Chern numbers associated with two- and three-level semi-quantum systems with symmetry

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Chern numbers and symmetry

This talk is based on the joint works with B. Zhilinskii and my talk given at RIMS 2011 together with a further study in progress:

- Energy bands: Chern numbers and symmetry,
  Ann. Phys., vol.**326** (2011), 3013-3066,

- Rearrangement of energy bands:
  Chern numbers in the presence of cubic symmetry,
  to appear in Acta Appl. Math. (2012),

- Chern numbers associated with semi-quantum systems
  with symmetry,
  RIMS Kôkyûroku 1774, pp. 130-146, (2012).

Organization

1. Finite-level quantum systems

2. Finite-level semi-quantum systems

3. Chern numbrs associated with two-level semi-quantum systems with symmetry by $U(1), D_3, O$

4. Chern numbrs associated with three-level semi-quantum systems with symmetry by O

5. Linear approximation on a subspace

1. Finite-level quantum systems

- Clebsh-Gordan formula
- one-parameter Hamiltonians
- a view to rotation-vibration coupling
- redistribution of eigenvalues

1.1 The Clebsh-Gordan formula

(V_ℓ, D^ℓ): a unitary irreducible rep. of $SU(2)$ with $\ell \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$
J_k, S_k: $su(2)$ operators acting on V_j and V_s, respectively, $k = 1, 2, 3$.
$\boldsymbol{J} = (J_k)$: angular momentum operators,
$\boldsymbol{S} = (S_k)$: spin operators.
$SU(2)$ acts on $V_j \otimes V_s$ unitarily, and the infinitesimal generators are

$$\boldsymbol{N} = \boldsymbol{J} \otimes \mathbf{1} + \mathbf{1} \otimes \boldsymbol{S}.$$

Then, according to the Clebsh-Gordan formula,
the representation space $V_j \otimes V_s$ is decomposed into

$$V_j \otimes V_s \cong V_{j+s} \oplus \cdots \oplus V_{|j-s|}.$$

1.2 A coupling $\boldsymbol{J} \otimes \boldsymbol{S}$

The squared operator $\boldsymbol{N}^2$ is expressed as

$$\boldsymbol{N}^2 = \boldsymbol{J}^2 \otimes \mathbf{1} + 2\boldsymbol{J} \otimes \boldsymbol{S} + \mathbf{1} \otimes \boldsymbol{S}^2.$$

The V_n, $|j - s| \leq n \leq j + s$, are eigenspaces of $\boldsymbol{N}^2$ associated with the eigenvalue $n(n+1)$.
The V_n are also eigenspaces of $\boldsymbol{J} \otimes \boldsymbol{S}$ associated with the eigenvalue

$$\frac{1}{2}\big(n(n+1) - j(j+1) - s(s+1)\big).$$

1.3 A one-parameter quantum operator on $V_j \otimes V_s$

$$H_\tau = (1-\tau)\mathbf{1} \otimes S_z + \tau \boldsymbol{J} \otimes \boldsymbol{S}, \quad 0 \le \tau \le 1.$$

The eigenspace decomposition of $V_j \otimes V_s$ w.r.t. H_τ changes:

$$\begin{array}{c} V_j \otimes |s\rangle \oplus \cdots \oplus V_j \otimes |-s\rangle \\ \text{w.r.t. } H_0 = \mathbf{1} \otimes S_z \end{array} \longrightarrow \begin{array}{c} V_{j+s} \oplus \cdots \oplus V_{|j-s|} \\ \text{w.r.t. } H_1 = \boldsymbol{J} \otimes \boldsymbol{S} \end{array},$$

where $|r\rangle := |s\ r\rangle$, $|r| \le s$ are the eigenvectors of S_z, and $j > s$.

The Clebsch-Gordan decomposition provides

$$\dim(V_j \otimes V_s) = \sum_{n=|j-s|}^{j+s} \dim V_n, \quad \dim V_n = \dim V_j + 2r, \quad |r| \le s.$$

Boris claims: $2r$ can be interpreted as Chern numbers, if the quantum system is transformed into a semi-quantum system by averaging $\boldsymbol{J}$.

1.4 An example, $V_1 \otimes |\frac{1}{2}\rangle \oplus V_1 \otimes |-\frac{1}{2}\rangle \;\rightarrow\; V_{\frac{1}{2}} \oplus V_{\frac{3}{2}}$

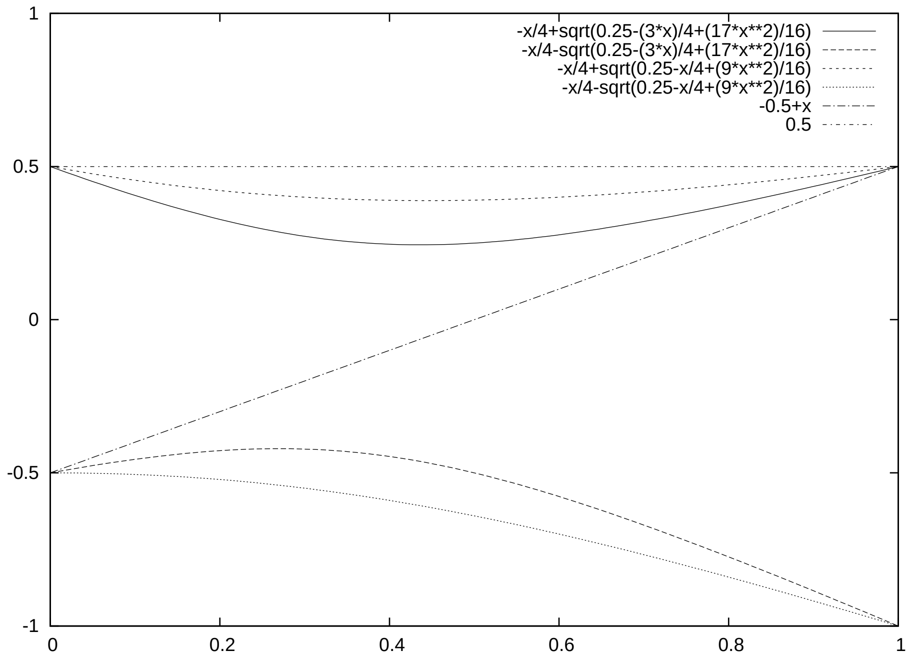

Figure 1: The redistribution of eigenvalues

$$\mathrm{dim} V_{\frac{1}{2}} = \mathrm{dim} V_1 + 2 \cdot (-\frac{1}{2}), \quad \mathrm{dim} V_{\frac{3}{2}} = \mathrm{dim} V_1 + 2 \cdot \frac{1}{2}$$

The associated Chern numbers will be $2r = -1, 1$.

1.5 A view to rotation-vibration coupling

Since the $SU(2)$ is viewed as the symmetry group of the harmonic oscillator of two degrees of freedom, the basis of the representation space V_s may be viewed as spanning a vibrational energy band. From this point of view, the coupling $V_j \otimes V_s$ may be thought of as the angular momentum (or rotation) and vibration coupling. Hence, the change in the eigenspace decomposition

$$V_j \otimes |s\rangle \oplus \cdots \oplus V_j \otimes |-s\rangle \longrightarrow V_{j+s} \oplus \cdots \oplus V_{|j-s|}$$

is interpreted as a a reorganization of vibrational energy bands, $\{|r\rangle\}_{|r|\leq s}$, through the interaction with rotation, and $\dim V_n = \dim V_j + 2r$ allows of the interpretation that the number $2r$ is characteristic of rovibration in each molecular band (V_n, $n = j+s, \cdots, |j-s|$).

2. Finite-level semi-quantum systems

- a view to rotation-vibration coupling
- averaging with coherent states
- one-parameter Hamiltonians
- Chern numbers of eigen-line bundles

2.1 Averaging with coherent states

$\boldsymbol{J}$: viewed as rotational variables, being made into classical ones,
$\boldsymbol{S}$: viewed as vibrational variables, being still quantum ones.

The $SU(2)$ coherent states are defined to be the $SU(2)$ orbit of the lowest weight vector of the representation D^j,

$$\mathbf{J} = D^j(g)|j\rangle, \quad g \in SU(2).$$

Averaging J_k with coherent states results in

$$\begin{aligned}
\langle \mathbf{J}|J_x|\mathbf{J}\rangle &= \langle j|D^j(g)^* J_x D^j(g)|j\rangle \;=\; j\cos\phi\sin\theta \;=\; x,\\
\langle \mathbf{J}|J_y|\mathbf{J}\rangle &= \langle j|D^j(g)^* J_y D^j(g)|j\rangle \;=\; j\sin\phi\sin\theta \;=\; y,\\
\langle \mathbf{J}|J_z|\mathbf{J}\rangle &= \langle j|D^j(g)^* J_z D^j(g)|j\rangle \;=\; j\cos\theta \;=\; z,
\end{aligned}$$

where

$$D^j(g) = e^{-i\phi J_z} e^{-i\theta J_y} e^{-i\psi J_z}, \quad \sum_{k=1}^{3} x_k^2 = \rho^2,\ \rho = j.$$

2.2 A one-parameter semi-quantum operator on V_s

Averaged with coherent states, the operator H_τ is made into a semi-quantum operator,

$$\overline{H}_\tau(\boldsymbol{x}) := \langle \mathbf{J}|H_\tau|\mathbf{J}\rangle = (1-\tau)S_z + \tau \sum_k x_k S_k, \quad \boldsymbol{x} \in S^2(\rho) \subset \mathbf{R}^3.$$

By associating the eigenspace of $\overline{H}_\tau(\boldsymbol{x})$ with $\boldsymbol{x} \in S^2(\rho)$, one may determine a one-parameter family of complex line bundles over $S^2(\rho)$. We denote by $S^2(\rho) \times \mathbf{C}|r\rangle$ and $L^{(r)}$ the complex line bundles associated with the eigenvalues r of S_z and ρr of $\boldsymbol{x} \cdot \boldsymbol{S}$, respectively. When t passes a critical value, the bundle structure changes;

$$\begin{array}{ccc} \displaystyle\sum_{|r|\le s}{}^{\oplus} S^2(\rho) \times \mathbf{C}|r\rangle & \longrightarrow & \displaystyle\sum_{|r|\le s}{}^{\oplus} L^{(r)} \\ \text{w.r.t. } \overline{H}_0(\boldsymbol{x}) = S_z & & \text{w.r.t. } \overline{H}_1(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{S} \end{array}$$

A question: $\mathrm{Ch}(L^{(r)}) = 2r$?

2.3 An example of a semi-quantum operator with $s = \frac{1}{2}$

The one-parameter semi-quantum Hamiltonian on $V_{\frac{1}{2}}$ is given by

$$\overline{H}_{\tau}(\boldsymbol{x}) = \frac{1}{2}\begin{pmatrix} 1-\tau+\tau z & \tau(x-iy) \\ \tau(x+iy) & -1+\tau-\tau z \end{pmatrix}, \quad \boldsymbol{x} \in S^2(1) \subset \mathbf{R}^3.$$

The eigenvalues are

$$\lambda(\tau) = \pm\sqrt{\frac{1}{4} - \frac{1-z}{2}(\tau-\tau^2)},$$

which are not degenerate if $z \neq -1$, but degenerate at $z = -1$ for $\tau = \frac{1}{2}$,

$$\lambda(\tau) = \pm|\tau - \frac{1}{2}|.$$

The splitting of $S^2 \times \mathbb{C}^2$ into line bundles associated with eigenvalues should change at $\tau = \frac{1}{2}$.

2.4 Examples eigen-line bundles, $L^{(\pm 1)} \to S^2(1)$

$$\overline{H}_1(\boldsymbol{x}) = \frac{1}{2}\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}, \quad \boldsymbol{x} \in S^2(1) \subset \mathbf{R}^3.$$

has normalized eigenvectors associated with the eigenvalues $1/2$, $-1/2$,

$$\boldsymbol{u}_+^{(-\frac{1}{2})} := \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \quad \boldsymbol{u}_+^{(\frac{1}{2})} := \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \quad \boldsymbol{x} \in U_+,$$

$$\boldsymbol{u}_-^{(-\frac{1}{2})} := \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}, \quad \boldsymbol{u}_-^{(\frac{1}{2})} := \begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \quad \boldsymbol{x} \in U_-,$$

where $U_+ = \{\boldsymbol{x} \in S^2(1) |\, \theta \neq \pi\}$, $U_- = \{\boldsymbol{x} \in S^2(1) |\, \theta \neq 0\}$.
The transformation rules on $U_+ \cap U_-$ are

$$\boldsymbol{u}_-^{(-\frac{1}{2})} = e^{i\phi}\boldsymbol{u}_+^{(-\frac{1}{2})}, \quad \boldsymbol{u}_-^{(\frac{1}{2})} = e^{-i\phi}\boldsymbol{u}_+^{(\frac{1}{2})} \quad \text{on } U_+ \cap U_-.$$

2.5 The eigen-line bundle $L^{(r)} \to S^2(\rho)$ with $|r| \leq s$

The matrix $\boldsymbol{x} \cdot \boldsymbol{S} = \sum_k x_k S_k$ with $\boldsymbol{x} \in S^2(\rho)$, which acts linearly on V_s, has the normalized eigenvector associated with the eigenvalue $r\rho$ and expressed as

$$\begin{aligned} \boldsymbol{u}_+^{(r)} &:= e^{-i\phi S_z} e^{-i\theta S_y} e^{i\phi S_z} |r\rangle \quad \text{on } U_+, \\ \boldsymbol{u}_-^{(r)} &:= e^{-i\phi S_z} e^{-i\theta S_y} e^{-i\phi S_z} |r\rangle \text{ on } U_-, \end{aligned}$$

where $|r| \leq s$, and where

$$U_+ = \{\boldsymbol{x} \in S^2(\rho) |, z \neq -\rho\}, \quad U_- = \{\boldsymbol{x} \in S^2(\rho) |, z \neq \rho\}.$$

The transformation rule on $U_+ \cap U_-$ are given by

$$e^{-2ir\phi} \boldsymbol{u}_+^{(r)} = \boldsymbol{u}_-^{(r)}.$$

Thus defined is the eigen-line bundle $L^{(r)} \to S^2(\rho)$.

2.6 The connection and the curvature of $L^{(r)}$

The connection form of the line bundle $L^{(r)} \to S^2(\rho)$ is defined by and expressed as

$$\omega_+^{(r)} := \langle r|D^s(g)^* dD^s(g)|r\rangle|_{\psi=-\phi} \;=\; -ir(-d\phi + \cos\theta d\phi) \quad \text{on } U_+,$$
$$\omega_-^{(r)} := \langle r|D^s(g)^* dD^s(g)|r\rangle|_{\psi=\phi} \;=\; -ir(d\phi + \cos\theta d\phi) \quad \text{on } U_-.$$

On the intersection $U_+ \cap U_-$, the $\omega_+^{(r)}$ and $\omega_-^{(r)}$ are related by

$$\omega_+^{(r)} - 2ird\phi = \omega_-^{(r)}.$$

Since $d\omega_+^{(r)} = d\omega_-^{(r)}$ on $U_+ \cap U_-$, the curvature form is defined globally on $S^2(\rho)$,

$$\Omega = \begin{cases} d\omega_+^{(r)} & \text{on } \; U_+, \\ d\omega_-^{(r)} & \text{on } \; U_-. \end{cases}$$

2.7 The Chern number of $L^{(r)}$

Let C denote the equator of $S^2(\rho)$, and $S^2_+(\rho), S^2_-(\rho)$ the northern and southern hemispheres, respectively. Then, by applying Stokes' theorem, one obtains

$$\begin{aligned}\int_{S^2(\rho)} \Omega &= \int_{S^2_+(\rho)} d\omega_+^{(r)} + \int_{S^2_-(\rho)} d\omega_-^{(r)} \\ &= \int_C \omega_+^{(r)} - \int_C \omega_-^{(r)} = 2ir \int_C d\phi = 4\pi ir\end{aligned}$$

Proposition 1. The Chern number of $L^{(r)}$ is given by

$$\frac{i}{2\pi} \int_{S^2(\rho)} \Omega = -2r.$$

3. Chern numbers associated with two-level semi-quantum systems with symmetry by $U(1)$, D_3, O

- $U(1)$, D_3, O as subgroups of $SO(3)$
- semi-quantum systems with $U(1)$, D_3, or O symmetry
- invariant Hamiltonians with control parameters
- Chern numbers of eigen-line bundles depending on parameters

3.1 $U(1)$ invariance of H_τ

We consider the $U(1)$ action on $V_j \otimes V_s$;

$$e^{-itJ_z} \otimes e^{-itS_z}.$$

Since J_k and S_k transform like the vector x_k,

$$e^{-itJ_z} J_k (\text{rep.}\ S_k) e^{itJ_z} = \sum_\ell a_{k\ell} J_\ell (\text{resp.}\ S_\ell), \quad (a_{k\ell}) = e^{-t\widehat{\boldsymbol{e}}_3},$$

Both $\mathbf{1} \otimes S_z$ and $\sum_k J_k \otimes S_k$ are invariant under this $U(1)$ action, so that the Hamiltonian $H_\tau = (1-\tau)\mathbf{1} \otimes S_z + \tau \sum_k J_k \otimes S_k$ is invariant under $U(1)$ as well.

3.2 $U(1)$ invariance of $\overline{H}_\tau(\boldsymbol{x})$

Since J_k transform like the vector x_k, the averages of J_k transform also in the same manner;

$$\langle \mathbf{J}|e^{-itJ_z}J_k e^{itJ_z}|\mathbf{J}\rangle = \sum_\ell a_{k\ell}\langle \mathbf{J}|J_\ell|\mathbf{J}\rangle = \sum_\ell a_{k\ell}x_\ell, \quad (a_{k\ell}) = e^{-t\widehat{\boldsymbol{e}}_3}$$

which defines an $SO(2)$ action on $S^2(\rho)$. Thus, the induced $U(1)$ action on the semi-quantum system is described as

$$x_k \mapsto \sum a_{k\ell}x_\ell, \quad S_k \mapsto \sum a_{k\ell}S_\ell.$$

Since S_z and $\sum_k x_k S_k$ are both invariant under the $U(1)$ action, $\overline{H}_\tau(\boldsymbol{x}) = (1-\tau)S_z + \tau\sum_k x_k S_k$ is invariant as well.

3.3 $U(1)$-invariant Hamiltonians

In place of $\sum_k x_k S_k$, we consider a Hamiltonian $H(\boldsymbol{x}) = \sum_k f_k(\boldsymbol{x}) S_k$. Since $H(\boldsymbol{x})$ transforms according to

$$\sum_k f_k(\boldsymbol{x}) S_k \mapsto \sum_{k,\ell} f_k(A_t \boldsymbol{x}) a_{k\ell} S_\ell, \quad A_t = (a_{k\ell}) = e^{-t\widehat{\boldsymbol{e}}_3},$$

the $U(1)$ invariance condition for $H(\boldsymbol{x})$ yields

$$f_k(A_t \boldsymbol{x}) = \sum_\ell a_{k\ell} f_\ell(\boldsymbol{x}),$$

which implies that the $\mathbf{R}^3$-valued function $\boldsymbol{F}(\boldsymbol{x}) = \sum_k f_k(\boldsymbol{x}) \boldsymbol{e}_k$ is $SO(2)$-equivariant; $\boldsymbol{F}(A_t \boldsymbol{x}) = A_t \boldsymbol{F}(\boldsymbol{x})$. Equivalently, one has

$$H(e^{-t\widehat{\boldsymbol{e}}_3} \boldsymbol{x}) = D(e^{-it/2}) H(\boldsymbol{x}) D(e^{it/2}),$$

$$D(e^{-it/2}) = \mathrm{diag}(e^{-it/2}, e^{it/2}).$$

3.4 Weighted $U(1)$ symmetry

If we start with the extended $U(1)$ action on $V_j \otimes V_s$ expressed as

$$e^{-itJ_z} \otimes e^{-itKS_z}, \quad K \in \{0, 1, 2, \cdots\},$$

the invariance condition,

$$H(e^{-t\widehat{\boldsymbol{e}}_3}\boldsymbol{x}) = D(e^{-iKt/2})H(\boldsymbol{x})D(e^{iKt/2}),$$

results in

$$h(e^{-it}w, e^{it}\overline{w}, z) = e^{-iKt}h(w, \overline{w}, z), \quad f_3(e^{-it}w, e^{it}\overline{w}, z) = f_3(w, \overline{w}, z),$$

so that

$$w\frac{\partial h}{\partial w} = Kh, \quad \frac{\partial h}{\partial \overline{w}} = 0, \quad \frac{\partial f_3}{\partial w} = \frac{\partial f_3}{\partial \overline{w}} = 0.$$

We then obtain, for example,

$$h(w, \overline{w}, z) = h(z)(x + iy)^K, \quad f_3(w, \overline{w}, z) = f(z).$$

3.5 Examples of Hamiltonians with weighted $U(1)$ symmetry

$$\sum_k f_k(\boldsymbol{x}) S_k = \begin{pmatrix} f(z) & h(z)(x-iy)^K \\ h(z)(x+iy)^K & -f(z) \end{pmatrix} \quad \text{for} \quad s = \frac{1}{2},$$

and

$$\begin{pmatrix} f(z) & \frac{1}{\sqrt{2}}\Big(h(z)(x-iy)^K\Big) & 0 \\ \frac{1}{\sqrt{2}}\Big(h(z)(x+iy)^K\Big) & 0 & \frac{1}{\sqrt{2}}\Big(h(z)(x-iy)^K\Big) \\ 0 & \frac{1}{\sqrt{2}}\Big(h(z)(x+iy)^K\Big) & -f(z) \end{pmatrix}$$

for $s = 1$, where $\boldsymbol{x} \in S^2(\rho)$, $\rho = j$.

3.6 Chern number in the presence of weighted $U(1)$ symmetry

Proposition 2. Let K and f_{ij} be an integer and any real polynomial in $z = \cos\theta$, respectively. Suppose that in the weighted $SO(2)$ invariant Hamiltonian

$$H = \begin{pmatrix} f_{11}(\cos\theta) & f_{12}(\cos\theta)\ \sin^K\theta\ \exp(iK\phi) \\ f_{12}(\cos\theta)\ \sin^K\theta\ \exp(-iK\phi) & -f_{11}(\cos\theta) \end{pmatrix},$$

the matrix elements f_{11} and $\sin^K\theta f_{12}$ do not share zeros. Then H has two eigenvalues, positive and negative, without degeneracy. For $K \neq 0$, the complex line bundle associated with each eigenvalue is defined over the two-sphere S^2. The first Chern number, which characterizes each line bundle, is equal to 0 or $\pm K$, depending on whether the number of zeros of the diagonal element, counted with their multiplicities, is even or odd.

3.7 A dihedral group D_3

The D_3 is a symmetry group of an equilateral triangle, which is known to be isomorphic with the symmetric group S_3.

$$\pi_1 = (1), \quad \pi_2 = (1\ 2\ 3),\ \pi_3 = (1\ 3\ 2),$$
$$\pi_4 = (1\ 2),\ \pi_5 = (2\ 3), \quad \pi_6 = (1\ 3).$$

As is well known, the E representation of D_3 is given by

$$D^E(\pi_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ D^E(\pi_2) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},\ D^E(\pi_3) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

$$D^E(\pi_4) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},\ D^E(\pi_5) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ D^E(\pi_6) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

and A_2 representation by

$$D^{A_2}(\pi_j) = 1, \quad j = 1, 2, 3, \quad D^{A_2}(\pi_k) = -1, \quad k = 4, 5, 6.$$

3.8 A D_3 action on $\mathbb{R}^3$

The E representation of D_3 acts on the set $\mathcal{H}_0(2)$ of 2×2 traceless Hermitian matrices by the adjoint action, which proves to induce the representation equivalent to $E \oplus A_1$. Since $\mathcal{H}_0(2) \cong \mathbb{R}^3$, D_3 acts on $\mathbb{R}^3$ in this manner.

Taking the Pauli basis $\sigma_1, \sigma_2, \sigma_3$ of $\mathcal{H}_0(2)$ as $\sigma'_y, \sigma'_z, \sigma'_x$, respectively, we identify the $\mathbb{R}^2$ spanned by σ'_x, σ'_y with the x-y plane as the representation space for E and $\mathbb{R}$ with the z axis as the representation space for A_2.

For example, one has

$$D^{E\oplus A_2}(\pi_2) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D^{E\oplus A_2}(\pi_4) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

3.9 D_3 equivariant functions

The sets of functions

$$\begin{pmatrix} y^2 - x^2 \\ 2xy \end{pmatrix}, \quad \begin{pmatrix} zy \\ -zx \end{pmatrix}$$

are E-equivariant and the functions

$$z, \quad y(y^2 - 3x^2)$$

are A_2-equivariant.

The functions

$$z^2, \quad x(x^2 - 3y^2)$$

are known to be A_1-equivariant or simply invariant.

3.10 Hamiltonians with D_3 symmetry

Since the equivariance of the above-mentioned functions and the invariannce of the Hamiltonian $D^E(g)H(\boldsymbol{x})D^E(g)^{-1} = H(D^{E\oplus A_2}(g)\boldsymbol{x})$, $g \in D_3$, are equivalent, we obtain the invariant Hamiltonian of the form

$$H(\boldsymbol{x}) = \begin{pmatrix} X & Y + iZ \\ Y - iZ & -X \end{pmatrix}, \quad \boldsymbol{x} \in S^2(1) \subset \mathbb{R}^3,$$

where

$$\begin{aligned} X(\boldsymbol{x}) =& b_1(y^2 - x^2) + b_2 zy, \\ Y(\boldsymbol{x}) =& 2b_1 yx - b_2 zx, \\ Z(\boldsymbol{x}) =& -(a_1 z + a_2 y(y^2 - 3x^2)), \end{aligned}$$

and where (a_1, a_2, b_1, b_2) are real constants.
We assume that $(a_1, a_2) \neq (0,0)$ and $(b_1, b_2) \neq (0,0)$.

3.11 Chern numbers in the presence of D_3 symmetry

Proposition 3. For the D_3 invariant Hamiltonian $H(\boldsymbol{x})$, owing to the invariance of the Chern numbers with respect the scaling of the parameters (a_1, a_2, b_1, b_2), the parameter space $\mathbb{R}^4 - \{0\}$ reduces to the two-torus T^2 determined by $a_1 = \cos\phi_1, a_2 = \sin\phi_1$ and $b_1 = \cos\phi_2, b_2 = \sin\phi_2$, on which the iso-Chern diagram for the eigen-line bundle associated with positive eigenvalue is described in the following figure. The iso-Chern diagram for the eigen-line bundle associated with negative eigenvalue is obtained by opposing the sign of the Chern number assigned to each iso-Chern domain.

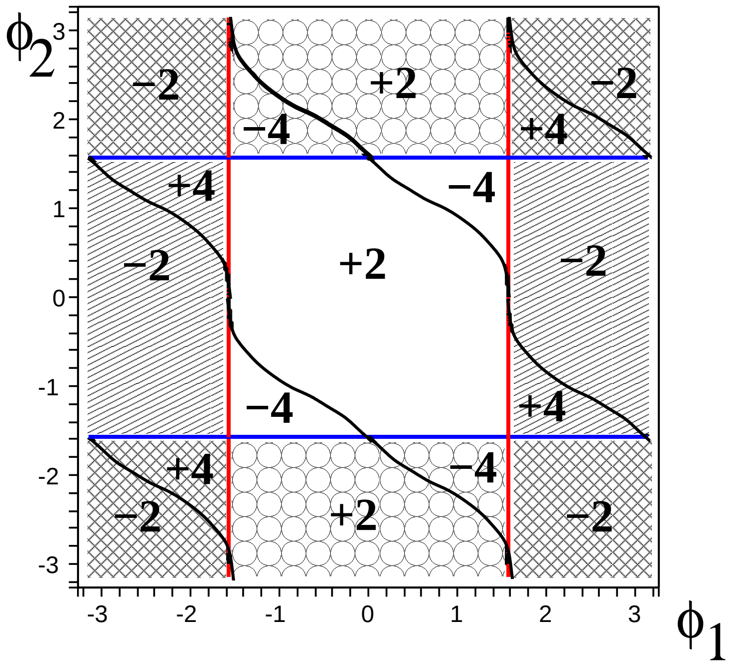

Figure 2: The iso-Chern diagram for the D_3 invariant Hamiltonian

In each iso-Chern domain, the Chern number for eigen-line bundle with positive eigenvalue is indicated. The read and blue lines ($\phi_1 = \pm\frac{\pi}{2}, \phi_2 = \pm\frac{\pi}{2}$) and black curves ($\cos\phi_1 \cos\phi_2 = \sin\phi_1 \sin^3\phi_2$) are the sets of degeneracy points.

3.12 The octahedral group O

The octahedral group O is the orientation-preserving symmetry group for the regular octahedron, which is known to be isomorphic to the symmetric group S_4 and further to be generated by

$$C_4^Z \mapsto \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \quad C_3^{[-1-1-1]} \mapsto \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \quad C_2^X \mapsto \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}.$$

This representation on $\mathbb{R}^3$ is known as the T_1 (or F_1) representation.

The two-dimensional representation E is generated by

$$C_4^Z \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad C_3^{[-1-1-1]} \mapsto \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad C_2^X \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

3.13 Actions of the group O

The E representation of the group O acts on the set $\mathcal{H}_0(2)$ of traceless 2×2 Hermitian matrices, which induces the reducible representation $E\oplus A_2$, where the representation space for E is spanned by σ_3, σ_1 and that for A_2 by σ_2.

The functions

$$\begin{pmatrix} 2z^2-x^2-y^2 \\ \sqrt{3}(x^2-y^2) \end{pmatrix}, \quad xyz$$

are known as E-equivariant and A_2-equivariant, respectively, where the group O acts on $\mathbb{R}^3$ by T_1 (or F_1) representation.

3.14 Hamiltonians with O symmetry

Let

$$\phi_1 = 2z^2 - x^2 - y^2, \quad \phi_2 = \sqrt{3}(x^2 - y^2), \quad \phi_3 = xyz.$$

Then, the Hamiltonian

$$H(\boldsymbol{x}) = \begin{pmatrix} a\phi_1 & a\phi_2 - ib\phi_3 \\ a\phi_2 + ib\phi_3 & -a\phi_1 \end{pmatrix}$$

proves to be invariant under the O group action, $D^E(g)H(\boldsymbol{x})D^E(g)^{-1} = H(D^{T_1}(g)\boldsymbol{x})$, $g \in O$, where a, b are real parameters with $(a, b) \neq (0, 0)$.

3.15 Chern numbers in the presence of O symmetry

Proposition 4. The parameter space $\mathbb{R}^2 - \{0\}$ for the O-invariant Hamiltonian $H(\boldsymbol{x})$ reduces to a circle, and the degeneracy points on this circle are $(a, b) = (\pm 1, 0), (0, \pm 1)$. The Chern numbers are shown in the figure,

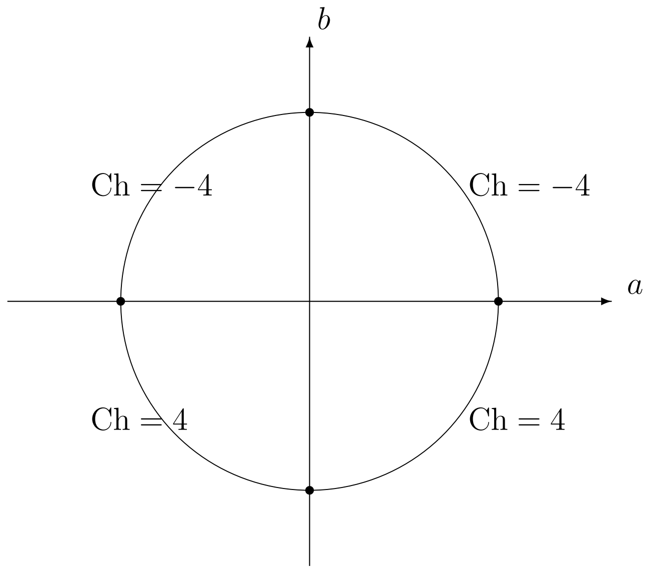

Figure 3: Chen numbers on the unit circle

3.16 A sketch of the proof

The condition of the degeneracy is described as

$$\det H(\boldsymbol{x}) = 0 \quad \Leftrightarrow \quad a^2(\phi_1^2 + \phi_2^2) = 0,\ b^2\phi_3^2 = 0.$$

Since the condition is scale invariant, we may restrict the parameters to the circle $a^2 + b^2 = 1$. There are four degeneracy points $(\pm 1, \pm 1)$ on this circle, for which the eigenvalues of $H(\boldsymbol{x})$ are degenerate on some points of S^2. For regular values of the parameter, line bundles are associated with each eigenvalue. The exceptional points at which the normalized eigenvector for the positive eigenvalue is not defined are

$$\boldsymbol{n}_\pm = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}, \quad \boldsymbol{a}_\pm = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \boldsymbol{b}_\pm = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

3.17 A sketch of the proof, continued

In the case of $a > 0$, the domains of normalized eigenvectors $\boldsymbol{v}_\pm$ are

$$U_+ = S^2 - \{\boldsymbol{n}_\pm\}, \quad U_- = S^2 - \{\boldsymbol{a}_\pm, \boldsymbol{b}_\pm\},$$

respectively. The $\boldsymbol{v}_\pm$ are related by

$$\boldsymbol{v}_+ = \Phi \boldsymbol{v}_-, \quad \Phi = \frac{a\phi_2 - ib\phi_3}{\sqrt{a^2\phi^2 + b^2\phi_3^2}} \quad \text{on} \quad U_+ \cap U_-.$$

The local connection form are defined to be

$$\omega_+ = \boldsymbol{v}_+^\dagger d\boldsymbol{v}_+, \quad \omega_- = \boldsymbol{v}_-^\dagger d\boldsymbol{v}_-,$$

and related by

$$\omega_+ = \Phi^{-1} d\Phi + \omega_- \quad \text{on} \quad U_+ \cap U_-.$$

3.18 A sketch of the proof, continued further

Let C_1 and C_2 be two circles at the levels $z = \pm h$ with $0 < h < 1$. Let S^2_+ and S^2_- be regions separated by C_1 and C_2. The S^2 is the region containing the equator and S^2_- is the union of two regions containing either of the north or the south pole. The orientation of C_i is in keeping with that of S^2_+.

The Chedrn number is then evaluated as

$$c_1 = \frac{i}{2\pi} \int_{S^2} \Omega = -\frac{1}{2\pi i} \int_{C_1+C_2} \Phi^{-1} d\Phi.$$

The right-hand side is minus the sum of the winding numbers of the maps $C_k \to U(1)$ by Φ with $k = 1, 2$. The winding numbers are computable directly. A linearization method is applicable if the circles are deformed suitably without changing the value of the contour integrals.

4. Chern numbers associatd wtih three-level semi-quantum systems with O symmetry

- the octahedral group $O \cong S_4$
- a choice of the action of O and
  the irreducible representations T_1 and E.
- O invariant semi-quantum Hamiltonians
- the iso-Chern diagram

4.1 The space of traceless 3×3 traceless Hermitian matrices as a (reducible) representation space of the O group

$$
\begin{aligned}
\mathcal{H}_0(3) \; = \; & \left\{ \begin{pmatrix} 0 & c_1 & b_1 \\ c_1 & 0 & a_1 \\ b_1 & a_1 & 0 \end{pmatrix} \right\} \\
& \oplus \left\{ \begin{pmatrix} 0 & -ic_2 & ib_2 \\ ic_2 & 0 & -ia_2 \\ -ib_2 & ia_2 & 0 \end{pmatrix} \right\} \\
& \oplus \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \right\},
\end{aligned}
$$

where $a_k, b_k, c_k \in \mathbb{R}$, $d_j \in \mathbb{R}$ with $d_1 + d_2 + d_3 = 0$.
Each subspace carries two- or three dimensional irreducible (E or T_1) representation of O under the adjoint action in the T_1 matrix form.

4.2 O-Invariant Hamiltonians

Let $H(\boldsymbol{x}) \in \mathcal{H}_0(3)$ with $\boldsymbol{x} \in S^2 \subset \mathbb{R}^3$.
The $H(\boldsymbol{x})$ is O-invariant if and only if

$$gH(\boldsymbol{x})g^{-1} = H(g\boldsymbol{x}) \quad \text{for} \quad g \in \boldsymbol{O},$$

where g is represented in the T_1 matrix form.

A simple example of O-invariant Hamiltonian is

$$H(\boldsymbol{x}) = \begin{pmatrix} 0 & -iz & iy \\ iz & 0 & -ix \\ -iy & ix & 0 \end{pmatrix}.$$

4.3 Examples of O-Invariant Hamiltonians

$$H(\boldsymbol{x}) = \begin{pmatrix} 0 & -iz(z^2 - \frac{3}{5}r^2) & iy(y^2 - \frac{3}{5}r^2) \\ iz(z^2 - \frac{3}{5}r^2) & 0 & -ix(x^2 - \frac{3}{5}r^2) \\ -iy(y^2 - \frac{3}{5}r^2) & ix(x^2 - \frac{3}{5}r^2) & 0 \end{pmatrix},$$

where $r^2 = x^2 + y^2 + z^2$.

$$H(\boldsymbol{x}) = \begin{pmatrix} 0 & xy & zx \\ xy & 0 & yz \\ zx & yz & 0 \end{pmatrix}, \quad H(\boldsymbol{x}) = \begin{pmatrix} 0 & z(x^2 - y^2) & y(z^2 - x^2) \\ z(x^2 - y^2) & 0 & x(y^2 - z^2) \\ y(x^2 - x^2) & x(y^2 - z^2) & 0 \end{pmatrix}.$$

$$H(\boldsymbol{x}) = \begin{pmatrix} 2x^2 - y^2 - z^2 & & \\ & 2y^2 - z^2 - x^2 & \\ & & 2z^2 - x^2 - y^2 \end{pmatrix}.$$

4.4 A remark

The Hamiltonian of the diagonal matrix form is expressed as

$$\begin{pmatrix} 2x^2 - y^2 - z^2 & & \\ & 2y^2 - z^2 - x^2 & \\ & & 2z^2 - x^2 - y^2 \end{pmatrix}$$

$$= (2z^2 - x^2 - y^2) \begin{pmatrix} -\frac{1}{2} & & \\ & -\frac{1}{2} & \\ & & 1 \end{pmatrix} + \sqrt{3}(x^2 - y^2) \begin{pmatrix} \frac{\sqrt{3}}{2} & & \\ & -\frac{\sqrt{3}}{2} & \\ & & 0 \end{pmatrix},$$

where $\phi_1 = 2z^2 - x^2 - y^2$ and $\phi_2 = \sqrt{3}(x^2 - y^2)$ are known to form an E-equivariant vector-valued function.

4.5 A model Hamiltonian with O symmetry

We consider the Hamiltonian of the form

$$H(\boldsymbol{x}) = \begin{pmatrix} 0 & -iZ & iY \\ iZ & 0 & -iX \\ -iY & iX & 0 \end{pmatrix}, \quad \boldsymbol{x} \in S^2 \subset \mathbb{R}^3,$$

where X, Y, Z are functions given by

$$\begin{aligned} X =& ax + bx(x^2 - \frac{3}{5}r^2), \\ Y =& ay + by(y^2 - \frac{3}{5}r^2), \\ Z =& az + bz(z^2 - \frac{3}{5}r^2), \end{aligned}$$

respectively, where $r^2 = x^2 + y^2 + z^2$, and where a, b are real parameters with $(a, b) \neq (0, 0)$. The constraint $r = 1$ is imposed, of course.

4.6 Degeneracy

The eigenvalues of $H(\boldsymbol{x})$ are $\lambda = 0, \pm R$ with $R^2 = X^2 + Y^2 + Z^2$.

Degeneracy occurs iff $R = 0$, which provides degeneracy points,

$$\begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{if and only if} \quad \frac{a}{2} = -\frac{b}{5}.$$

$$\begin{pmatrix} 0 \\ \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} \pm\frac{1}{\sqrt{2}} \\ 0 \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \text{if and only if} \quad \frac{a}{1} = \frac{b}{10}.$$

$$\begin{pmatrix} \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \end{pmatrix}, \quad \text{if and only if} \quad \frac{a}{4} = \frac{b}{15}.$$

4.7 Chern numbers in the presence of O symmetry

Proposition 4. The parameter space $\mathbb{R}^2 - \{0\}$ for the O-invariant Hamiltonian $H(\boldsymbol{x})$ reduces to the unit circle. In association with the positive eigenvalue $\lambda = R$, an eigen-line bundle is determined on each arc between consecutive degeneracy points on the unit circle. The Chern numbers of the eigen-line bundles are shown as follows:

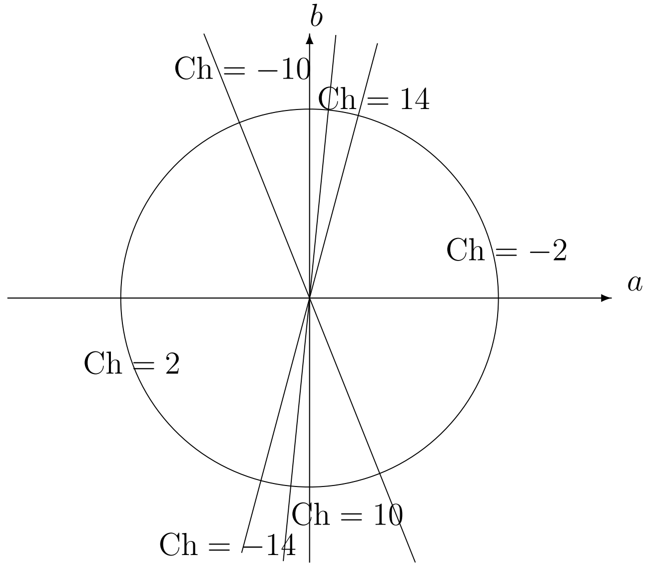

Figure 4: Chern numbers on the unit cycle: Chern numbers are assigned to arcs separated by degeneracy points.

5. Linearization method on a subspace

In order to work with eigen-line bundles, we need to know eigenvalues and eigenvectors by solving eigenvalue equation.
However, algebraic equations of degree greater than two are not easy to solve.

Is it possible to evaluate a Chern number by linear approximation on a two-dimensional subspace assigned to a degeneracy point on S^2 ?

5.1 A more general model of O-invariant Hamiltonian

$$
\begin{aligned}
& H(\boldsymbol{x}) \\
= & \begin{pmatrix} 0 & iz & -iy \\ -iz & 0 & ix \\ iy & -ix & 0 \end{pmatrix} \\
& +a \begin{pmatrix} y^2+z^2-2x^2 & 0 & 0 \\ 0 & z^2+x^2-2y^2 & 0 \\ 0 & 0 & x^2+y^2-2z^2 \end{pmatrix} \\
& +b \begin{pmatrix} 0 & xy & zx \\ xy & 0 & yz \\ zx & yz & 0 \end{pmatrix} .
\end{aligned}
$$

5.2 Toward the iso-Chern diagram

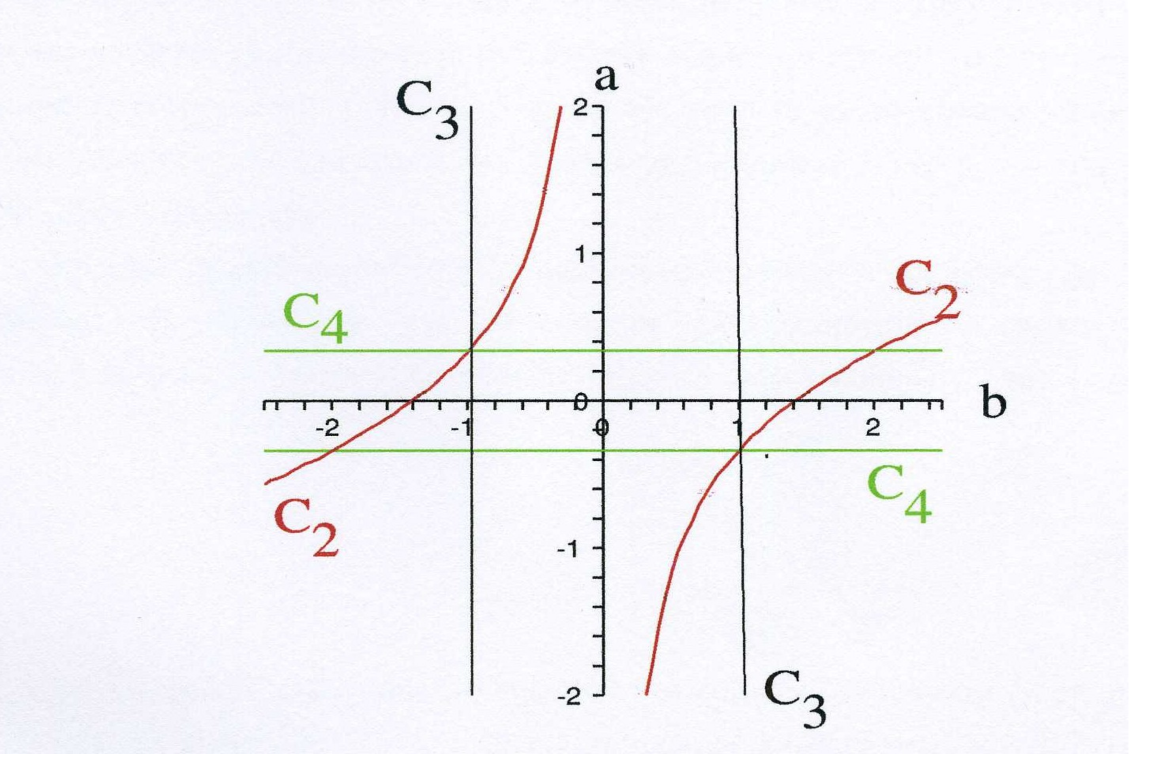

Figure 5: The iso-Chern diagaram

To each point of a degeneracy curve, there corresponds a set of degeneracy points on S^2, which forms an orbit of the group O. The symbol C_k attached to each degeneracy curve denote the associated isotropy subgroup. Hence the order of the orbit in question is $24/\#C_k$. $\Delta_k\text{Chern} = \pm 24/\#C_k$?

5.3 Eigenspace decomposition at a degeneracy point

Let $|e_k(\boldsymbol{x})\rangle$ denote the normalized eigenvector associated with the disjoint eigenvalue $\lambda_k = \lambda_k(\boldsymbol{x})$ of the Hamiltonian $H(\boldsymbol{x})$. Then,

$$\mathbb{C}^3 = \mathrm{span}\{|e_1(\boldsymbol{x})\rangle\} \oplus \mathrm{span}\{|e_2(\boldsymbol{x})\rangle\} \oplus \mathrm{span}\{|e_3(\boldsymbol{x})\rangle\}.$$

If $\lambda_1(\boldsymbol{x}_0) \neq \lambda_2(\boldsymbol{x}_0) = \lambda_3(\boldsymbol{x}_0)$ at $\boldsymbol{x}_0$, the decomposition becomes

$$\mathbb{C}^3 = \mathrm{span}\{|e_1(\boldsymbol{x}_0)\rangle\} \oplus \mathrm{span}\{|e_2(\boldsymbol{x}_0)\rangle, |e_3(\boldsymbol{x}_0)\rangle\},$$

where the basis $\{|e_2(\boldsymbol{x}_0)\rangle, |e_3(\boldsymbol{x}_0)\rangle\}$ is determined up to $U(2)$.

Can one use the subspace $\mathrm{span}\{|e_2(\boldsymbol{x}_0)\rangle, |e_3(\boldsymbol{x}_0)\rangle\}$ with $\boldsymbol{x}_0$ being isolated, in order to evaluate a change in Chern numbers against the variation of control parameters by means of the linear approximation of the Hamiltonian on this subspace ?

5.4 The Hamiltonian at a degeneracy point

For $a = \frac{1}{3}$, the model Hamiltonian have a degeneracy point $\boldsymbol{x}_0 = (0,0,1)^T$, at which the Hamiltonian takes the form

$$H(\boldsymbol{x}_0) = \begin{pmatrix} \frac{1}{3} & i & 0 \\ -i & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix},$$

and has the eigenvalues and the associated eigenvectors,

$$\lambda_1 = \frac{4}{3}, \quad \lambda_2 = \lambda_3 = -\frac{2}{3},$$

$$|e_1(\boldsymbol{x}_0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}, \quad |e_2(\boldsymbol{x}_0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}, \quad |e_3(\boldsymbol{x}_0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

respectively. Note that the orbit of $\boldsymbol{x}_0$ by the action of the O group forms a set of degeneracy points.

5.5 Linearization at the degeneracy point $\boldsymbol{x}_0$

With respect to the basis $|e_k(\boldsymbol{x}_0)\rangle$, $k = 1, 2, 3$, the Hamiltonian is linearized at $\boldsymbol{x}_0$ to be

$$H_{\mathrm{fl}}(q) = \begin{pmatrix} a+1 & 0 & \frac{b-1}{\sqrt{2}}(y - ix) \\ 0 & a-1 & \frac{b+1}{\sqrt{2}}(y + ix) \\ \frac{b-1}{\sqrt{2}}(y + ix) & \frac{b+1}{\sqrt{2}}(y - ix) & -2a \end{pmatrix},$$

where q is a point of the tangent plane Π_0 to S^2 at $\boldsymbol{x}_0$, which is endowed with the Cartesian coordinates (x, y).

The $H_{\mathrm{fl}}(q)$ does not take a block diagonal form, *i.e.*, does not fit the decomposition $\mathrm{span}\{|e_1(\boldsymbol{x}_0)\rangle\} \oplus \mathrm{span}\{|e_2(\boldsymbol{x}_0)\rangle, |e_3(\boldsymbol{x}_0)\rangle\}$.

5.6 Restriction on a subspace

Restricted on $V_2 := \mathrm{span}\{|e_2(\boldsymbol{x}_0)\rangle, |e_3(\boldsymbol{x}_0)\rangle\}$, the $H_{\mathrm{fl}}(q)$ reduces to

$$H_{2\mathrm{l}}(q) = \begin{pmatrix} a-1 & \frac{1+b}{\sqrt{2}}(y+ix) \\ \frac{1+b}{\sqrt{2}}(y-ix) & -2a \end{pmatrix},$$

which has the eigenvalues

$$\lambda_{\pm} := -\frac{a+1}{2} \pm \frac{1}{2}\sqrt{(3a-1)^2 + 2(1+b)^2(x^2+y^2)}.$$

Does $\lambda_{\pm}$ approximate to eigenvalues of $H_{\mathrm{fl}}(q)$?
If so, the linearization on the subspace V_2 may work.

5.7 Linearizable on the subspace V_2 ?

Let $F(\lambda) = \det(H_{\text{fl}}(q) - \lambda I_3)$. Then, for $\lambda_\pm$, one has

$$F(\lambda_\pm) = (\lambda_\pm - a + 1)\Big(\frac{b-1}{\sqrt{2}}\Big)^2 (x^2 + y^2).$$

If the factor $\lambda_\pm - a + 1$ is small enough for a sufficiently close to $\frac{1}{3}$, the $\lambda_\pm$ will approximate to eigenvalues of $H_{\text{fl}}(q)$.

Let $a = \frac{1}{3} + t$. We can show that if $(1+b)^2(x^2+y^2) \ll 9t^2$ then

$$\lambda_+ - a + 1 \approx -\frac{3}{2}t + \frac{3}{2}|t| = \begin{cases} 0 & \text{for} \quad t > 0, \\ -\frac{4}{3}t & \text{for} \quad t < 0, \end{cases}$$

$$\lambda_- - a + 1 \approx -\frac{3}{2}t - \frac{3}{2}|t| = \begin{cases} -\frac{4}{3}t & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t < 0, \end{cases}$$

which means that λ_+ approximates to an eigenvalue of $H_{\text{fl}}(q)$ for $t > 0$, but not for $t < 0$.

5.8 Normalized eigenvector of $H_{\mathrm{fl}}(q)$

For the eigenvalue λ sufficiently close to λ_+, if $(x,y) \neq (0,0)$, the $H_{\mathrm{fl}}(q)$ has the normalized eigenvector expressed in two ways as

$$|v_{\mathrm{up}}(q)\rangle = \frac{1}{N_{\mathrm{up}}} \begin{pmatrix} (a-1-\lambda)Y \\ (a+1-\lambda)X \\ -(a+1-\lambda)(a-1-\lambda) \end{pmatrix},$$

$$|v_{\mathrm{down}}(q)\rangle = \frac{1}{N_{\mathrm{down}}} \begin{pmatrix} -\overline{X}Y \\ |Y|^2 + (a+1-\lambda)(\lambda+2a) \\ (a+1-\lambda)\overline{X} \end{pmatrix},$$

where

$$X = \frac{b+1}{\sqrt{2}}(y+ix), \quad Y = \frac{b-1}{\sqrt{2}}(y-ix),$$

$$N_{\mathrm{up}}^2 = (a-1-\lambda)^2|Y|^2 + (a+1-\lambda)^2|X|^2 + (a+1-\lambda)^2(a-1-\lambda)^2,$$

$$N_{\mathrm{down}}^2 = |X|^2|Y|^2 + \left(|Y|^2 + (a+1-\lambda)(\lambda+2a)\right)^2 + (a+1-\lambda)^2|X|^2.$$

5.9 An eigen-line bundle on the tangent plane Π_0

For $t > 0$, $N_{\rm up}$ vanishes at $(x, y) = (0, 0)$, but $N_{\rm down}$ does not.

Outside of $(x, y) = (0, 0)$, $|v_{\rm up}(q)\rangle$ and $|v_{\rm down}(q)\rangle$ are related by

$$|v_{\rm up}(q)\rangle = \Phi_{\rm fl}|v_{\rm down}(q)\rangle, \quad \Phi_{\rm fl} = \frac{N_{\rm down}}{N_{\rm up}} \frac{(a+1-\lambda)X}{|Y|^2 + (a+1-\lambda)(\lambda+2a)}.$$

Since $a+1-\lambda$, $\lambda+2a$, $N_{\rm up}$, and $N_{\rm down}$ are positive if $t > 0$ for the eigenvalue λ sufficiently close to λ_+, the winding number associated with $\Phi_{\rm fl}$ for a circle enclosing the origin is the same as that associated with $X/|X|$.

The winding number for $X/|X|$ is associated with the eigen-line bundle corresponding to the eigenvalue λ_+ of $H_{2\rm l}(q)$.

5.10 A linearization method on the subspace V_2 for Chern number

A generic setting for an eigen-line bundle over S^2:
For a non-degenerate eigenvalue, let S^2_+ and S^2_- be open subsets of S^2 in which the normalized eigenvectors $|v_+\rangle$ and $|v_-\rangle$ are defined, respectively. These eigenvector are related with each other by a transition function Φ on the intersection $S^2_+ \cap S^2_-$; $|v_+\rangle = \Phi|v_-\rangle$.

The (first) Chern number of the eigen-line bundle is equal to minus the sum of the winding number for a small circle centered at each exceptional point at which the normalized eigenvector, say $|v_+\rangle$, is not defined.

The linearization method on the subspce V_2:
If the circle enclosing an exceptional point for Φ is small enough, the winding number (or mapping degree) assigned to the exceptional point is equal to that for Φ_{fl} and then to that for $X/|X|$.
Then, the linearization method serves the Chern number calculation.

5.11 A Change in eigen-line bundles

For a regular point of the parameter space, we have the eigen-line bundles L_k, $k = 1, 2, 3$, associated with eigenvalues λ_k, $k = 1, 2, 3$, for the full Hamiltonian. We here assume that $\lambda_1 > \lambda_2 > \lambda_3$ for simplicity. If we cross the line $a = \frac{1}{3}$ with $|b| < 1$ from the domain with $a < \frac{1}{3}$ to the domain with $a > \frac{1}{3}$ in the parameter space (see Fig.5), the direct sum of the bundles $L_1 \oplus L_2 \oplus L_3$ changes into $L_1 \oplus L_2' \oplus L_3'$ after the crossing. This is because in crossing the line $a = \frac{1}{3}$ the degeneracy in eigenvalues occurs in the form $\lambda_1 \neq \lambda_2 = \lambda_3$ at $\boldsymbol{x}_0 = (0, 0, 1)^T$ and at the orbit of $\boldsymbol{x}_0$ by the group O with $\lambda_1 = \frac{4}{3}, \lambda_2 = \lambda_3 = -\frac{2}{3}$. The eigen-line bundle L_1 do not undergo a topological change.

5.12 A Change in Chern numbers

We are interested in the eigen-line bundle associated with the eigenvalue which approximates to λ_+. The change we will observe is a change in the Chern number according to the topological change of $L_2 \to L_2'$. According to the linearization method, we are allowed to consider the exceptional point for $|v_{\mathrm{up}}(p)\rangle$ with $t > 0$. Since the circle C_0 enclosing the exceptional point in question is clockwise oriented, the map $X/|X| = \sin t + i \cos t : C_0 \to U(1)$ has the winding number -1. Hence, the Chern number components to be assigned is $+1$. To obtain a full change in the Chern number, we have to sum up all the components attached to all the exceptional points concerned, the number of which is $\#\mathcal{O} = 24/\#C_4 = 6$.