$$

with $\eta : T(C) \twoheadrightarrow \operatorname{Tan} X$ natural projection,

$$C_0 := \text{(the locus of pts of contact)} \subseteq T(C) \text{ a section of } \pi,$$



- Then, ...

Step 1: rephrase "tangential degeneration".

- Let $C \to X$ the normal of $X \subseteq \mathbb{P}^N$, $\iota : C \to \mathbb{P}^N$ the induced morph.
- One can assign any $P \in C$ to a 'tangent line' T_P to X at $\iota(P)$. (just extend a rational map $C \dashrightarrow \mathbb{G}(1, \mathbb{P}^N); P \mapsto T_P$, to a morphism)
- Set

$$T(C) := \text{(projective tangent bundle)} = \coprod_{P \in C} T_P \subseteq C \times \mathbb{P}^N$$

with $\pi : T(C) \to C$ canonical projection,

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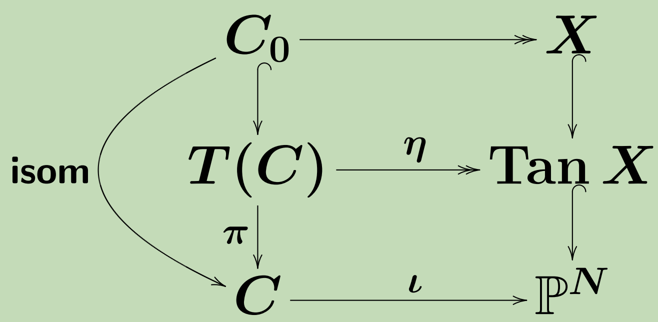

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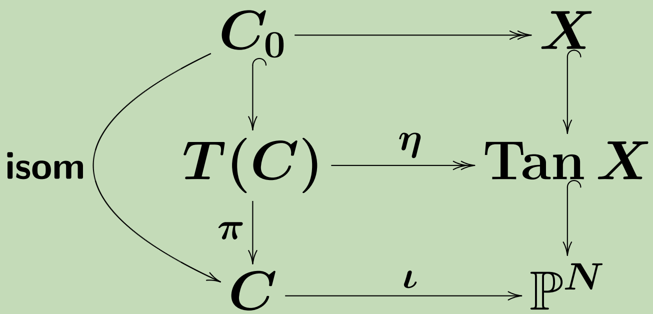
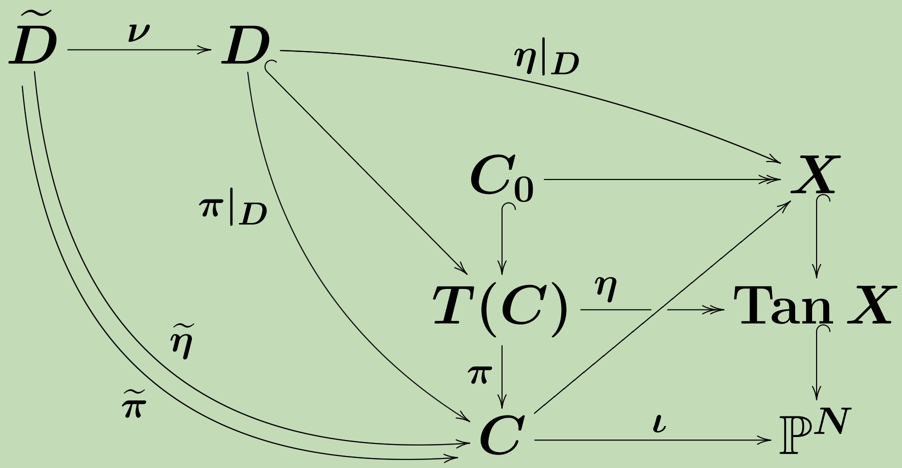


- Then, X tangentially degenerate $\Leftrightarrow$ $\dim \eta^{-1} X \setminus C_0 = 1$.
- Assume $X \subseteq \mathbb{P}^N$ $(N \geq 3)$ tangentially degenerate.
$\rightsquigarrow \exists$ 1-dim irred comp in $\eta^{-1} X \setminus C_0$.

Step 2: parametrize the pts of contact and of intersection.

- Consider
  - $\boxed{D}$ a 1-dim irred component of $\overline{\eta^{-1}X \setminus C_0}$ with reduced str,
    $\rightsquigarrow$ D is not a fibre of $\pi : T(C) \to C$,
  - $\nu : \widetilde{D} \to D$ the normalization,
  - $\widetilde{\pi} := \pi\nu : \widetilde{D} \to C$, and
  - $\widetilde{\eta} : \widetilde{D} \to C$ the natural morphism s.t. $\eta\nu = \iota\widetilde{\eta}$.
    $\leftsquigarrow$ $\eta\nu : \widetilde{D} \to X$ factors thru the normalization $C \to X$.



- Then for each $Q \in \widetilde{D}$, $\iota\widetilde{\eta}(Q) \in T_{\iota\widetilde{\pi}(Q)}$
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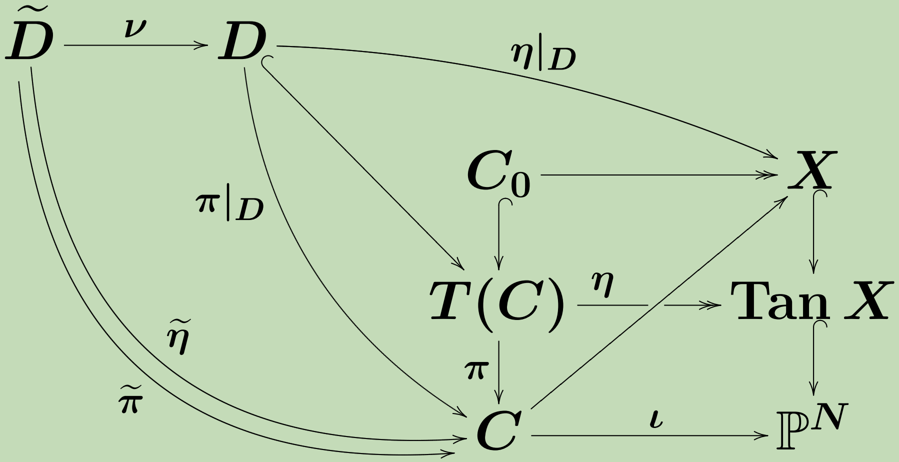

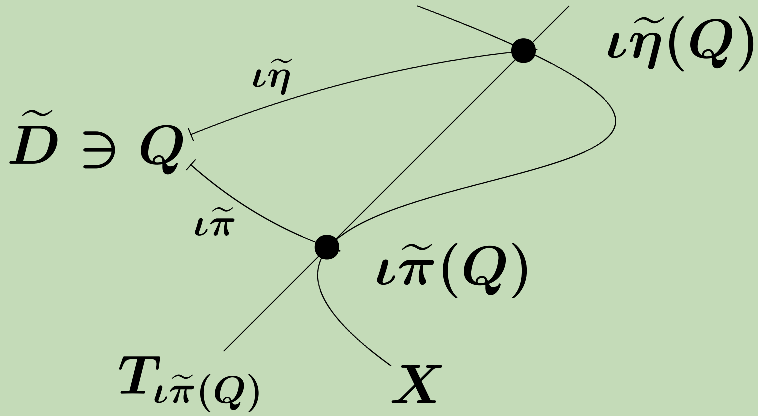

- Then for each $Q \in \widetilde{D}$, $\iota\widetilde{\eta}(Q) \in T_{\iota\widetilde{\pi}(Q)}$

  $\because$ $\pi\nu(Q) = \widetilde{\pi}(Q)$
  $\rightsquigarrow$ $\nu(Q) \in \pi^{-1}\widetilde{\pi}(Q)$
  $\rightsquigarrow$ $\iota\widetilde{\eta}(Q) = \eta\nu(Q) \in \eta(\pi^{-1}\widetilde{\pi}(Q)) = T_{\iota\widetilde{\pi}(Q)}$.

Step 3: $D \cap C_0 \neq \emptyset$ (i.e., $\exists$ inflection pt).

- **Let $\mathcal{P}^1_C(\mathcal{O}_C(1))$ the bdle of prin parts of $\mathcal{O}_C(1) := \iota^*\mathcal{O}_{\mathbb{P}^N}(1)$ of 1st ord, with natural homo a^1: $H^0(C, \mathcal{O}_C(1)) \otimes \mathcal{O}_C \to \mathcal{P}^1_C(\mathcal{O}_C(1))$ and the canonical exact sequence:**
  (ξ) $$0 \to \Omega^1_C \otimes \mathcal{O}_C(1) \to \mathcal{P}^1_C(\mathcal{O}_C(1)) \to \mathcal{O}_C(1) \to 0.$$
- **...**

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- Set $\mathcal{P} := \operatorname{Im} \mathbf{a}^1$, locally free of rk 2.
  $\because \iota : C \to \mathbb{P}^N$ gener unramified $\rightsquigarrow d\iota$ gener surj $\rightsquigarrow \mathbf{a}^1$ gener surj.
- Note that $T_C = \mathbb{P}(\mathcal{P})$, and (sect $T(C) \hookleftarrow C_0$) $\leftrightsquigarrow$ (1-quot $\mathcal{P} \twoheadrightarrow \mathcal{O}_C(1)$).
- ...

$$\begin{array}{ccccccccc}
 & 0 & \to & \iota^*\Omega^1_{\mathbb{P}^N} \otimes \mathcal{O}_C(1) & \to & H^0(C, \mathcal{O}_C(1)) \otimes \mathcal{O}_C & \to & \mathcal{O}_C(1) & \to 0 \ \text{(exact)} \\
 & & & \downarrow d\iota \otimes 1_{\mathcal{O}_C(1)} & & \downarrow \mathbf{a}^1 & & \| & \\
(\xi) & 0 & \to & \Omega^1_C \otimes \mathcal{O}_C(1) & \to & \mathcal{P}^1_C(\mathcal{O}_C(1)) & \to & \mathcal{O}_C(1) & \to 0 \ \text{(exact)}
\end{array}$$

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  $\rightsquigarrow \mathcal{P} \twoheadrightarrow \mathcal{O}_C(1)$ itself splits by the assumption $p = 0$.
  In fact, $\widetilde{\pi}$ is separable.
  $\rightsquigarrow$ can surj $\mathcal{P}^1_C(\mathcal{O}_C(1)) \twoheadrightarrow \mathcal{O}_C(1)$ splits, i.e., (ξ) would split.
- ...

$$\begin{array}{cccccccccc}
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 & & & & \downarrow & & \| & & & \\
(\xi) & 0 & \to & \Omega^1_C \otimes \mathcal{O}_C(1) & \to \mathcal{P}^1_C(\mathcal{O}_C(1)) & \to & \mathcal{O}_C(1) & \to & 0 & \text{(exact)}
\end{array}$$

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  $\rightsquigarrow$ can surj $\mathcal{P}^1_C(\mathcal{O}_C(1)) \twoheadrightarrow \mathcal{O}_C(1)$ splits, i.e., (ξ) would split.
- But (ξ) does not splits: Indeed, according to a theorem of Atiyah,

$$\begin{array}{ccccc} (\xi) & & \leftrightarrow & c_1(\mathcal{O}_C(1)) = \deg \mathcal{O}_C(1) \cdot 1_k \\ \Cap & & & \Cap \\ \mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_C(1), \Omega^1_C \otimes \mathcal{O}_C(1)) & = & H^1(C, \Omega^1_C) & \simeq k \end{array}$$

  where $\boxed{\deg \mathcal{O}_C(1) \cdot 1_k \neq 0}$ by $p = 0$.
- Therefore $D \cap C_0 \neq \emptyset$. $\square$

> **The First Chern Class**
> $c_1 : \mathrm{Pic}\, C \to H^1(C, \Omega^1_C)$
> $\leftsquigarrow d\log : \mathcal{O}_C^\times \to \Omega^1_C;\ f \mapsto \frac{df}{f}$

Step 4: local study around $C_0 \cap D$.

- Take a point $P_0 \in C_0 \cap D$,
  assume $x_1, \ldots, x_N \in \mathcal{O}_{P_0,C}$ defines $\iota : C \dashrightarrow \mathbb{A}^N \subseteq \mathbb{P}^N$ around P_0,
  set $\mathsf{x} := (x_1, \ldots, x_N)$, and fix a point $Q_0 \in \widetilde{D}$ s.t. $\widetilde{\pi}(Q_0) = \widetilde{\eta}(Q_0) = P_0$.

$\rightsquigarrow$ $\iota\widetilde{\pi}, \iota\widetilde{\eta} : \widetilde{D} \dashrightarrow \mathbb{A}^N$ resp given locally by

$$\widetilde{\pi}^*\mathsf{x} = (\widetilde{\pi}^*x_1, \ldots, \widetilde{\pi}^*x_N),\ \widetilde{\eta}^*\mathsf{x} = (\widetilde{\eta}^*x_1, \ldots, \widetilde{\eta}^*x_N) \text{ around } Q_0.$$

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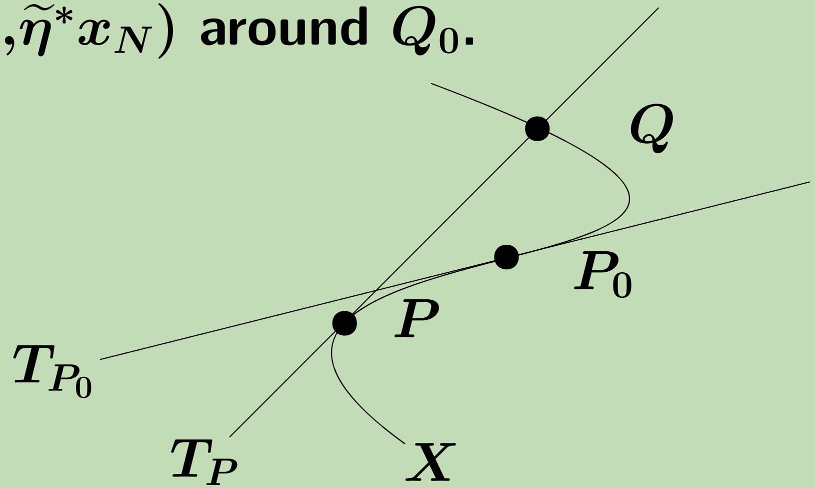

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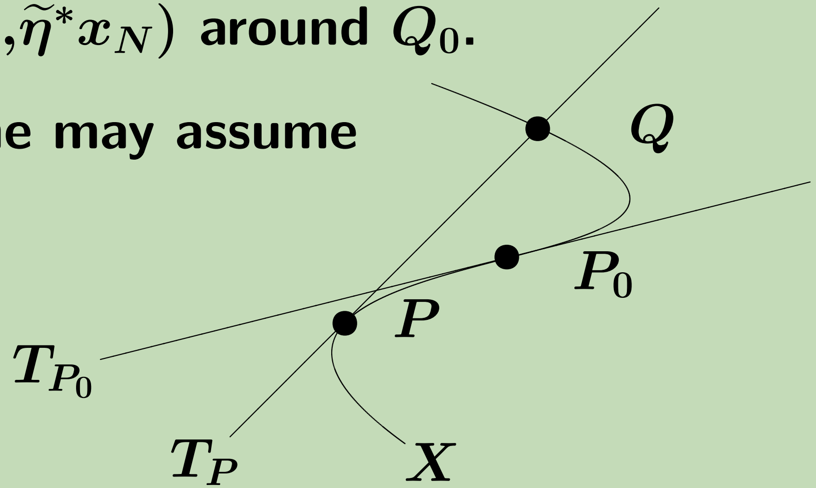
- Choosing a suitable change of coordinates, one may assume

$$\begin{cases} x_1 = t^{b_1} + \cdots \\ x_2 = t^{b_2} + \cdots \\ \quad \vdots \\ x_N = t^{b_N} + \cdots \end{cases}$$



  in the completion $\widehat{\mathcal{O}_{C,P_0}} \simeq k[[t]]$ with some reg para t of C at P_0,
  where $b_i := b_i(P_0)$ the orders at P_0.
- ...

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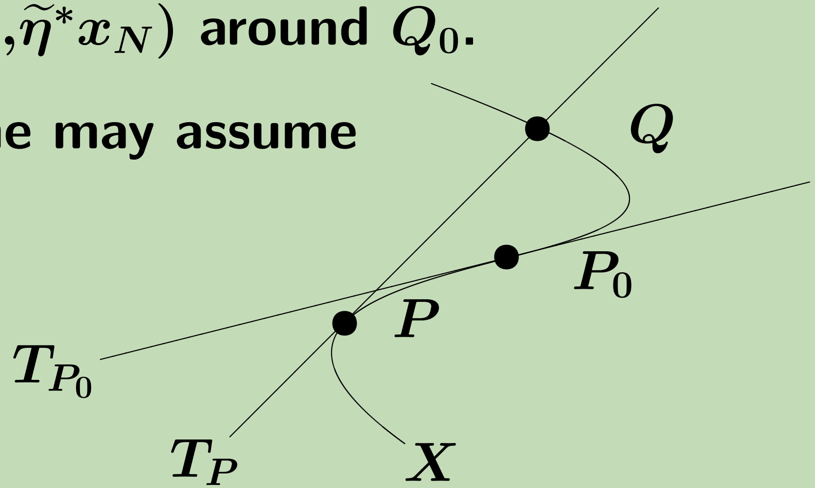

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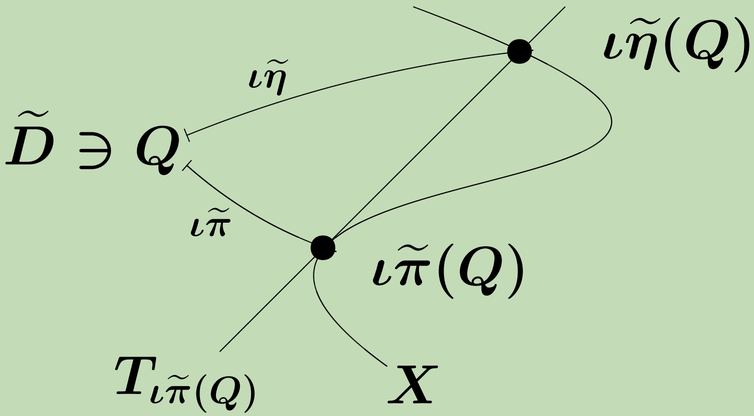
- Moreover may assume that

$$\widetilde{\pi}^*t = u^d + \cdots, \qquad \widetilde{\eta}^*t = \xi u^{d'} + \cdots \qquad \text{in } \widehat{\mathcal{O}_{\widetilde{D},Q_0}} \simeq k[[u]]$$

  with some $d \geq 1$, $d' \geq 1$, $\xi \in k^\times$ and some reg para u of $\widetilde{D}$ at Q_0.

Step 4: local study around $C_0 \cap D$ (continued).

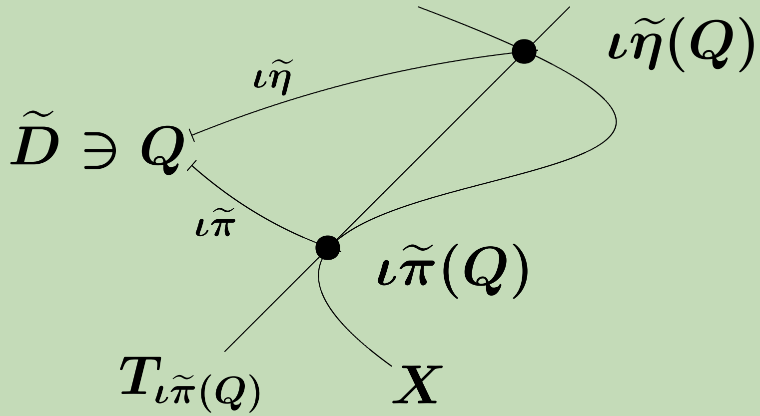


- $\boxed{\iota\widetilde{\eta}(Q) \in T_{\iota\widetilde{\pi}(Q)} \text{ for each } Q \in \widetilde{D}}$

$\rightsquigarrow$ $\widetilde{\pi}^*\dot{\mathbf{x}} \parallel \widetilde{\eta}^*\mathbf{x} - \widetilde{\pi}^*\mathbf{x}$ as vectors in $\mathbb{A}^N$,
where $\mathbf{x} := (x_1, \ldots, x_N)$ and $\dot{x}_i := dx_i/dt$.

$\rightsquigarrow$ ...

Step 4: local study around $C_0 \cap D$ (continued).



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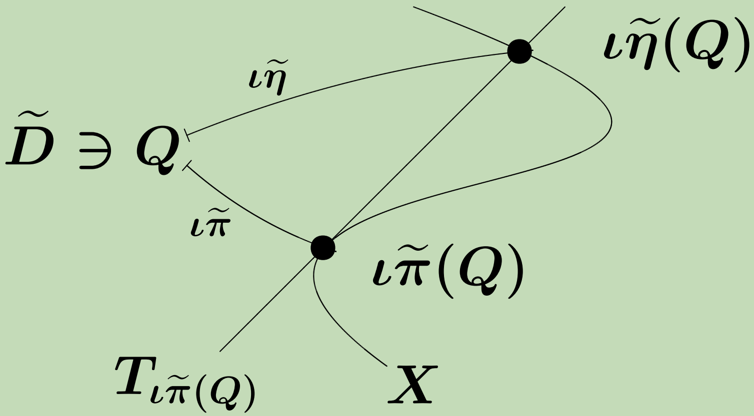
$\rightsquigarrow$ $\Gamma_{ij} = 0$ in $k[[u]]$ $(1 \leq i < j \leq N)$, where

$\Gamma_{ij} := \det \begin{bmatrix} \widetilde{\pi}^*\dot{x}_i & \widetilde{\eta}^*x_i - \widetilde{\pi}^*x_i \\ \widetilde{\pi}^*\dot{x}_j & \widetilde{\eta}^*x_i - \widetilde{\pi}^*x_j \end{bmatrix}$: (i,j)-minor of $[\widetilde{\pi}^*\dot{\mathbf{x}}, \widetilde{\eta}^*\mathbf{x} - \widetilde{\pi}^*\mathbf{x}]$

...

Step 4: local study around $C_0 \cap D$ (continued).



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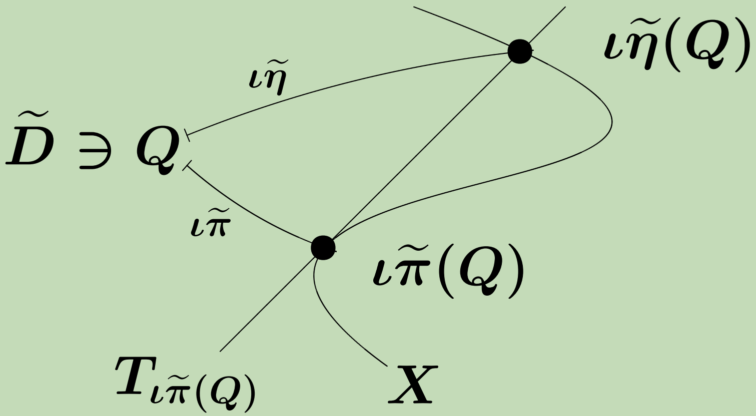
$$= \det \begin{bmatrix} b_i(u^d + \cdots)^{b_i - 1} + \cdots & \{(\xi u^{d'} + \cdots)^{b_i} + \cdots\} - \{(u^d + \cdots)^{b_i} + \cdots\} \\ b_j(u^d + \cdots)^{b_j - 1} + \cdots & \{(\xi u^{d'} + \cdots)^{b_j} + \cdots\} - \{(u^d + \cdots)^{b_j} + \cdots\} \end{bmatrix}$$

$$= \begin{cases} \xi^{b_j} u^{d(b_i - 1) + d'b_j} + \cdots, & \text{if } d' < d, \\ (b_i(\xi^{b_j} - 1) - b_j(\xi^{b_i} - 1)) u^{d(b_i + b_j + 1)} + \cdots, & \text{if } d' = d, \\ (b_i - b_j) u^{d(b_i + b_j - 1)} + \cdots, & \text{if } d' > d. \end{cases}$$

$\rightsquigarrow$ ...

Step 4: local study around $C_0 \cap D$ (continued).



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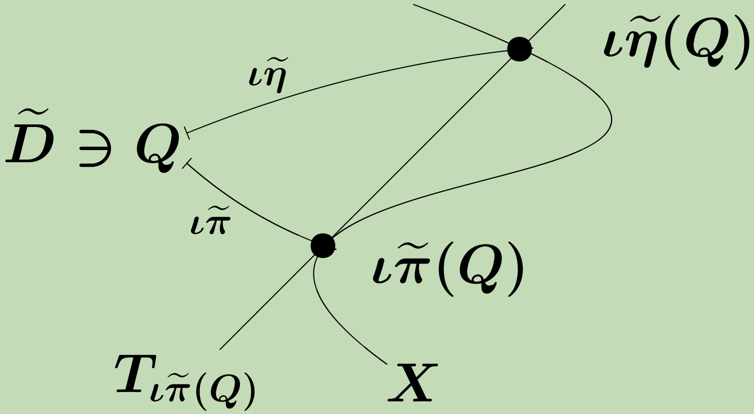
$\rightsquigarrow$ $d = d'$ and $b_i(\xi^{b_j} - 1) - b_j(\xi^{b_i} - 1) = 0$,
where $\{b_i := b_i(P_0)\}$ the orders at P_0 and $\widetilde{\eta}^*t = \xi u^d + \cdots$.

- ...

Step 4: local study around $C_0 \cap D$ (continued).



- $\iota\widetilde{\eta}(Q) \in T_{\iota\widetilde{\pi}(Q)}$ for each $Q \in \widetilde{D}$

$\rightsquigarrow$ $\widetilde{\pi}^*\dot{\mathbf{x}} \parallel \widetilde{\eta}^*\mathbf{x} - \widetilde{\pi}^*\mathbf{x}$ as vectors in $\mathbb{A}^N$, where $\mathbf{x} := (x_1, \ldots, x_N)$ and $\dot{x}_i := dx_i/dt$.

$\rightsquigarrow$ $\Gamma_{ij} = 0$ in $k[[u]]$ $(1 \le i < j \le N)$, where

$$\Gamma_{ij} := \det \begin{bmatrix} \widetilde{\pi}^*\dot{x}_i & \widetilde{\eta}^*x_i - \widetilde{\pi}^*x_i \\ \widetilde{\pi}^*\dot{x}_j & \widetilde{\eta}^*x_i - \widetilde{\pi}^*x_j \end{bmatrix} : (i,j)\text{-minor of } [\widetilde{\pi}^*\dot{\mathbf{x}}, \widetilde{\eta}^*\mathbf{x} - \widetilde{\pi}^*\mathbf{x}]$$

$$= \det \begin{bmatrix} b_i(u^d + \cdots)^{b_i-1} + \cdots & \{(\xi u^{d'} + \cdots)^{b_i} + \cdots\} - \{(u^d + \cdots)^{b_i} + \cdots\} \\ b_j(u^d + \cdots)^{b_j-1} + \cdots & \{(\xi u^{d'} + \cdots)^{b_j} + \cdots\} - \{(u^d + \cdots)^{b_j} + \cdots\} \end{bmatrix}$$

$$= \begin{cases} \xi^{b_j} u^{d(b_i-1)+d'b_j} + \cdots, & \text{if } d' < d, \\ (b_i(\xi^{b_j} - 1) - b_j(\xi^{b_i} - 1))u^{d(b_i+b_j+1)} + \cdots, & \text{if } d' = d, \\ (b_i - b_j)u^{d(b_i+b_j-1)} + \cdots, & \text{if } d' > d. \end{cases}$$

$\rightsquigarrow$ $d = d'$ and $b_i(\xi^{b_j} - 1) - b_j(\xi^{b_i} - 1) = 0$,

where $\{b_i := b_i(P_0)\}$ the orders at P_0 and $\widetilde{\eta}^*t = \xi u^d + \cdots$.

- Now, set $F_{ab}(X) := b(X^a - 1) - a(X^b - 1) \in \mathbb{Q}[X]$ for $a > b \ge 1$.

Step 5: deduce contradiction.

$$F_{ab}(X) := b(X^a - 1) - a(X^b - 1) \in \mathbb{Q}[X]$$

- The polynomials $\{F_{b_j b_i}(X)\}_{1\leq i<j\leq N}$ in X ($b_i := b_i(P_0)$) have
  - irrelevant common root $X = 1$ with mult ≥ 2 ⇝ C_0 the pts P of contact,

    and
  - other common roots $X = \xi$ ⇝ D the pts Q of intersection of T_P and X.

  (Note: ξ might be equal to 1.)
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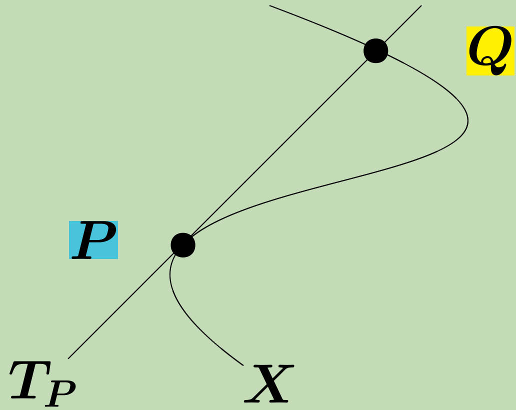

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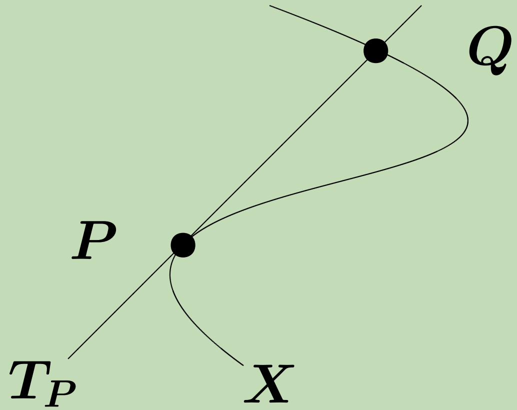

- But this contradicts to (our assumption: $(b_i, b_j, b_k) = 1$ $(\exists i < j < k)$.)

Lemma If $a > b > c \ge 1$ are relatively prime, then F_{ab}, F_{ac}, F_{bc} have a unique common root $X = 1$ in $\mathbb{C}$ and its multiplicity is exactly equal to 2.

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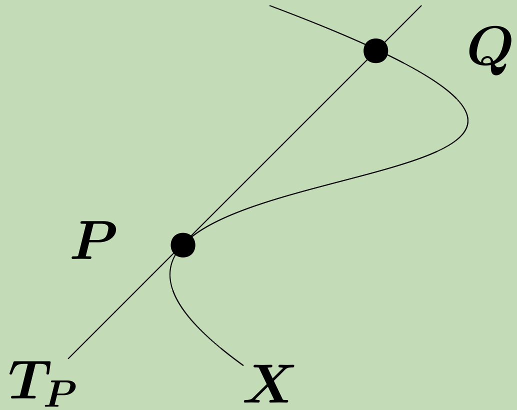

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Proof

- According to a lemma by Bolognesi-Pirola,
  F_{ab}, F_{ac}, F_{bc} have a unique common root $X = 1$ in $\mathbb{C}$.
  (elementary calculus (Rolle's theorem) with a clever argument)
- On the other hand, $X = 1$ is a root of $F_{ab}(X)$ of multiplicity exactly 2 since $F_{ab}(1) = F'_{ab}(1) = 0$ and $F''_{ab}(1) = ab(a - b) \neq 0$. $\square$

Case: $b_1 = 1$ ($\Leftrightarrow X$ smooth or nordal $\Leftrightarrow \iota$ unramified) [K(1986)]

Claim: $F_{a1}(X)$ and $F_{b1}(X)$ $(a > b > c = 1)$ have a unique common root $X = 1$ in $\mathbb{C}$ and its multiplicity is exactly equal to 2.

$$\begin{aligned} F_{a1}(X) &= (X^a - 1) - a(X-1) \\ &= (X-1)^2(X^{a-1} + 2X^{a-2} + \cdots + (a-2)X + (a-1)) \end{aligned}$$

- Set $f_a(X) := X^{a-1} + 2X^{a-2} + \cdots + (a-2)X + (a-1)$. $[X^{a-1} f_a(1/X) = \frac{d}{dX}(\frac{X^a-1}{X-1})]$

Claim $\Leftrightarrow f_a(X)$ and $f_b(X)$ $(a > b > 1)$ have no common root.
$\Leftrightarrow \nexists \xi \in \mathbb{C}$ s.t. $f_a(\xi) - \xi^{a-b} f_b(\xi) = f_b(\xi) = 0$ $(a > b > 1)$.

Here

$$\begin{aligned} f_a(X) - X^{a-b} f_b(X) &= bX^{a-b-1} + (b+1)X^{a-b-2} + \cdots + (a-1), \\ f_b(X) &= X^{b-1} + 2X^{b-2} + \cdots + (b-1). \end{aligned}$$

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- According to Kakeya's theorem, if $\boxed{f_a(\xi) - \xi^{a-b} f_b(\xi) = f_b(\zeta) = 0}$ $(\xi, \zeta \in \mathbb{C})$, then

$$\frac{1}{2} \leq |\zeta| \leq \frac{b-2}{b-1} < \frac{b}{b+1} \leq |\xi| \leq \frac{a-2}{a-1}.$$

$\rightsquigarrow$ $\xi \neq \zeta$. Thus the claim is proved. $\square$

Fact (Kakeya's theorem (掛谷の定理))

Let $f(X) = c_0 + c_1 X + \cdots + c_n X^n \in \mathbb{R}[X]$ with $c_i > 0$ $(\forall i)$.
If $f(\xi) = 0$ $(\xi \in \mathbb{C})$, then

$$\min\left\{\frac{c_0}{c_1}, \frac{c_1}{c_2}, \ldots, \frac{c_{n-1}}{c_n}\right\} \leq |\xi| \leq \max\left\{\frac{c_0}{c_1}, \frac{c_1}{c_2}, \ldots, \frac{c_{n-1}}{c_n}\right\}.$$

Problem Is $f_a(X) \in \mathbb{Z}[X]$ irreducible over $\mathbb{Q}$?

5 Conjectures

$$F_{ab}(X) = b(X^a - 1) - a(X^b - 1)$$

Observation (Esteves-Homma's example, revisited)

Assume $p > 3$ and set $X := \overline{\varphi(\mathbb{A}^1)} \subseteq \mathbb{P}^3$, where $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$ is defined by

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- the point at infinity $P_0 := \varphi(\infty) \in X$ is a unique inflection point.
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$F_{b_2(P_0)b_1(P_0)}(X) = F_{p,1}(X) = (X^p - 1) - p(X-1) = (X-1)^p$,
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This observation leads to the following ...

Conjecture

For any non-deg proj curve $X \subseteq \mathbb{P}^N$ with $N \geq 3$ in arbitrary char p,
if for any $P \in C$ there exist distinct $i, j, k > 0$ s.t.
none of $b_i(P)$, $b_j(P)$ and $b_k(P)$ is divisible by p,
then X is not tangentially degenerate.

Compare with

Theorem (K (2014), tangential trisecant lemma)

For any non-deg proj curve $X \subseteq \mathbb{P}^N$ with $N \geq 3$ in $p = 0$,
if for any $P \in C$ there exist distinct $i, j, k > 0$ s.t.
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In particular, ...

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In particular, under the condition $p = 0$, the following should hold:

My Belief

For any (possibly singular) projective curve $X \subseteq \mathbb{P}^N$ in $p = 0$,
if X is tangentially degenerate, then X is planar.

Thank you for your attention!

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Generic projection:

The existence of good plane-curve models follows from the trisecant lemma, by using general linear projections.
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Corollary **For a proj curve $X \subseteq \mathbb{P}^N$ with normalization C, assume that**

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- **the induced morphism $\iota : C \to \mathbb{P}^N$ is unramified.**

Then $\exists\ P \in X$ s.t. $\pi_P \iota : C \to \mathbb{P}^{N-1}$ is unramified,
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This consequence is one of keys in a nice result due to L.Ein, as follows:

Theorem (Ein (1987))

Let $H_{d,g,n}$ the open subscheme of the Hilbert scheme corresponding to smooth irreducible curves of degree d and genus g in $\mathbb{P}^n$.
Then $H_{d,g,4}$ is irreducible if $d \geq g + 4$.

Remark

- Severi's assertion (1921): "$H_{d,g,n}$ irreducible if $d \geq g + n$."
- Ein (1986): Assume $n \geq 6$. Then $H_{16n-35,8n+6,n}$ is reducible.

$\rightsquigarrow$ Severi's assertion is not correct.