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グラスマン束の次数公式
Degree Formulae for Grassmann Bundles

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——幾何，数理物理，そして量子論——

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0 Introduction of Introduction

What is degree?

(a) For an algebraic curve C in $\mathbb{P}^2$, the degree of C is given as follows:

$$\begin{aligned}\deg C &= \#[C \cap (\text{general line})] \\ &= \deg F(X, Y, Z),\end{aligned}$$

where $C = \{(x : y : z) \in \mathbb{P}^2 | F(x, y, z) = 0\}$ with homogeneous polynomial $F(X, Y, Z)$.
(b) More generally for a hypersurface H in $\mathbb{P}^N$, the degree of H is given as follows:

$$\begin{aligned}\deg H &= \#[H \cap (\text{general line})] \\ &= \deg F(X_0, \ldots, X_N),\end{aligned}$$

where $H = \{(x_0, \ldots, x_N) \in \mathbb{P}^N | F(x_0, \ldots, x_N) = 0\}$ with homog poly $F(X_0, \ldots, X_N)$.
(c) For an algebraic curve C in $\mathbb{P}^N$, the degree of C is given as follows:

$$\deg C = \#[C \cap (\text{general hyperplane})].$$

(d) How about the case of a projective variety X in $\mathbb{P}^N$?

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For a projective variety X in $\mathbb{P}^N$ of dimension n, the degree of X is given as follows:

(1) For general linear subspace L of dimension $N - n = \operatorname{codim}_{\mathbb{P}^N} X$,

$$\deg X = \#[X \cap L],$$

(2) Set $\varphi_X(l) := \dim_{\mathbb{C}}[\mathbb{C}[X_0, \ldots, X_N]/I(X)]_l$ $(l \in \mathbb{Z})$ the Hilbert function of $X \subseteq \mathbb{P}^N$.
$\rightsquigarrow \exists$ a polynomial $P_X(z) \in \mathbb{Q}[z]$ s.t. $\varphi_X(l) = P_X(l)$ for all $l \gg 0$. Then

$$\deg X := (\dim X)!(\text{the leading coefficient of } P_X \text{ in } z).$$

In fact,

$$P_X(z) = \frac{\deg X}{(\dim X)!} z^{\dim X} + (\text{terms of lower degree in } z).$$

(3) Let ω be the standard Kähler form on $\mathbb{P}^N$ (ass to the Fubini-Study metric). Then

$$\deg X = \int_X \omega^{\dim X} \overset{\text{Wirtinger}}{=} (\dim X)! \operatorname{Vol}(X).$$

(4) Let $\mathcal{O}_{\mathbb{P}^N}(1)$ be the hyperplane section bundle on $\mathbb{P}^N$, and consider the first Chern class $c_1(\mathcal{O}_{\mathbb{P}^N}(1)|_X) \in A^1(X)$ of its restriction to X. Then

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What is Grassmann bundle?

(a) First of all, the Grassmann variety is

$$\mathbb{G}(d,r) := \{S \subset E : \text{corank } d \text{ subspace of } E\},$$

where E a vector space of dimension r with $0 < d < r$.

(b) For a vector bundle $\mathcal{E}$ over a variety X, the Grassmann bundle is a relative version of Grassmann variety, precisely speaking, the Grassmann bundle $\mathbb{G}_X(d,\mathcal{E})$ is a variety parametrising corank d subbundles of $\mathcal{E}$:

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$$\begin{array}{ccccc} \mathbb{G}_X(d,\mathcal{E}) & \hookleftarrow & \pi^{-1}(x) & \simeq \mathbb{G}(d,r) \\ \pi\downarrow & & \downarrow & \\ X & \hookleftarrow & \{x\}. & \end{array}$$

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What is Grassmann bundle? (continued)

(c) Consider a functor from the category of schemes over X to the category of sets:

$$\underline{\mathbb{G}_X(d,\mathcal{E})} : \mathsf{Sch}_X \to \mathsf{Sets},$$

defined by

$$\underline{\mathbb{G}_X(d,\mathcal{E})}(X') := \{S' \subset \mathcal{E} \times_X X' : \text{corank } d \text{ subbundle of } \mathcal{E}\} \quad (X' \in \mathsf{Sch}_X).$$

Then $\underline{\mathbb{G}_X(d,\mathcal{E})}$ is represented by a scheme G over X, i.e., $\exists$ a scheme G over X s.t.

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for any $X' \in \mathsf{Sch}_X$. The above G is denoted by $\mathbb{G}_X(d,\mathcal{E})$, and
called the Grassmann bundle over X parametrising corank d subbundles of $\mathcal{E}$.
In fact, set $X' := G$, and let

$$[1_G] \leftrightarrow [\mathcal{S} \subset \mathcal{E} \times_X G],$$

where $\mathcal{S}$ a corank d subbdle of $\mathcal{E} \times_X G$ over G, called the universal subbundle on G.
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$G := \mathbb{G}(2,4)$ **the Grassmann variety parametrising codim 2 subspaces of a 4-dim vector space, with respect to the Plücker embedding:**

$$G \hookrightarrow \mathbb{P}^5.$$

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Proof Well-known as a classical fact, and also easily verified. $\square$

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For instance, a projective geometric proof is here:

[3] J.Harris: Algebraic Geometry. A First Course. GTM 133. Springer-Verlag, New York, 1992.

Quiz 2 $\deg G = ???$

if

$G := \mathbb{G}_C(2, \mathcal{L}^{\oplus 4})$ **the Grassmann bundle over a proj curve C parametr cork 2 subbdles of $\mathcal{L}^{\oplus 4}$ w/ very ample $\mathcal{L} \in \operatorname{Pic} C$, embedded in a proj space $\mathbb{P}^M$, as follows:**

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e.g., $\deg \mathbb{G}_{\mathbb{P}^1}(2, \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 4}) = ???$

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Answer $\deg G = 20 \deg \mathcal{L}$, **e.g.,** $\deg \mathbb{G}_{\mathbb{P}^1}(2, \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 4}) = 20.$

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How to calculate it?

Proposition $\deg G = 20 \deg \mathcal{L}$, if $G = \mathbb{G}_C(2, \mathcal{L}^{\oplus 4})$.

Proof ...

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Proof

$$\begin{array}{ccccccc}
\mathbb{G}(2,4) \times C & \overset{\text{Plücker embedding} \times 1_C}{\hookrightarrow} & & \mathbb{P}^5 \times C & & & \\
\wr\| & & & \wr\| & \searrow^{\left|\mathcal{O}_{\mathbb{P}^5}(1) \boxtimes \mathcal{L}^{\otimes 2}\right|} & & \\
G & \hookrightarrow & \mathbb{P}_C(\wedge^2 \mathcal{L}^{\oplus 4}) & = & \mathbb{P}_C((\mathcal{L}^{\otimes 2})^{\oplus 6}) & \hookrightarrow & \mathbb{P}^M. \\
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\end{array}$$

Proposition $\deg G = 20 \deg \mathcal{L}$, if $G = \mathbb{G}_C(2, \mathcal{L}^{\oplus 4})$.

Proof

$$\begin{array}{ccccccc}
\mathbb{G}(2,4) \times C & & \overset{\text{Plücker embedding} \times 1_C}{\hookrightarrow} & & \mathbb{P}^5 \times C & & \\
\wr\| & & & & \wr\| & \searrow^{|\mathcal{O}_{\mathbb{P}^5}(1) \boxtimes \mathcal{L}^{\otimes 2}|} & \\
G & \hookrightarrow & \mathbb{P}_C(\wedge^2 \mathcal{L}^{\oplus 4}) & = & \mathbb{P}_C((\mathcal{L}^{\otimes 2})^{\oplus 6}) & \hookrightarrow & \mathbb{P}^M. \\
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$\rightsquigarrow \mathcal{O}_G(1) = \mathcal{O}_{\mathbb{P}^M}(1)|_G \simeq \mathcal{O}_{\mathbb{P}^5}(1) \boxtimes \mathcal{L}^{\otimes 2}|_{\mathbb{G}(2,4)\times C} = \mathcal{O}_{\mathbb{G}(2,4)}(1) \boxtimes \mathcal{L}^{\otimes 2}$, and

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&= \binom{5}{4} (\mathcal{O}_{\mathbb{G}(2,4)}(1))^4_{\mathbb{G}(2,4)} (\mathcal{L}^{\otimes 2})^1_C \\
&= 5 \cdot \deg \mathbb{G}(2,4) \cdot 2 \deg \mathcal{L} \\
\boxed{\deg \mathbb{G}(2,4) = 2} \rightsquigarrow \quad &= 5 \cdot 2 \cdot 2 \deg \mathcal{L} \\
&= 20 \deg \mathcal{L}. \qquad \square
\end{aligned}$$

Quiz 3 $\quad \deg G =$ **???**

if

$G := \mathbb{G}_{\mathbb{P}^4}(2, T_{\mathbb{P}^4})$ **the Grassmann bundle over** $\mathbb{P}^4$ **parametrising corank** 2 **subbundles of the tangent bundle** $T_{\mathbb{P}^4}$**, embedded in a proj space** $\mathbb{P}^M$**, as follows:**

$$\begin{array}{ccccc} & \text{relative Plücker embedding}/\mathbb{P}^4 & & \left|\mathcal{O}_{\mathbb{P}_{\mathbb{P}^4}(\wedge^2 T_{\mathbb{P}^4})}(1)\right| & \\ G & \hookrightarrow & \mathbb{P}_{\mathbb{P}^4}(\wedge^2 T_{\mathbb{P}^4}) & \hookrightarrow & \mathbb{P}^M. \\ & \searrow & \downarrow & & \\ & & \mathbb{P}^4 & & \end{array}$$

Note: $\wedge^2 T_{\mathbb{P}^4}$ **is very ample with** $h^0 - 1 = 125 = M$**.**

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Answer $\deg G = 5040.$

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How to calculate it?

More generally,

X a proj variety over a field k, $\mathcal{E}$ a vector bundle over X,

$\mathbb{G}_X(d, \mathcal{E})$ the Grassmann bundle of corank d subbundles of $\mathcal{E}$ over X,

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Question How to calculate the degree of $G \subset \mathbb{P}^M$? i.e.,

How to calculate the self-intersection number as follows:

$$\deg G = \int_G c_1\left(\mathcal{O}_{\mathbb{P}^M}(1)|_G\right)^{\dim G}, \quad c_1\left(\mathcal{O}_{\mathbb{P}^M}(1)|_G\right) = c_1(\mathcal{Q}),$$

where $\mathcal{Q} \leftarrow \pi^*\mathcal{E}$ the universal quotient bundle of rank d on G.

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After my talk, you will be able to do this calculation!

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a joint work with Tomohide TERASOMA (寺杣友秀)

2 Main Results

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Remark ...

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Remark If X is a proj variety of dim n and $\wedge^d \mathcal{E}$ is very ample, then

$$\deg \mathbb{G}_X(d, \mathcal{E}) = \int_{\mathbb{G}_X(d,\mathcal{E})} c_1(\mathcal{Q})^N = \int_X \pi_*(c_1(\mathcal{Q})^N),$$

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formulae for $\pi_* \operatorname{ch}(\det \mathcal{Q})$ $\rightsquigarrow$ degree formulae for Grassmann bundles.

Theorem 1 (monomial type)

$$\pi_* \operatorname{ch}(\det \mathcal{Q}) = \sum_{k} \frac{\prod_{1 \le i < j \le d} (k_i - k_j + j - i)}{\prod_{1 \le i \le d} (r + k_i - i)!} \prod_{1 \le i \le d} s_{k_i}(\mathcal{E}),$$

where

$k = (k_1, \ldots, k_d) \in \mathbb{Z}^d_{\ge 0}$ a non-negative integer vector,

$s_i(\mathcal{E})$ the i-th Segre class of $\mathcal{E}$, and

$1/m! := 0$ if $m < 0$.

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Remark (definition of Segre classes)

The Segre classes $s_i(\mathcal{E})$ here are the ones satisfying

$$s(\mathcal{E}, t)c(\mathcal{E}, -t) = 1,$$

where

$s(\mathcal{E}, t) = \sum_i s_i(\mathcal{E})t^i$ the Segre series, and
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Our $s_i(\mathcal{E})$ differs by the sign $(-1)^i$ from the one in the textbook "Intersection Theory" by Fulton.

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Theorem 2 (Schur polynomial type)

$$\pi_* \operatorname{ch}(\det \mathcal{Q}) = \sum_{\lambda} \frac{\prod_{1\le i<j\le d} (\lambda_i - \lambda_j + j - i)}{\prod_{1\le i\le d} (r + \lambda_i - i)!} \Delta_\lambda(s(\mathcal{E})),$$

where

$\lambda = (\lambda_1, \ldots, \lambda_d)$ a partition, i.e., integers, $\lambda_1 \ge \cdots \ge \lambda_d \ge 0$, and
$\Delta_\lambda(s(\mathcal{E})) := \det[s_{\lambda_i+j-i}(\mathcal{E})]_{1\le i,j\le d} \in A^{|\lambda|}(X)$ with $|\lambda| = \sum_i \lambda_i$,
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Corollary (degree formulae for Grassmann bundle)

X a proj variety of dim n, $\mathcal{E}$ very ample, $N := \dim \mathbb{G}_X(d, \mathcal{E}) = d(r-d) + n$.

$$\deg \mathbb{G}_X(d,\mathcal{E}) = N! \sum_{|k|=n} \frac{\prod_{1\le i<j\le d}(k_i - k_j + j - i)}{\prod_{1\le i\le d}(r + k_i - i)!} \int_X \prod_{1\le i\le d} s_{k_i}(\mathcal{E})$$
$$= N! \sum_{|\lambda|=n} \frac{\prod_{1\le i<j\le d}(\lambda_i - \lambda_j + j - i)}{\prod_{1\le i\le d}(r + \lambda_i - i)!} \int_X \Delta_\lambda(s(\mathcal{E})),$$

where $|k| := \sum_i k_i$, and $|\lambda| := \sum_i \lambda_i$.

Remark (1) If $n = 0$, then one recovers a classical fact:

$$\deg \mathbb{G}(d,r) = \frac{(d(r-d))! \prod_{1\le k\le d-1} k!}{\prod_{1\le k\le d}(r-k)!},$$

where $\mathbb{G}(d,r)$ the Grassmann variety of codim d subspace of an r-dim vec space.

(2) If $d = 1$, then the definition of the top Segre class follows:

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where $\mathbb{P}_X(\mathcal{E})$ the proj space bdle ass to $\mathcal{E}$ on a proj variety X of dim n.

Corollary (degree formulae for Grassmann bundle)

X a proj variety of dim n, $\mathcal{E}$ very ample, $N := \dim \mathbb{G}_X(d, \mathcal{E}) = d(r-d)+n$.

$$\deg \mathbb{G}_X(d,\mathcal{E}) = N! \sum_{|k|=n} \frac{\prod_{1\le i<j\le d}(k_i - k_j + j - i)}{\prod_{1\le i\le d}(r+k_i-i)!} \int_X \prod_{1\le i\le d} s_{k_i}(\mathcal{E})$$

$$= N! \sum_{|\lambda|=n} \frac{\prod_{1\le i<j\le d}(\lambda_i - \lambda_j + j - i)}{\prod_{1\le i\le d}(r+\lambda_i-i)!} \int_X \Delta_\lambda(s(\mathcal{E})),$$

where $|k| := \sum_i k_i$, and $|\lambda| := \sum_i \lambda_i$.

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Proposition **(Answer to Quiz 3)**

$$\deg G = 5040, \text{ if } G = \mathbb{G}_{\mathbb{P}^4}(2, T_{\mathbb{P}^4}).$$

Proof

...

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Formula 1 **(monomial type)**

$\deg G = \cdots$

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Formula 1 (monomial type) ($4 = \dim X$, $4 = \operatorname{rk} \mathcal{E}$ and $2 =$corank)

$$\deg G = (2(4-2)+4)! \sum_{\substack{k_1+k_2=4 \\ k_i \geq 0}} \frac{\prod_{1\leq i<j\leq 2}(k_i - k_j + j - i)}{\prod_{1\leq i\leq 2}(4 + k_i - i)} \prod_{1\leq i\leq 2} \binom{i+4}{4}$$

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$$= 8! \sum_{0\leq k\leq 4} \frac{2k-3}{(3+k)!(6-k)!} \binom{k+4}{4} \binom{8-k}{4} \qquad (\leftsquigarrow k := k_1)$$

$$= \cdots$$

$$= 5040. \qquad \square$$

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$$\begin{aligned}
&\deg G \\
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&= 8! \sum_{2\leq\lambda\leq 4} \frac{2\lambda - 3}{(3+\lambda)!(6-\lambda)!} \det\begin{bmatrix} \binom{\lambda+4}{4} & \binom{\lambda+5}{4} \\ \binom{7-\lambda}{4} & \binom{8-\lambda}{4} \end{bmatrix} \qquad (\leftsquigarrow \lambda := \lambda_1) \\
&= \cdots \\
&= 5040. \qquad \square
\end{aligned}$$

Contents

a joint work with Tomohide TERASOMA (寺杣友秀)

3 Sketch of Proof

$\pi_* \left(c_1(\mathcal{Q})^N\right) = ? \in A^*(X)$, in terms of $s_i(\mathcal{E})$

(a) Set-up —"The double structure" of flag bundles—

$$G_i = \mathbb{F}^i_G(\mathcal{Q}) := \{[Q_i \subset Q_{i-1} \subset \cdots \subset Q_1(\subset \mathcal{Q})] \,|\, \operatorname{cork} Q_j = j\}$$

partial flag bundle of $\mathcal{Q}$ over G, parametrising flags of subbundles with corank 1 up to i in $\mathcal{Q}$, and

$$X_i = \mathbb{F}^i_X(\mathcal{E}) := \{[E_i \subset E_{i-1} \subset \cdots \subset E_1(\subset \mathcal{E})] \,|\, \operatorname{cork} E_j = j\}$$

partial flag bundle of $\mathcal{E}$ over X, parametrising flags of subbundles with corank 1 up to i in $\mathcal{E}$.

X a scheme of finite type over a field k,

$\mathcal{E}$ a vector bundle of rank r on X,

$\mathbb{G}_X(d, \mathcal{E})$ the Grassmann bundle of codim d subbundles of $\mathcal{E}$ over X,

$\pi : \mathbb{G}_X(d, \mathcal{E}) \to X$ the projection,

$\mathcal{Q} \leftarrow \pi^*\mathcal{E}$, the universal quotient bundle of rank d,

$\operatorname{ch}(\det \mathcal{Q}) := \sum_{N \geq 0} \frac{1}{N!} c_1(\mathcal{Q})^N$ the Chern character of $\det \mathcal{Q}$,

$A^*(\dots)$ the intersection ring of $\dots$,

$\pi_* : A^*(\mathbb{G}_X(d, \mathcal{E})) \otimes \mathbb{Q} \to A^{*-d(r-d)}(X) \otimes \mathbb{Q}$ the push-forward.

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Tower of proj space bundles

$\pi_*\left(c_1(\mathcal{Q})^N\right) = ? \in A^*(X)$, **in terms of** $s_i(\mathcal{E})$

$$\begin{array}{ll} \vdots & \\ \downarrow & \\ X_2 & := \mathbb{P}_{X_1}(\mathcal{E}_1) \\ \downarrow & p_1 \\ X_1 & := \mathbb{P}_X(\mathcal{E}) \\ \downarrow & p_0 \\ X & \end{array}$$

$$0 \to \mathcal{E}_1 \to p_0^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1) \to 0,$$

$\boxed{\mathcal{E}}$

$\wr$

Claim: $X_2 = \mathbb{F}^2_X(\mathcal{E})$

$\mathbb{F}^i_*(*)$ **flags of subbdles of cork** 1 **up to** i

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$$\begin{array}{llll}
\vdots & & & \\
\downarrow & & & \\
X_2 & := \mathbb{P}_{X_1}(\mathcal{E}_1) & \ni & y_2 \\
\downarrow & p_1 & & \downarrow \\
X_1 & := \mathbb{P}_X(\mathcal{E}) & \ni & y_1 \\
\downarrow & p_0 & & \downarrow \\
X & & \ni & y_0
\end{array}$$

$$0 \to \mathcal{E}_1 \to p_0^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1) \to 0,$$

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$$\begin{array}{lcccc}
 & \vdots & & & y_2 \in \mathbb{F}^2_X(\mathcal{E}) \\
 & \downarrow & & & \updownarrow \\
0 \to \mathcal{E}_2 \to p_1^*\mathcal{E}_1 \to \mathcal{O}_{\mathbb{P}_{X_1}(\mathcal{E}_1)}(1) \to 0, & X_2 := \mathbb{P}_{X_1}(\mathcal{E}_1) & \ni & y_2 & \leftrightarrow [\mathcal{E}_2(y_2) \subset \mathcal{E}_1(y_1)(\subset \mathcal{E}(y_0))] \\
 & \downarrow p_1 & & \downarrow & \\
0 \to \mathcal{E}_1 \to p_0^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1) \to 0, & X_1 := \mathbb{P}_X(\mathcal{E}) & \ni & y_1 & \leftrightarrow [\mathcal{E}_1(y_1) = E_1 \subset \mathcal{E}(y_0)] \\
 & \downarrow p_0 & & \downarrow & \\
\boxed{\mathcal{E}} & X & \ni & y_0 & \\
 & \wr & & &
\end{array}$$

$\mathbb{P}_X(\mathcal{E}) \times_X k(y_0) = \{$cork 1 subspaces E_1 of $\mathcal{E}(y_0) := \mathcal{E} \otimes k(y_0)\}$

Claim: $X_2 = \mathbb{F}^2_X(\mathcal{E})$

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Tower of proj space bundles

$$\boxed{\pi_*\left(c_1(\mathcal{Q})^N\right) = ? \in A^*(X), \text{ in terms of } s_i(\mathcal{E})}$$

$$\begin{array}{ccccccccc}
 & \vdots & & & & & y_2 \in \mathbb{F}^2_X(\mathcal{E}) \\
 & \downarrow & & & & & \wr\uparrow \\
0 \to \mathcal{E}_2 \to p_1^*\mathcal{E}_1 \to \mathcal{O}_{\mathbb{P}_{X_1}(\mathcal{E}_1)}(1) \to 0, & X_2 & := \mathbb{P}_{X_1}(\mathcal{E}_1) & \ni & y_2 & \leftrightarrow & [\mathcal{E}_2(y_2) \subset \mathcal{E}_1(y_1)(\subset \mathcal{E}(y_0))] \\
 & \downarrow\ p_1 & & & \downarrow & & \\
0 \to \mathcal{E}_1 \to p_0^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1) \to 0, & X_1 & := \mathbb{P}_X(\mathcal{E}) & \ni & y_1 & \leftrightarrow & [\mathcal{E}_1(y_1) = E_1 \subset \mathcal{E}(y_0)] \\
 & \downarrow\ p_0 & & & \downarrow & & \\
\boxed{\mathcal{E}} & X & & \ni & y_0 & &
\end{array}$$

$$\wr\downarrow \qquad \boxed{\mathbb{P}_X(\mathcal{E}) \times_X k(y_0) = \{\text{cork 1 subspaces } E_1 \text{ of } \mathcal{E}(y_0) := \mathcal{E} \otimes k(y_0)\}}$$

$$X_2 = \mathbb{F}^2_X(\mathcal{E}) \qquad \boxed{\mathbb{F}^i_*(*) \text{ flags of subbdles of cork 1 up to } i}$$

$$\wr\downarrow$$

inductively, set $X_{i+1} := \underline{\mathbb{P}_{X_i}(\mathcal{E}_i) = \mathbb{F}^i_X(\mathcal{E})} \quad (0 \le i \le d-1)$

$$\downarrow\ p_i$$

$$0 \to \mathcal{E}_i \to p_{i-1}^*\mathcal{E}_{i-1} \to \mathcal{O}_{X_i}(1) \to 0, \quad X_i$$

$\pi_*\left(c_1(\mathcal{Q})^N\right) = ? \in A^*(X)$, in terms of $s_i(\mathcal{E})$

$$
\begin{array}{cccc}
0 \to \mathcal{L}'_{d-1} \to q_{d-2}^*\mathcal{Q}_{d-2} \to \mathcal{L}'_{d-2} \to 0, & G_{d-1} := \mathbb{P}_{G_{d-2}}(\mathcal{Q}_{d-2}) = \mathbb{F}_G^{d-1}(\mathcal{Q}) & \mathbb{F}_X^d(\mathcal{E}) = \mathbb{P}_{X_{d-1}}(\mathcal{E}_{d-1}) =: X_d & 0 \to \mathcal{E}_d \to p_{d-1}^*\mathcal{E}_{d-1} \to \mathcal{L}_{d-1} \to 0, \\
 & \downarrow q_{d-2} & \downarrow p_{d-1} & \\
0 \to \mathcal{Q}_{d-2} \to q_{d-3}^*\mathcal{Q}_{d-3} \to \mathcal{L}'_{d-3} \to 0, & G_{d-2} := \mathbb{P}_{G_{d-3}}(\mathcal{Q}_{d-3}) = \mathbb{F}_G^{d-2}(\mathcal{Q}) & \mathbb{F}_X^{d-1}(\mathcal{E}) = \mathbb{P}_{X_{d-2}}(\mathcal{E}_{d-2}) =: X_{d-1} & 0 \to \mathcal{E}_{d-1} \to p_{d-2}^*\mathcal{E}_{d-2} \to \mathcal{L}_{d-2} \to 0, \\
 & \downarrow q_{d-3} & \downarrow p_{d-2} & \\
\vdots & \vdots & \vdots & \vdots \\
 & \downarrow q_1 & \downarrow p_2 & \\
0 \to \mathcal{Q}_1 \to q_0^*\mathcal{Q} \to \mathcal{L}'_0 \to 0, & G_1 := \mathbb{P}_G(\mathcal{Q}) = \mathbb{F}_G^1(\mathcal{Q}) & \mathbb{F}_X^2(\mathcal{E}) = \mathbb{P}_{X_1}(\mathcal{E}_1) =: X_2 & 0 \to \mathcal{E}_2 \to p_1^*\mathcal{E}_1 \to \mathcal{L}_1 \to 0, \\
 & \downarrow q_0 & \downarrow p_1 & \\
\boxed{0 \to \mathcal{S} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0,} & G := \mathbb{G}_X(d, \mathcal{E}) & \mathbb{F}_X^1(\mathcal{E}) = \mathbb{P}_X(\mathcal{E}) =: X_1 & 0 \to \mathcal{E}_1 \to p_0^*\mathcal{E} \to \mathcal{L}_0 \to 0, \\
 & \searrow^{\pi} & \downarrow p_0 & \\
 & & X & \boxed{\mathcal{E}}
\end{array}
$$

- $\mathcal{L}'_i := \mathcal{O}_{\mathbb{P}_{G_i}(\mathcal{Q}_i)}(1) \in \operatorname{Pic} G_{i+1}$, $\quad \mathcal{L}_i := \mathcal{O}_{\mathbb{P}_{X_i}(\mathcal{E}_i)}(1) \in \operatorname{Pic} X_{i+1}$.
- ...

$\pi_*\left(c_1(\mathcal{Q})^N\right) = ? \in A^*(X)$, in terms of $s_i(\mathcal{E})$

$$
\begin{array}{cccc}
0 \to \mathcal{L}'_{d-1} \to q^*_{d-2}\mathcal{Q}_{d-2} \to \mathcal{L}'_{d-2} \to 0, & G_{d-1} := \mathbb{P}_{G_{d-2}}(\mathcal{Q}_{d-2}) = \mathbb{F}^{d-1}_G(\mathcal{Q}) & \mathbb{F}^d_X(\mathcal{E}) = \mathbb{P}_{X_{d-1}}(\mathcal{E}_{d-1}) =: X_d & 0 \to \mathcal{E}_d \to p^*_{d-1}\mathcal{E}_{d-1} \to \mathcal{L}_{d-1} \to 0, \\
 & \downarrow q_{d-2} & \downarrow p_{d-1} & \\
0 \to \mathcal{Q}_{d-2} \to q^*_{d-3}\mathcal{Q}_{d-3} \to \mathcal{L}'_{d-3} \to 0, & G_{d-2} := \mathbb{P}_{G_{d-3}}(\mathcal{Q}_{d-3}) = \mathbb{F}^{d-2}_G(\mathcal{Q}) & \mathbb{F}^{d-1}_X(\mathcal{E}) = \mathbb{P}_{X_{d-2}}(\mathcal{E}_{d-2}) =: X_{d-1} & 0 \to \mathcal{E}_{d-1} \to p^*_{d-2}\mathcal{E}_{d-2} \to \mathcal{L}_{d-2} \to 0, \\
 & \downarrow q_{d-3} & \downarrow p_{d-2} & \\
\vdots & \vdots & \vdots & \vdots \\
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 & \downarrow q_0 & \downarrow p_1 & \\
0 \to \mathcal{S} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0, & G := \mathbb{G}_X(d, \mathcal{E}) & \mathbb{F}^1_X(\mathcal{E}) = \mathbb{P}_X(\mathcal{E}) =: X_1 & 0 \to \mathcal{E}_1 \to p^*_0\mathcal{E} \to \mathcal{L}_0 \to 0, \\
 & \searrow \pi & \downarrow p_0 & \\
 & & X & \mathcal{E}
\end{array}
$$

($q: G_{d-1} \to G$; $\pi: G \to X$)

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- $q^* c_1(\mathcal{Q}) = c_1(\mathcal{L}'_0) + \cdots + c_1(\mathcal{L}'_{d-1}) \in A^1(\mathbb{F}^{d-1}_G(\mathcal{Q}))$.
- ...

$\mathbb{F}^i_*(*)$ flags of subbdles of cork 1 up to i

$$\pi_*\left(c_1(\mathcal{Q})^N\right) = ? \in A^*(X), \text{ in terms of } s_i(\mathcal{E})$$

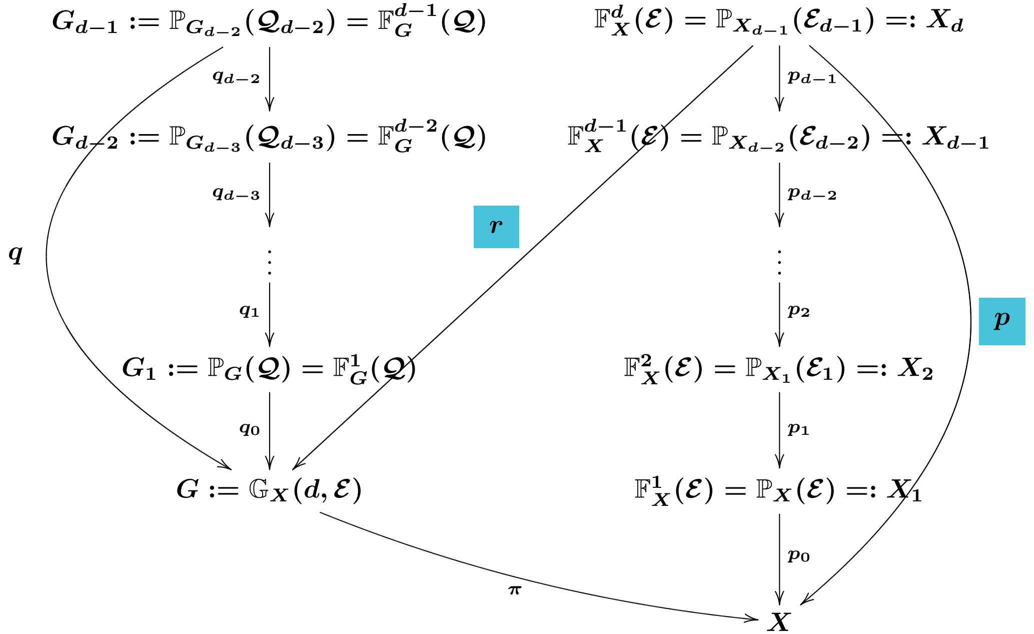
$$0 \to \mathcal{L}'_{d-1} \to q^*_{d-2}\mathcal{Q}_{d-2} \to \mathcal{L}'_{d-2} \to 0,$$

$$0 \to \mathcal{Q}_{d-2} \to q^*_{d-3}\mathcal{Q}_{d-3} \to \mathcal{L}'_{d-3} \to 0,$$

$$\vdots$$

$$0 \to \mathcal{Q}_1 \to q^*_0\mathcal{Q} \to \mathcal{L}'_0 \to 0,$$

$$0 \to \mathcal{S} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0,$$



$$0 \to \mathcal{E}_d \to p^*_{d-1}\mathcal{E}_{d-1} \to \mathcal{L}_{d-1} \to 0,$$

$$0 \to \mathcal{E}_{d-1} \to p^*_{d-2}\mathcal{E}_{d-2} \to \mathcal{L}_{d-2} \to 0,$$

$$\vdots$$

$$0 \to \mathcal{E}_2 \to p^*_1\mathcal{E}_1 \to \mathcal{L}_1 \to 0,$$

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$\mathbb{F}^i_*(*)$ flags of subbdles of cork 1 up to i

$\pi_* \left(c_1(\mathcal{Q})^N \right) = ? \in A^*(X)$, in terms of $s_i(\mathcal{E})$

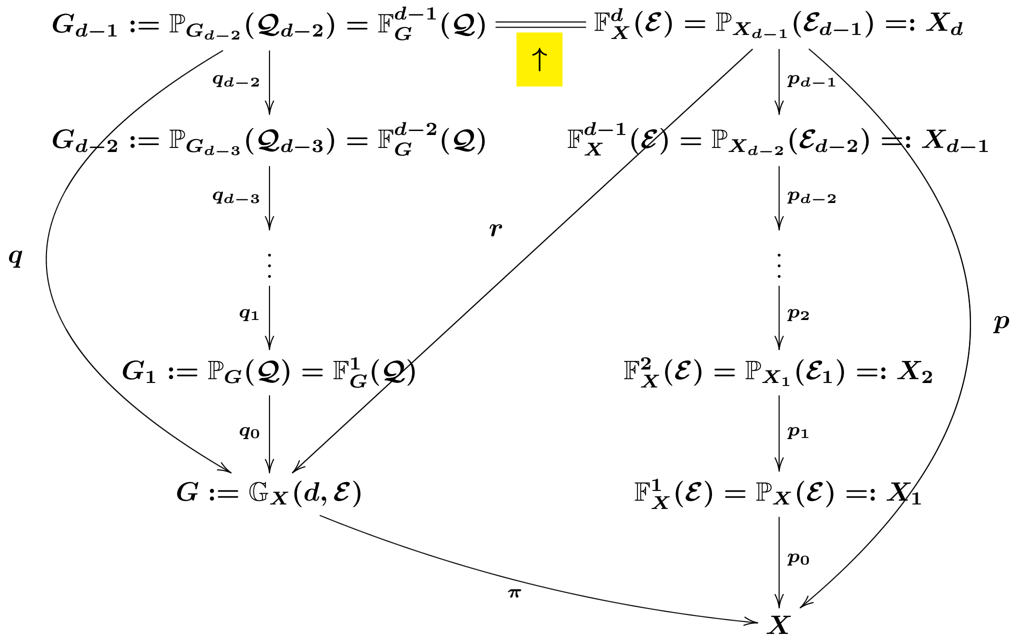
$0 \to \mathcal{L}'_{d-1} \to q^*_{d-2}\mathcal{Q}_{d-2} \to \mathcal{L}'_{d-2} \to 0,$

$0 \to \mathcal{Q}_{d-2} \to q^*_{d-3}\mathcal{Q}_{d-3} \to \mathcal{L}'_{d-3} \to 0,$

$\vdots$

$0 \to \mathcal{Q}_1 \to q^*_0\mathcal{Q} \to \mathcal{L}'_0 \to 0,$

$0 \to \mathcal{S} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0,$



$0 \to \mathcal{E}_d \to p^*_{d-1}\mathcal{E}_{d-1} \to \mathcal{L}_{d-1} \to 0,$

$0 \to \mathcal{E}_{d-1} \to p^*_{d-2}\mathcal{E}_{d-2} \to \mathcal{L}_{d-2} \to 0,$

$\vdots$

$0 \to \mathcal{E}_2 \to p^*_1\mathcal{E}_1 \to \mathcal{L}_1 \to 0,$

$0 \to \mathcal{E}_1 \to p^*_0\mathcal{E} \to \mathcal{L}_0 \to 0,$

$\mathcal{E}$

- $\mathcal{L}'_i := \mathcal{O}_{\mathbb{P}_{G_i}(\mathcal{Q}_i)}(1) \in \operatorname{Pic} G_{i+1}, \quad \mathcal{L}_i := \mathcal{O}_{\mathbb{P}_{X_i}(\mathcal{E}_i)}(1) \in \operatorname{Pic} X_{i+1}.$
- $q^* c_1(\mathcal{Q}) = c_1(\mathcal{L}'_0) + \cdots + c_1(\mathcal{L}'_{d-1}) \in A^1(\mathbb{F}^{d-1}_G(\mathcal{Q})).$

$\mathbb{F}^i_*(*)$ flags of subbdles of cork 1 up to i

- r the induced morphism s.t. $[\mathcal{E}_d \rightarrowtail p^*\mathcal{E}\,] = r^*[\mathcal{S} \rightarrowtail \pi^*\mathcal{E}].$
- $\mathbb{F}^{d-1}_G(\mathcal{Q}) = \mathbb{F}^d_X(\mathcal{E})$ over G, $\quad c_1(\mathcal{L}'_i) = c_1(\mathcal{L}_i) =: \xi_i.$

$\because$ ...

$\pi_*\left(c_1(\mathcal{Q})^N\right) = ? \in A^*(X)$, in terms of $s_i(\mathcal{E})$

$$0 \to \mathcal{L}'_{d-1} \to q_{d-2}^*\mathcal{Q}_{d-2} \to \mathcal{L}'_{d-2} \to 0, \qquad G_{d-1} := \mathbb{P}_{G_{d-2}}(\mathcal{Q}_{d-2}) = \mathbb{F}_G^{d-1}(\mathcal{Q}) \xlongequal{\uparrow} \mathbb{F}_X^d(\mathcal{E}) = \mathbb{P}_{X_{d-1}}(\mathcal{E}_{d-1}) =: X_d \qquad 0 \to \mathcal{E}_d \to p_{d-1}^*\mathcal{E}_{d-1} \to \mathcal{L}_{d-1} \to 0,$$

$$q_{d-2}\downarrow \qquad\qquad \downarrow p_{d-1}$$

$$0 \to \mathcal{Q}_{d-2} \to q_{d-3}^*\mathcal{Q}_{d-3} \to \mathcal{L}'_{d-3} \to 0, \qquad G_{d-2} := \mathbb{P}_{G_{d-3}}(\mathcal{Q}_{d-3}) = \mathbb{F}_G^{d-2}(\mathcal{Q}) \qquad \mathbb{F}_X^{d-1}(\mathcal{E}) = \mathbb{P}_{X_{d-2}}(\mathcal{E}_{d-2}) =: X_{d-1} \qquad 0 \to \mathcal{E}_{d-1} \to p_{d-2}^*\mathcal{E}_{d-2} \to \mathcal{L}_{d-2} \to 0,$$

$$q_{d-3}\downarrow \qquad r \qquad \downarrow p_{d-2}$$

$$\vdots \qquad q \qquad \vdots \qquad\qquad \vdots \qquad p \qquad \vdots$$

$$q_1\downarrow \qquad\qquad \downarrow p_2$$

$$0 \to \mathcal{Q}_1 \to q_0^*\mathcal{Q} \to \mathcal{L}'_0 \to 0, \qquad G_1 := \mathbb{P}_G(\mathcal{Q}) = \mathbb{F}_G^1(\mathcal{Q}) \qquad \mathbb{F}_X^2(\mathcal{E}) = \mathbb{P}_{X_1}(\mathcal{E}_1) =: X_2 \qquad 0 \to \mathcal{E}_2 \to p_1^*\mathcal{E}_1 \to \mathcal{L}_1 \to 0,$$

$$q_0\downarrow \qquad\qquad \downarrow p_1$$

$$\boxed{0 \to \mathcal{S} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0,} \qquad G := \mathbb{G}_X(d, \mathcal{E}) \qquad \mathbb{F}_X^1(\mathcal{E}) = \mathbb{P}_X(\mathcal{E}) =: X_1 \qquad 0 \to \mathcal{E}_1 \to p_0^*\mathcal{E} \to \mathcal{L}_0 \to 0,$$

$$\pi \searrow \qquad\qquad \downarrow p_0$$

$$X \qquad\qquad \boxed{\mathcal{E}}$$

- $\mathcal{L}'_i := \mathcal{O}_{\mathbb{P}_{G_i}(\mathcal{Q}_i)}(1) \in \operatorname{Pic} G_{i+1}, \quad \mathcal{L}_i := \mathcal{O}_{\mathbb{P}_{X_i}(\mathcal{E}_i)}(1) \in \operatorname{Pic} X_{i+1}.$
- $q^* c_1(\mathcal{Q}) = c_1(\mathcal{L}'_0) + \cdots + c_1(\mathcal{L}'_{d-1}) \in A^1(\mathbb{F}_G^{d-1}(\mathcal{Q}))$.

$\mathbb{F}_*^i(*)$ flags of subbdles of cork 1 up to i

- r the induced morphism s.t. $[\mathcal{E}_d \rightarrowtail p^*\mathcal{E}\,] = r^*[\mathcal{S} \rightarrowtail \pi^*\mathcal{E}]$.
- $\mathbb{F}_G^{d-1}(\mathcal{Q}) = \mathbb{F}_X^d(\mathcal{E})$ over G, $\quad c_1(\mathcal{L}'_i) = c_1(\mathcal{L}_i) =: \xi_i$.

$\because$ A subbdle $\mathcal{E}' \subset p^*\mathcal{E}$ containing $\mathcal{E}_d$ corresponds bijectively to a subbbdle $\mathcal{Q}' \subset r^*\mathcal{Q}$ by

$$\begin{array}{ccccccccc} 0 & \to & r^*\mathcal{S} & \to & p^*\mathcal{E} & \to & r^*\mathcal{Q} & \to & 0 \\ & & \| & & \cup & & \cup & & \\ 0 & \to & \mathcal{E}_d & \to & \mathcal{E}' & \to & \mathcal{Q}' & \to & 0, \end{array}$$

and $\mathcal{L}'_i = \mathcal{Q}_i/\mathcal{Q}_{i+1} \simeq \mathcal{E}_i/\mathcal{E}_{i+1} = \mathcal{L}_i$. $\square$

$\pi_*\left(c_1(\mathcal{Q})^N\right) = ? \in A^*(X)$, in terms of $s_i(\mathcal{E})$

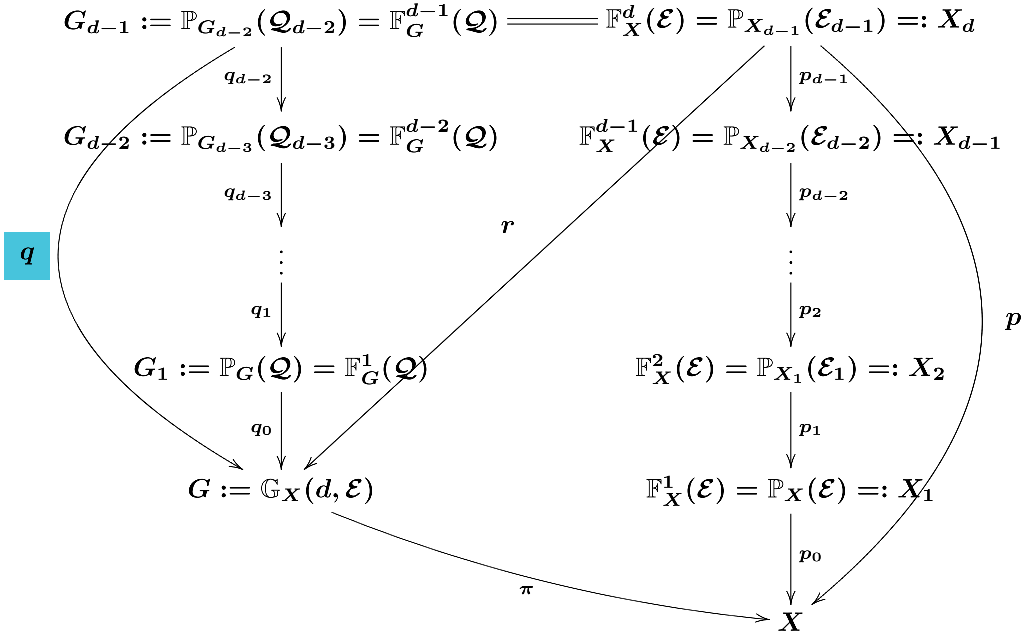
$$0 \to \mathcal{L}'_{d-1} \to q^*_{d-2}\mathcal{Q}_{d-2} \to \mathcal{L}'_{d-2} \to 0,$$

$$0 \to \mathcal{Q}_{d-2} \to q^*_{d-3}\mathcal{Q}_{d-3} \to \mathcal{L}'_{d-3} \to 0,$$

$$\vdots$$

$$0 \to \mathcal{Q}_1 \to q^*_0\mathcal{Q} \to \mathcal{L}'_0 \to 0,$$

$$0 \to \mathcal{S} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0,$$



$$0 \to \mathcal{E}_d \to p^*_{d-1}\mathcal{E}_{d-1} \to \mathcal{L}_{d-1} \to 0,$$

$$0 \to \mathcal{E}_{d-1} \to p^*_{d-2}\mathcal{E}_{d-2} \to \mathcal{L}_{d-2} \to 0,$$

$$\vdots$$

$$0 \to \mathcal{E}_2 \to p^*_1\mathcal{E}_1 \to \mathcal{L}_1 \to 0,$$

$$0 \to \mathcal{E}_1 \to p^*_0\mathcal{E} \to \mathcal{L}_0 \to 0,$$

$$\mathcal{E}$$

- $\mathcal{L}'_i := \mathcal{O}_{\mathbb{P}_{G_i}(\mathcal{Q}_i)}(1) \in \operatorname{Pic} G_{i+1}, \quad \mathcal{L}_i := \mathcal{O}_{\mathbb{P}_{X_i}(\mathcal{E}_i)}(1) \in \operatorname{Pic} X_{i+1}.$
- $q^*\, c_1(\mathcal{Q}) = c_1(\mathcal{L}'_0) + \cdots + c_1(\mathcal{L}'_{d-1}) \in A^1(\mathbb{F}^{d-1}_G(\mathcal{Q}))$.

$\mathbb{F}^i_*(*)$ flags of subbdles of cork 1 up to i

- r the induced morphism s.t. $[\mathcal{E}_d \rightarrowtail p^*\mathcal{E}\,] = r^*[\mathcal{S} \rightarrowtail \pi^*\mathcal{E}]$.
- $\mathbb{F}^{d-1}_G(\mathcal{Q}) = \mathbb{F}^d_X(\mathcal{E})$ over G, $\quad c_1(\mathcal{L}'_i) = c_1(\mathcal{L}_i) =: \xi_i$.
- $q^* c_1(\mathcal{Q}) = \xi_0 + \cdots + \xi_{d-1}$.
- ...

$\pi_*\left(c_1(\mathcal{Q})^N\right) = \,? \in A^*(X)$, in terms of $s_i(\mathcal{E})$

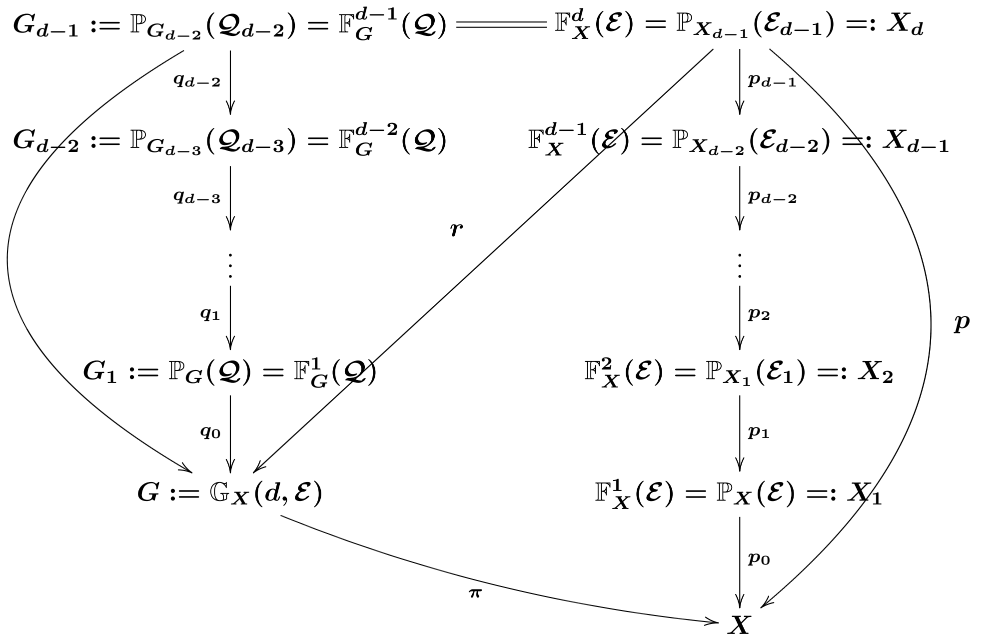
$0 \to \mathcal{L}'_{d-1} \to q_{d-2}^*\mathcal{Q}_{d-2} \to \mathcal{L}'_{d-2} \to 0,$

$0 \to \mathcal{Q}_{d-2} \to q_{d-3}^*\mathcal{Q}_{d-3} \to \mathcal{L}'_{d-3} \to 0,$

$\vdots$

$0 \to \mathcal{Q}_1 \to q_0^*\mathcal{Q} \to \mathcal{L}'_0 \to 0,$

$0 \to \mathcal{S} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0,$



$0 \to \mathcal{E}_d \to p_{d-1}^*\mathcal{E}_{d-1} \to \mathcal{L}_{d-1} \to 0,$

$0 \to \mathcal{E}_{d-1} \to p_{d-2}^*\mathcal{E}_{d-2} \to \mathcal{L}_{d-2} \to 0,$

$\vdots$

$0 \to \mathcal{E}_2 \to p_1^*\mathcal{E}_1 \to \mathcal{L}_1 \to 0,$

$0 \to \mathcal{E}_1 \to p_0^*\mathcal{E} \to \mathcal{L}_0 \to 0,$

$\mathcal{E}$

- $\mathcal{L}'_i := \mathcal{O}_{\mathbb{P}_{G_i}(\mathcal{Q}_i)}(1) \in \operatorname{Pic} G_{i+1}$, $\quad \mathcal{L}_i := \mathcal{O}_{\mathbb{P}_{X_i}(\mathcal{E}_i)}(1) \in \operatorname{Pic} X_{i+1}$.
- $q^*\, c_1(\mathcal{Q}) = c_1(\mathcal{L}'_0) + \cdots + c_1(\mathcal{L}'_{d-1}) \in A^1(\mathbb{F}^{d-1}_G(\mathcal{Q}))$.

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- $\mathbb{F}^{d-1}_G(\mathcal{Q}) = \mathbb{F}^d_X(\mathcal{E})$ over G, $\quad c_1(\mathcal{L}'_i) = c_1(\mathcal{L}_i) =: \xi_i$.
- $q^*\, c_1(\mathcal{Q}) = \xi_0 + \cdots + \xi_{d-1}$.
- $\pi_*(c_1(\mathcal{Q})^N) = \pi_* q_* (\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}^1 q^*(c_1(\mathcal{Q})^N))$ $\left(\because\ q_i : \mathbb{P}^{d-i-1}\text{-bdle w/ tautlgcl cls } \xi_i \rightsquigarrow \alpha = q_{i*}(\xi_i^{d-i} q_i^*\alpha)\right)$

$= \cdots$

$\pi_*\left(c_1(\mathcal{Q})^N\right) = ? \in A^*(X)$, in terms of $s_i(\mathcal{E})$

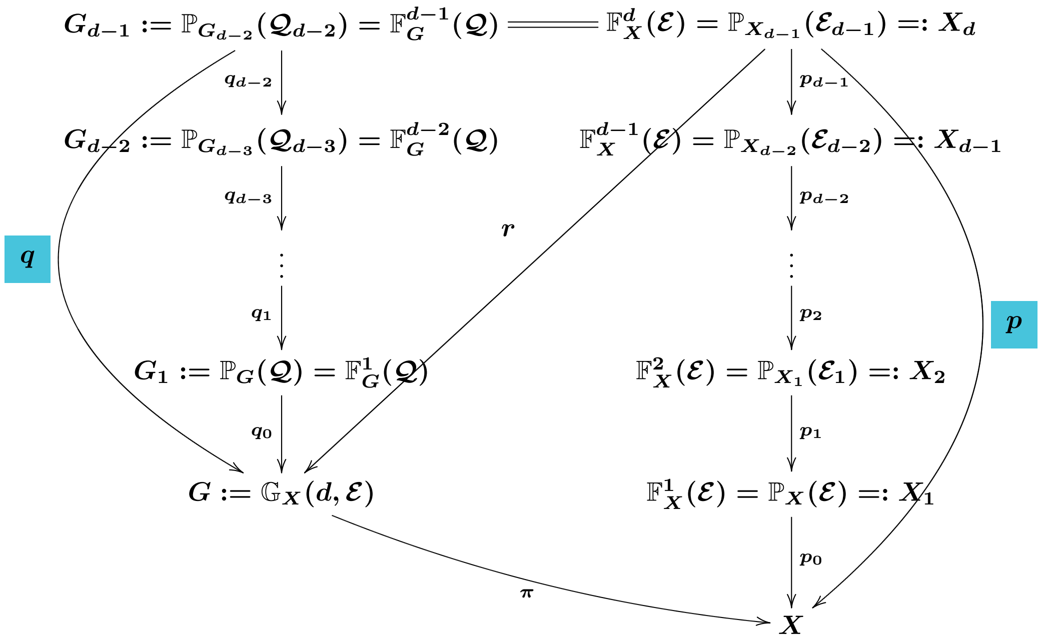
$0 \to \mathcal{L}'_{d-1} \to q^*_{d-2}\mathcal{Q}_{d-2} \to \mathcal{L}'_{d-2} \to 0,$

$0 \to \mathcal{Q}_{d-2} \to q^*_{d-3}\mathcal{Q}_{d-3} \to \mathcal{L}'_{d-3} \to 0,$

$\vdots$

$0 \to \mathcal{Q}_1 \to q^*_0\mathcal{Q} \to \mathcal{L}'_0 \to 0,$

$0 \to \mathcal{S} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0,$



$0 \to \mathcal{E}_d \to p^*_{d-1}\mathcal{E}_{d-1} \to \mathcal{L}_{d-1} \to 0,$

$0 \to \mathcal{E}_{d-1} \to p^*_{d-2}\mathcal{E}_{d-2} \to \mathcal{L}_{d-2} \to 0,$

$\vdots$

$0 \to \mathcal{E}_2 \to p^*_1\mathcal{E}_1 \to \mathcal{L}_1 \to 0,$

$0 \to \mathcal{E}_1 \to p^*_0\mathcal{E} \to \mathcal{L}_0 \to 0,$

$\mathcal{E}$

- $\mathcal{L}'_i := \mathcal{O}_{\mathbb{P}_{G_i}(\mathcal{Q}_i)}(1) \in \operatorname{Pic} G_{i+1}, \quad \mathcal{L}_i := \mathcal{O}_{\mathbb{P}_{X_i}(\mathcal{E}_i)}(1) \in \operatorname{Pic} X_{i+1}.$
- $q^*c_1(\mathcal{Q}) = c_1(\mathcal{L}'_0) + \cdots + c_1(\mathcal{L}'_{d-1}) \in A^1(\mathbb{F}^{d-1}_G(\mathcal{Q})).$

$\mathbb{F}^i_*(*)$ flags of subbdles of cork 1 up to i

- r the induced morphism s.t. $[\mathcal{E}_d \rightarrowtail p^*\mathcal{E}] = r^*[\mathcal{S} \rightarrowtail \pi^*\mathcal{E}].$
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- $q^*c_1(\mathcal{Q}) = \xi_0 + \cdots + \xi_{d-1}.$
- $\pi_*(c_1(\mathcal{Q})^N) = \pi_* q_*(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}^1 q^*(c_1(\mathcal{Q})^N)) \quad \left(\because \; q_i : \mathbb{P}^{d-i-1}\text{-bdle w/ tautlgcl cls } \xi_i \rightsquigarrow \alpha = q_{i*}(\xi_i^{d-i} q_i^*\alpha)\right)$

$$= \pi_* q_*(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}^1(\xi_0 + \cdots + \xi_{d-1})^N)$$
$$= p_*(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}^1(\xi_0 + \cdots + \xi_{d-1})^N) \in A^*(X).$$

How to calculate the push-forward in the RHS below?:

$$\pi_*(c_1(\mathcal{Q})^N) = p_*(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}^1(\xi_0+\cdots+\xi_{d-1})^N).$$

- $X_{i+1} = \mathbb{P}_{X_i}(\mathcal{E}_{i+1}) \xrightarrow{p_i} X_i$ is a $\mathbb{P}^{r-i-1}$-bundle,

$$\rightsquigarrow A^*(X_{i+1}) = A^*(X_i)[\xi_i]/(\xi_i^{r-i} - c_1(\mathcal{E}_i)\xi_i^{r-i-1} + \cdots + (-1)^{r-i}c_{r-i}(\mathcal{E}_i))$$
$$= \bigoplus_{0\le j\le r-i-1} A^*(X_i)\overline{\xi_i}^j.$$

- **Therefore,**

How to calculate the push-forward in the RHS below?:

$$\pi_*(c_1(\mathcal{Q})^N) = p_*(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}^1(\xi_0+\cdots+\xi_{d-1})^N).$$

- $X_{i+1} = \mathbb{P}_{X_i}(\mathcal{E}_{i+1})$ $\xrightarrow{p_i}$ X_i **is a** $\mathbb{P}^{r-i-1}$**-bundle,**

$$\rightsquigarrow A^*(X_{i+1}) = A^*(X_i)[\xi_i]/(\xi_i^{r-i} - c_1(\mathcal{E}_i)\xi_i^{r-i-1} + \cdots + (-1)^{r-i}c_{r-i}(\mathcal{E}_i))$$
$$= \bigoplus_{0\le j\le r-i-1} A^*(X_i)\overline{\xi_i}^j.$$

- **Therefore,**

$$A^*(X_d) = \frac{A^*(X)[\xi_0,\xi_1,\ldots,\xi_{d-1}]}{\begin{pmatrix}\xi_0^r - c_1(\mathcal{E})\xi_0^{r-1} + \cdots + (-1)^r c_r(\mathcal{E}),\\ \xi_1^{r-1} - c_1(\mathcal{E}_1)\xi_1^{r-2} + \cdots + (-1)^{r-1}c_{r-1}(\mathcal{E}_1),\\ \vdots \\ \xi_{d-1}^{r-d+1} - c_1(\mathcal{E}_{d-1})\xi_{d-1}^{r-d} + \cdots + (-1)^{r-d+1}c_{r-d+1}(\mathcal{E}_{d-1})\end{pmatrix}}$$
$$= \bigoplus_{\substack{0\le i_l\le r-l-1\\(0\le l\le d-1)}} A^*(X)\overline{\xi_0}^{i_0}\overline{\xi_1}^{i_1}\cdots\overline{\xi_{d-1}}^{i_{d-1}}.$$

How to calculate the push-forward in the RHS below?:

$$\pi_*(c_1(\mathcal{Q})^N) = p_*(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}^1(\xi_0+\cdots+\xi_{d-1})^N).$$

- $X_{i+1} = \mathbb{P}_{X_i}(\mathcal{E}_{i+1})$ $\overset{p_i}{\downarrow}$ X_i is a $\mathbb{P}^{r-i-1}$-bundle,

$$\rightsquigarrow A^*(X_{i+1}) = A^*(X_i)[\xi_i]/(\xi_i^{r-i} - c_1(\mathcal{E}_i)\xi_i^{r-i-1} + \cdots + (-1)^{r-i}c_{r-i}(\mathcal{E}_i))$$
$$= \bigoplus_{0\le j\le r-i-1} A^*(X_i)\overline{\xi_i}^j.$$

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$$A^*(X_d) = \frac{A^*(X)[\xi_0,\xi_1,\ldots,\xi_{d-1}]}{\begin{pmatrix} \xi_0^r - c_1(\mathcal{E})\xi_0^{r-1} + \cdots + (-1)^r c_r(\mathcal{E}), \\ \xi_1^{r-1} - c_1(\mathcal{E}_1)\xi_1^{r-2} + \cdots + (-1)^{r-1}c_{r-1}(\mathcal{E}_1), \\ \vdots \\ \xi_{d-1}^{r-d+1} - c_1(\mathcal{E}_{d-1})\xi_{d-1}^{r-d} + \cdots + (-1)^{r-d+1}c_{r-d+1}(\mathcal{E}_{d-1}) \end{pmatrix}}$$
$$= \bigoplus_{\substack{0\le i_l\le r-l-1 \\ (0\le l\le d-1)}} A^*(X)\overline{\xi_0}^{i_0}\overline{\xi_1}^{i_1}\cdots\overline{\xi_{d-1}}^{i_{d-1}}.$$

- $p_* : A^*(X_d) \to A^{*-c}(X)$, the push-forward by $p : X_d \to \cdots \to X_1 \to X$ is given by

$$\alpha \mapsto \mathtt{coeff}_{\overline{\xi_0},\ldots,\overline{\xi_{d-1}}}(\alpha; r-1, r-2, \ldots, r-d)$$
$$:= \left(\text{the coefficient of } \alpha \text{ in the leading monomial, } \overline{\xi_0}^{r-1}\cdots\overline{\xi_{d-1}}^{r-d}\right),$$

where $c := \sum_{0\le i\le d-1}(r-i-1)$ the relative dimension of X_d/X.

How to calculate the push-forward in the RHS below?:

$$\pi_*(c_1(\mathcal{Q})^N) = p_*(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}^1(\xi_0+\cdots+\xi_{d-1})^N).$$

- $X_{i+1} = \mathbb{P}_{X_i}(\mathcal{E}_{i+1})$ $\quad p_i \downarrow$ $\quad X_i$ **is a** $\mathbb{P}^{r-i-1}$**-bundle,**

$$\rightsquigarrow A^*(X_{i+1}) = A^*(X_i)[\xi_i]/(\xi_i^{r-i} - c_1(\mathcal{E}_i)\xi_i^{r-i-1} + \cdots + (-1)^{r-i}c_{r-i}(\mathcal{E}_i))$$

$$= \bigoplus_{0\le j\le r-i-1} A^*(X_i)\overline{\xi_i}^j.$$

- **Therefore,**

$$A^*(X_d) = \frac{A^*(X)[\xi_0,\xi_1,\ldots,\xi_{d-1}]}{\begin{pmatrix} \xi_0^r - c_1(\mathcal{E})\xi_0^{r-1} + \cdots + (-1)^r c_r(\mathcal{E}), \\ \xi_1^{r-1} - c_1(\mathcal{E}_1)\xi_1^{r-2} + \cdots + (-1)^{r-1}c_{r-1}(\mathcal{E}_1), \\ \vdots \\ \xi_{d-1}^{r-d+1} - c_1(\mathcal{E}_{d-1})\xi_{d-1}^{r-d} + \cdots + (-1)^{r-d+1}c_{r-d+1}(\mathcal{E}_{d-1}) \end{pmatrix}}$$

$$= \bigoplus_{\substack{0\le i_l\le r-l-1 \\ (0\le l\le d-1)}} A^*(X)\overline{\xi_0}^{i_0}\overline{\xi_1}^{i_1}\cdots\overline{\xi_{d-1}}^{i_{d-1}}.$$

- $p_* : A^*(X_d) \to A^{*-c}(X)$**, the push-forward by** $p : X_d \to \cdots \to X_1 \to X$ **is given by**

$$\alpha \mapsto \mathtt{coeff}_{\overline{\xi_0},\ldots,\overline{\xi_{d-1}}}(\alpha; r-1, r-2, \ldots, r-d)$$

$$:= \left(\textbf{the coefficient of } \alpha \textbf{ in the leading monomial, } \overline{\xi_0}^{r-1}\cdots\overline{\xi_{d-1}}^{r-d}\right),$$

where $c := \sum_{0\le i\le d-1}(r-i-1)$ **the relative dimension of** X_d/X**.**

How to calculate the push-forward in the RHS below?:

$$\pi_*(c_1(\mathcal{Q})^N) = p_*(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}^1(\xi_0+\cdots+\xi_{d-1})^N).$$

- $X_{i+1} = \mathbb{P}_{X_i}(\mathcal{E}_{i+1}) \xrightarrow{p_i} X_i$ is a $\mathbb{P}^{r-i-1}$-bundle,

$$\rightsquigarrow A^*(X_{i+1}) = A^*(X_i)[\xi_i]/(\xi_i^{r-i} - c_1(\mathcal{E}_i)\xi_i^{r-i-1} + \cdots + (-1)^{r-i}c_{r-i}(\mathcal{E}_i))$$
$$= \bigoplus_{0\le j\le r-i-1} A^*(X_i)\overline{\xi_i}^j.$$

- **Therefore,**

$$A^*(X_d) = \frac{A^*(X)[\xi_0,\xi_1,\ldots,\xi_{d-1}]}{\begin{pmatrix} \xi_0^r - c_1(\mathcal{E})\xi_0^{r-1} + \cdots + (-1)^r c_r(\mathcal{E}), \\ \xi_1^{r-1} - c_1(\mathcal{E}_1)\xi_1^{r-2} + \cdots + (-1)^{r-1}c_{r-1}(\mathcal{E}_1), \\ \vdots \\ \xi_{d-1}^{r-d+1} - c_1(\mathcal{E}_{d-1})\xi_{d-1}^{r-d} + \cdots + (-1)^{r-d+1}c_{r-d+1}(\mathcal{E}_{d-1}) \end{pmatrix}}$$
$$= \bigoplus_{\substack{0\le i_l\le r-l-1 \\ (0\le l\le d-1)}} A^*(X)\overline{\xi_0}^{i_0}\overline{\xi_1}^{i_1}\cdots\overline{\xi_{d-1}}^{i_{d-1}}.$$

- $p_* : A^*(X_d) \to A^{*-c}(X)$, **the push-forward by** $p : X_d \to \cdots \to X_1 \to X$ **is given by**

$$\alpha \mapsto \mathtt{coeff}_{\overline{\xi_0},\ldots,\overline{\xi_{d-1}}}(\alpha; r-1, r-2, \ldots, r-d)$$
$$:= \left(\text{the coefficient of } \alpha \text{ in the leading monomial, } \overline{\xi_0}^{r-1}\cdots\overline{\xi_{d-1}}^{r-d}\right),$$

where $c := \sum_{0\le i\le d-1}(r-i-1)$ **the relative dimension of** X_d/X.

$$\therefore\ \pi_*(c_1(\mathcal{Q})^N) = p_*(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}^1(\xi_0+\cdots+\xi_{d-1})^N)$$
$$= \mathtt{coeff}_{\overline{\xi_0},\ldots,\overline{\xi_{d-1}}}(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}^1(\xi_0+\cdots+\xi_{d-1})^N; r-1, r-2, \ldots, r-d),$$

(b) Laurent series

$\pi_* \left(c_1(\mathcal{Q})^N\right) = ? \in A^*(X)$, in terms of $s_i(\mathcal{E})$

Proposition 3 For a non-negative integer N,

$$\pi_*(c_1(\mathcal{Q})^N) = \mathtt{const}_{\underline{t}}(P_N(\underline{t})),$$

where

$\pi_* : A^*(\mathbb{G}_X(d,\mathcal{E})) \to A^{*-d(r-d)}(X)$ the push-forward,

$\mathtt{const}_{\underline{t}}(\dots)$ the constant term in the Laurent expansion of $\dots$ w.r.t. $\underline{t} := (t_0, \dots, t_{d-1})$,

$s(\mathcal{E}, t)$ the Segre series of $\mathcal{E}$ in t, and

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Proof $\bigstar\cdots$

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$0\to\mathcal{E}_{i+1}\to\mathcal{E}_i\to\mathcal{L}_i\to 0$

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$c(*,t)s(*,-t) = 1$

$\rightsquigarrow \mathtt{const}_{t_i}\left(t_i^{-p_i}\mathtt{coeff}_{\overline{\xi_i}}\left(\frac{c(\mathcal{E}_{i+1},t_i)}{c(\mathcal{E}_i,t_i)}; r-i-1\right)\right) = \mathtt{const}_{t_i}(t_i^{-p_i}(-t_i)^{r-i-1}s(\mathcal{E}_i,-t_i)).$

• Thus, $\cdots$

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$$\rightsquigarrow \begin{cases} c_1(\mathcal{E}_{i+1}),\dots,c_{r-i-2}(\mathcal{E}_{i+1}) \in \bigoplus_{j=0}^{r-i-2} A^*(X_i)\overline{\xi_i}^j, \text{ and} \\ c_{r-i-1}(\mathcal{E}_{i+1}) \equiv (-\xi_i)^{r-i-1} \bmod \bigoplus_{j=0}^{r-i-2} A^*(X_i)\overline{\xi_i}^j \text{ in } A^*(X_{i+1}) = \bigoplus_{j=0}^{r-i-1} A^*(X_i)\overline{\xi_i}^j. \end{cases}$$

$\therefore\ \texttt{coeff}_{\overline{\xi_i}}\left(\frac{c(\mathcal{E}_{i+1},t_i)}{c(\mathcal{E}_i,t_i)}; r-i-1\right) = \texttt{coeff}_{\overline{\xi_i}}\left(\frac{(-1)^{r-i-1}\overline{\xi_i}^{r-i-1}t_i^{r-i-1}}{c(\mathcal{E}_i,t_i)}; r-i-1\right)$

$= \frac{(-t_i)^{r-i-1}}{c(\mathcal{E}_i,t_i)} = (-t_i)^{r-i-1}s(\mathcal{E}_i,-t_i).$

$c(*,t)s(*,-t) = 1$

$\rightsquigarrow \texttt{const}_{t_i}\left(t_i^{-p_i}\,\texttt{coeff}_{\overline{\xi_i}}\left(\frac{c(\mathcal{E}_{i+1},t_i)}{c(\mathcal{E}_i,t_i)}; r-i-1\right)\right) = \texttt{const}_{t_i}(t_i^{-p_i}(-t_i)^{r-i-1}s(\mathcal{E}_i,-t_i)).$

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Lemma 4 $\texttt{coeff}_{\overline{\xi_0},\dots,\overline{\xi_{d-1}}}\left(\xi_0^{p_0}\cdots\xi_{d-1}^{p_{d-1}}; r-1,\dots,r-d\right) = \texttt{const}_{\underline{t}}\left(\Delta(\underline{t})\prod_{0\le i\le d-1} t_i^{-p_i+r-d} s(\mathcal{E},t_i)\right).$

follows from an equality, $s(\mathcal{E}_{i+1},t_{i+1}) = (1-\xi_i t_{i+1})s(\mathcal{E}_i,t_{i+1})$, and

Lemma 5 $\texttt{coeff}_{\overline{\xi_i}}(\xi_i^{p_i}; r-i-1) = \texttt{const}_{t_i}(t_i^{-p_i+r-i-1}s(\mathcal{E}_i,t_i))\ (0\le i\le d-1).$

Proof $\bigstar\ \dfrac{1}{1+\xi_i t_i} = \dfrac{c(\mathcal{E}_{i+1},t_i)}{c(\mathcal{E}_i,t_i)}\left(= \dfrac{1+c_1(\mathcal{E}_{i+1})t_i+\cdots+c_{r-i-1}(\mathcal{E}_{i+1})t_i^{r-i-1}}{c(\mathcal{E}_i,t_i)}\right).$

$0\to\mathcal{E}_{i+1}\to\mathcal{E}_i\to\mathcal{L}_i\to 0$

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• Replace $-t_i \mapsto t_i$. $\square$

(c) An Expression of Constant Term

Theorem 1 (monomial type)

$$\pi_* \operatorname{ch}(\det \mathcal{Q}) = \sum_{k} \frac{\prod_{1 \le i < j \le d}(k_i - k_j + j - i)}{\prod_{1 \le i \le d}(r + k_i - i)!} \prod_{1 \le i \le d} s_{k_i}(\mathcal{E}).$$

Proof of Theorem 1

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Recall that $\operatorname{ch}(\det \mathcal{Q}) := \sum_{N\ge 0} \frac{1}{N!} c_1(\mathcal{Q})^N$, and

Proposition 3 For a non-negative integer N, $\pi_*(c_1(\mathcal{Q})^N) = \mathrm{const}_{\underline{t}}(P_N(\underline{t}))$,

where

$$P_N(\underline{t}) := \prod_{0\le i<j\le d-1} (t_i - t_j) \Big(\sum_{0\le i\le d-1} \frac{1}{t_i} \Big)^N \prod_{0\le i\le d-1} t_i^{-(d-1-i)+r-d} s(\mathcal{E}_0, t_i).$$

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We give two proofs for

Theorem 2 **(Schur polynomial type)**

$$\pi_* \operatorname{ch}(\det \mathcal{Q}) = \sum_{\lambda} \frac{\prod_{1 \le i < j \le d}(\lambda_i - \lambda_j + j - i)}{\prod_{1 \le i \le d}(r + \lambda_i - i)!} \Delta_\lambda(s(\mathcal{E})).$$

Proof 1 of Theorem 2

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Proof

$$\prod_{i=0}^{d-1} s(\mathcal{E}, t_i) = \prod_{i=0}^{d-1} \frac{1}{c(\mathcal{E}, -t_i)} = \prod_{i=0}^{d-1} \prod_{j=1}^{r} \frac{1}{1 - \alpha_j t_i} \overset{\text{Cauchy} \atop \downarrow}{=} \sum_{\lambda} s_\lambda(\underline{\alpha}) s_\lambda(\underline{t}) \overset{\text{Jacobi-Trudi} \atop \downarrow}{=} \sum_{\lambda} \Delta_\lambda(s(\mathcal{E})) s_\lambda(\underline{t}),$$

where

$\underline{\alpha} = \{\alpha_1, \ldots, \alpha_r\}$ **the Chern roots of the vector bundle $\mathcal{E}$.** □

Proof 2 of Theorem 2

symmetrisation w.r.t. $\underline{t}$
$\downarrow$

Our strategy to obtain the constant term of $P_N^{\mathsf{symm}}(\underline{t}) := S_{\underline{t}}(P_N(\underline{t}))$ is to calculate it as the "residue" of $P_N^{\mathsf{symm}}(\frac{1}{t_0}, \ldots, \frac{1}{t_{d-1}})\frac{1}{t_0 \cdots t_{d-1}}$.

$$P_N^{\mathsf{symm}}\Big(\frac{1}{t_0}, \ldots, \frac{1}{t_{d-1}}\Big)\frac{1}{t_0 \cdots t_{d-1}} = \frac{(-1)^{\frac{d(d-1)}{2}}}{d!} \prod_{0 \le i < j \le d-1} (t_i - t_j)^2 \Big(\sum_{0 \le i \le d-1} t_i\Big)^N \prod_{0 \le i \le d-1} t_i^{-r} s\Big(\mathcal{E}, \frac{1}{t_i}\Big).$$

$\uparrow$ symmetrisation w.r.t. $\underline{t}$ + taking the inverse of $\underline{t}$ $\times \frac{1}{t_0 \cdots t_{d-1}}$

$$P_N(\underline{t}) = \prod_{0 \le i < j \le d-1} (t_i - t_j) \Big(\sum_{0 \le i \le d-1} \frac{1}{t_i}\Big)^N \prod_{0 \le i \le d-1} t_i^{-(d-1-i)+r-d} s(\mathcal{E}_0, t_i).$$

Note the constant term is invariant under the symmetrisation w.r.t. $\underline{t} = (t_0, \ldots, t_{d-1})$ and taking the inverse of variables $\underline{t}$:

$$\mathtt{const}_{\underline{t}}(P_N(\underline{t})) = \mathtt{const}_{\underline{t}}(P_N^{\mathsf{symm}}(\underline{t})) = \mathtt{const}_{\underline{t}}\left(P_N^{\mathsf{symm}}\Big(\frac{1}{t_0}, \ldots, \frac{1}{t_{d-1}}\Big)\right).$$

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$x = 1 - r$

Then ...

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Then for a real number $x > 0$,

$$I_{\text{conf}}(\lambda, x) = d! \prod_{1 \le i < j \le d} (\lambda_i - \lambda_j + j - i) \prod_{i=1}^{d} \Gamma(x + d - i + \lambda_i).$$

Summary

$$\pi_* \operatorname{ch}(\det \mathcal{Q}) = \sum_{k} \frac{\Delta(k_i - i)}{\{r + k_i - i\}!} \prod_{1 \le i \le d} s_{k_i}(\mathcal{E})$$
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$\operatorname{ch}(\det \mathcal{Q}) := \sum_{N \ge 0} \frac{1}{N!}\theta^N$ the Chern character of $\det \mathcal{Q}$,
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$\pi_* : A^*(\mathbb{G}_X(d, \mathcal{E})) \otimes \mathbb{Q} \to A^{*-d(r-d)}(X) \otimes \mathbb{Q}$ the push-forward,
X a scheme of finite type over a field k,
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Thank you for your attention!

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