Duality of (2,3,5)-distributions and Lagrangian cone structures

(2,3,5)-分布とラグランジュ錐構造の双対性

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【 $(2, 3, 5)$-distributions】

Let Y be a 5-dimensional manifold and $D \subset TY$ a distribution of rank 2. Then D is called a $(2, 3, 5)$-distribution if it has small growth $(2, 3, 5)$, namely, if $\mathrm{rank}(\partial\mathcal{D}) = 3$ and $\mathrm{rank}(\partial^{(2)}\mathcal{D}) = 5$, where $\partial\mathcal{D} := [\mathcal{D}, \mathcal{D}]$ is the derived system and $\partial^{(2)}\mathcal{D} := [\mathcal{D}, \partial\mathcal{D}]$ $(= \mathcal{D} + [\mathcal{D}, \partial\mathcal{D}])$.

Today, I would like to speak some results obtained in the joint papers:

Goo Ishikawa, Yoshinori Machida, Masatomo Takahashi, *Singularities of tangent surfaces in Cartan's split* G_2*-geometry,* Hokkaido Univ. Preprint Series in Math. #1020 (2012), Asian J. of Math., **20–2**, (2016), 353–382.

Goo Ishikawa, Yumiko Kitagawa, Wataru Yukuno, *Duality of singular paths for* $(2,3,5)$*-distributions,* arXiv:1308.2501 [math.DG] (2013), J. of Dynamical and Control Systems, **21** (2015), 155–171.

Goo Ishikawa, Yumiko Kitagawa, Asahi Tsuchida, Wataru Yukuno, *Duality of* $(2,3,5)$*-distributions and Lagrangian cone structures,* in preparation.

【 Cartan prolongation 】

Let D be a $(2,3,5)$-distribution on a (5-dim.) manifold Y. Let $Z := PD = (D-0)/\mathbf{R}^\times$ be the space of tangential lines in D, $Z := \{(y,\ell) \mid y \in Y,\ \ell \subset D_y(\subset T_yY),\ \dim(\ell) = 1\}$. Then $\dim(Z) = 6$ and the projection $\pi_Y : Z \to Y$ is an $\mathbf{R}P^1$-bundle.

We define a subbundle $E \subset TZ$ of rank 2 (Cartan prolongation of $D \subset TY$) by setting for each $(y,\ell) \in Z$, $\ell \subset D_y$,

$$E_{(y,\ell)} := \pi_{Y*}^{-1}(\ell)\ (\subset T_{(y,\ell)}Z).$$

Then E is a distribution with (weak) growth $(2,3,4,5,6)$:

$$\mathrm{rank}(E) = 2,\ \mathrm{rank}(\partial E) = 3,\ \mathrm{rank}(\partial^{(2)}E) = 4,$$
$$\mathrm{rank}(\partial^{(3)}E) = 5,\ \mathrm{rank}(\partial^{(4)}E) = 6.$$

【 Pseudo product structure 】

Then we see that there exists an intrinsic decomposition

$$E = K \oplus L$$

of E with $L := \mathrm{Ker}(\pi_{Y*}) \subset E$ and a complementary line subbundle K of E (a pseudo-product structure in the sense of N. Tanaka).

We will explain this in terms of "geometric control theory".

【 Control systems 】

A control system $\mathbb{C} : \mathcal{U} \xrightarrow{F} TM \to M$ on a manifold M is given by a locally trivial fibration $\pi_{\mathcal{U}} : \mathcal{U} \to M$ over M and a map $F : \mathcal{U} \to TM$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{F} & TM \\ \pi_{\mathcal{U}} \searrow & & \swarrow \pi_{TM} \\ & M & \end{array}$$

Locally on M, a control system is given by a family of vector fields $f_u(x) = F(x, u)$ over M, $(x, u) \in \mathcal{U}, x \in M$.

Example. A distribution $D \subset TM$ (vector subbundle) is regarded as a control system $\mathbb{D} : D \hookrightarrow TM \longrightarrow M$, by the inclusion.

【 Equivalences of control systems 】

Two control systems $\mathbb{C} : \mathcal{U} \xrightarrow{F} TM \xrightarrow{\pi_{TM}} M$ and $\mathbb{C}' : \mathcal{U}' \xrightarrow{F'} TM' \xrightarrow{\pi_{TM'}} M'$ are called equivalent if the diagram

$$\begin{array}{ccccc} \mathcal{U} & \xrightarrow{F} & TM & \xrightarrow{\pi_{TM}} & M \\ \psi \downarrow & & \varphi_* \downarrow & & \downarrow \varphi \\ \mathcal{U}' & \xrightarrow{F'} & TM' & \xrightarrow{\pi_{TM'}} & M' \end{array}$$

commutes for some diffeomorphisms ψ and φ.

The pair (ψ, φ) of diffeomorphisms is called an equivalence of the control systems $\mathbb{C}$ and $\mathbb{C}'$.

【 Controls, trajectories and paths 】

Given a control system $\mathbb{C} : \mathcal{U} \xrightarrow{F} TM \to M$,
an L^∞ (measurable, essentially bounded) map $c : [a, b] \to \mathcal{U}$
is called an admissible control if the curve

$$\gamma := \pi_{\mathcal{U}} \circ c : [a, b] \to M$$

satisfies the differential equation

$$\dot{\gamma}(t) = F(c(t)) \quad \text{(a.e. } t \in [a, b]).$$

Then the Lipschitz curve γ is called a trajectory.
If we write $c(t) = (x(t), u(t))$, then $x(t) = \gamma(t)$ and

$$\dot{x}(t) = F(x(t), u(t)), \quad \text{(a.e. } t \in [a, b]).$$

We use the term "path" for a smooth (C^∞) immersive trajectory regarded up to parametrisation.

【 Endpoint mappings and singular controls 】

The totality $\mathcal{C}$ of admissible controls $c : [a, b] \to \mathcal{U}$ with a given initial point $q_0 \in M$ is a Banach manifold.
The endpoint mapping $\mathrm{End} : \mathcal{C} \to M$ is defined by

$$\mathrm{End}(c) := \pi_{\mathcal{U}} \circ c(b).$$

An admissible control $c : [a, b] \to \mathcal{U}$ with the initial point $\pi_{\mathcal{U}}(c(a)) = q_0$ is called singular or abnormal, if $c \in \mathcal{C}$ is a singular point of End, namely if the differential $\mathrm{End}_* : T_c\mathcal{C} \to T_{\mathcal{E}(c)}M$ is not surjective. If c is a singular control, then the trajectory $\gamma = \pi_{\mathcal{U}} \circ c$ is called a singular trajectory or an abnormal extremal.

【 Local characterisation of singular controls 】

We define the Hamiltonian function $H : \mathcal{U} \times_M T^*M \to \mathbf{R}$ of the control system $F : \mathcal{U} \to TM$ by

$$H(x, p, u) := \langle p, F(x, u)\rangle, \quad ((x, u), (x, p)) \in \mathcal{U} \times_M T^*M.$$

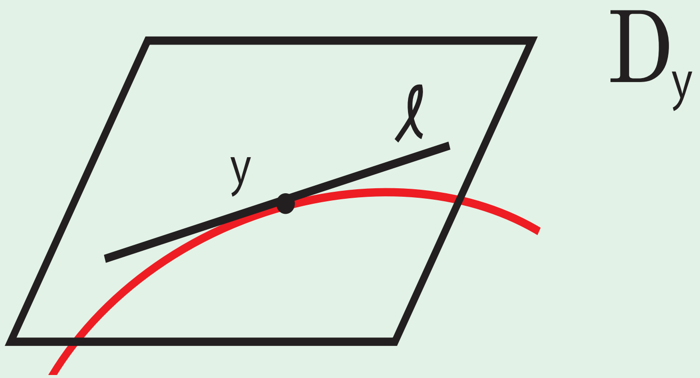
A singular control $(x(t), u(t))$ is characterised by the liftability to an abnormal bi-extremal $(x(t), p(t), u(t))$ satisfying the constrained Hamiltonian equation

$$\begin{cases} \dot{x}_i(t) = \dfrac{\partial H}{\partial p_i}(x(t), p(t), u(t)), \quad (1 \leq i \leq m) \\ \dot{p}_i(t) = -\dfrac{\partial H}{\partial x_i}(x(t), p(t), u(t)), \quad (1 \leq i \leq m) \\ \dfrac{\partial H}{\partial u_j}(x(t), p(t), u(t)) = 0, \quad (1 \leq j \leq r), \qquad p(t) \neq 0. \end{cases}$$

【The space X of singular paths】

Let $D \subset TY$ be a $(2, 3, 5)$-distribution.

Then, it is known that for any point y of Y and for any direction $\ell \subset D_y$, there exists uniquely a singular D-path (an immersed abnormal extremal for D) through y with the given direction ℓ.



Thus the singular D-paths form another five dimensional manifold X.

【The double fibrations】

Let $Z = PD = (D-0)/\mathbf{R}^\times$ be the space of tangential lines in D, $\dim(Z) = 6$. Then Z is naturally foliated by the liftings of singular D-paths, and we have locally double fibrations:

$$Y \xleftarrow{\pi_Y} Z \xrightarrow{\pi_X} X.$$

【Cartan prolongation】

Let $E \subset TZ$ be the Cartan prolongation of $D \subset TY$:

For each $(y, \ell) \in Z$, $\ell \subset T_yY$, $E_{(y,\ell)} := \pi_{Y*}^{-1}(\ell)$.

Then E is a distribution with growth $(2, 3, 4, 5, 6)$.

If we put $L = \mathrm{Ker}(\pi_{Y*}), K = \mathrm{Ker}(\pi_{X*})$, then we have a decomposition $E = K \oplus L$ by integrable sub-bundles.

【 Pseudo-product structures of G_2-type 】

Theorem. There exists a natural bijective correspondence:

$\{(2,3,5)\text{-distributions}\}/\cong \;\longleftrightarrow$

$$\left\{ \begin{array}{c} \text{pseudo-product structures of } G_2\text{-type (Z, E):} \\ (2,3,4,5,6)\text{-distributions } E \text{ with a decomposition} \\ E = K \oplus L, \;\; \mathrm{rank}(K) = \mathrm{rank}(L) = 1, \\ [\mathcal{K}, \mathcal{L}] = \partial\mathcal{E} \;\; (:= [\mathcal{E}, \mathcal{E}] = \mathcal{E} + [\mathcal{E}, \mathcal{E}]), \\ [\, \mathcal{K}, \partial\mathcal{E} \,] = \partial^{(2)}\mathcal{E}, \; [\, \mathcal{L}, \partial\mathcal{E} \,] = \partial\mathcal{E}, \\ [\, \mathcal{K}, \partial^{(2)}\mathcal{E} \,] = \partial^{(3)}\mathcal{E}, \; [\, \mathcal{L}, \partial^{(2)}\mathcal{E} \,] = \partial^{(2)}\mathcal{E}, \\ [\, \mathcal{K}, \partial^{(3)}\mathcal{E} \,] = \partial^{(3)}\mathcal{E}, \; [\, \mathcal{L}, \partial^{(3)}\mathcal{E} \,] = \partial^{(4)}\mathcal{E}. \end{array} \right\} / \cong$$

$$\begin{array}{ccccccccc} \mathcal{E} & \subset & \partial\mathcal{E} & \subset & \partial^{(2)}\mathcal{E} & \subset & \partial^{(3)}\mathcal{E} & \subset & \partial^{(4)}\mathcal{E} \\ 2 & & 3 & & 4 & & 5 & & 6 \end{array}$$

【 The symbol algebra 】

Taking the gradation of the filtration on TZ, we have the symbol algebra:

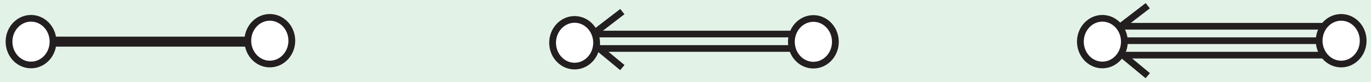
$$\begin{aligned} \mathfrak{m} &= \mathfrak{g}_{-5} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \\ &= \langle e_6 \rangle \oplus \langle e_5 \rangle \oplus \langle e_4 \rangle \oplus \langle e_3 \rangle \oplus \langle e_1, e_2 \rangle, \end{aligned}$$

$$\begin{aligned} &[e_1, e_2] = e_3, \\ &[e_1, e_3] = e_4, \ [e_2, e_3] = 0, \\ &[e_1, e_4] = e_5, \ [e_2, e_4] = 0, \\ &[e_1, e_5] = 0, \ [e_2, e_5] = e_6. \end{aligned}$$

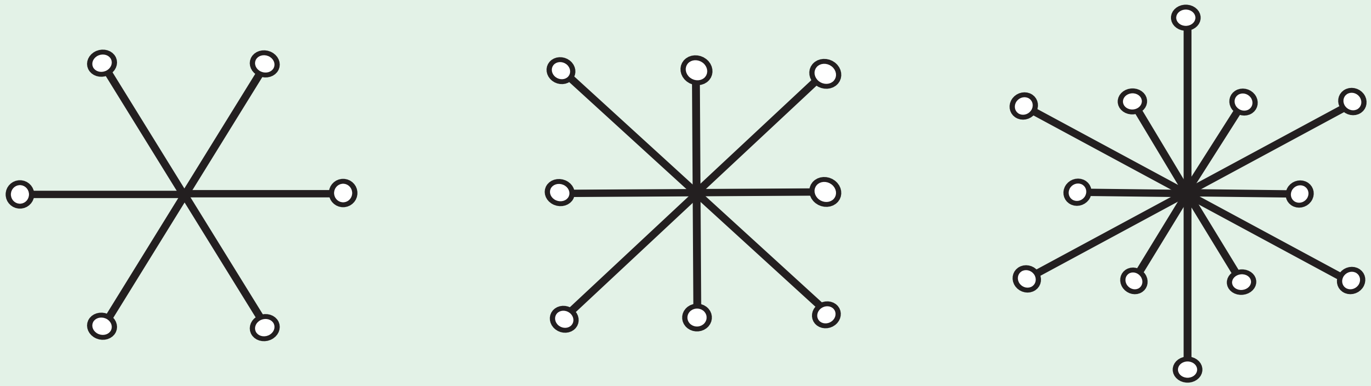
$$\mathfrak{k} = \langle e_1 \rangle = \mathrm{Ker}\{\mathfrak{g}_{-1} \to \mathrm{Hom}(\mathfrak{g}_{-4}, \mathfrak{g}_{-5})\},$$

$$\mathfrak{l} = \langle e_2 \rangle = \mathrm{Ker}\{\mathfrak{g}_{-1} \to \mathrm{Hom}(\mathfrak{g}_{-2}, \mathfrak{g}_{-3})\}.$$

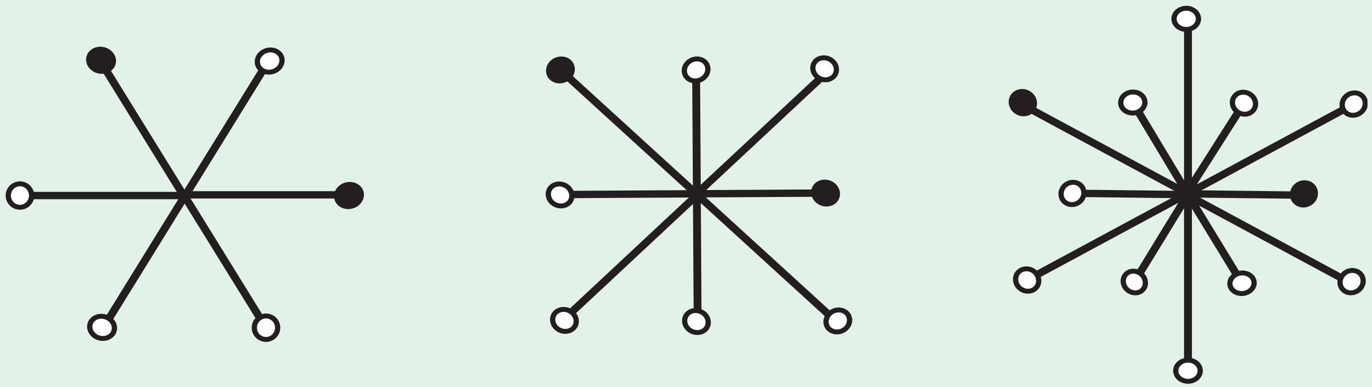
【 Simple Lie algebras of rank 2 】



A_2 B_2 G_2



Root systems of types A_2, B_2 and G_2



Fundamental roots for A_2, B_2 and G_2

【 A_2 geometry 】

$$Y^2 = P^2 \xleftarrow{\pi_Y} Z^3 = PT^*(P^2) = PT^*(P^{2*}) \xrightarrow{\pi_X} X^2 = P^{2*},$$

$E := \mathrm{Ker}(\pi_{Y*}) \oplus \mathrm{Ker}(\pi_{X*}) \subset TZ$: contact structure.

$$\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \langle e_3 \rangle \oplus \langle e_1, e_2 \rangle, \quad [e_1, e_2] = e_3.$$

There is no canonical decomposition of $\mathfrak{g}_{-1}$ (from geometric control theory nor from the theory of graded Lie algebras).

【 B_2 geometry 】

$$(Z^4,\ E = K \oplus L)$$

pseudo-product Engel structure

3rd order ODE $\swarrow$ $\searrow$

$$(Y^3, D)$$

projective contact structure

$$(X^3, C)$$

non-degenerate (strictly convex) cone structure

Classification of geometric structures, contact geometry of 3rd order ODE, by E. Cartan, S.-S. Chern(1940), N. Tanaka, ...

"Wünschmann invariant" $= 0 \iff C$: metric cone.

【 Engel Lie algebra 】

$$\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \langle e_4 \rangle \oplus \langle e_3 \rangle \oplus \langle e_1, e_2 \rangle,$$

$$[e_1, e_2] = e_3, \ \ [e_1, e_3] = e_4, \ \ [e_2, e_3] = 0.$$

$$\mathfrak{l} = \langle e_2 \rangle = \mathrm{Ker}\{\mathfrak{g}_{-1} \to \mathrm{Hom}(\mathfrak{g}_{-2}, \mathfrak{g}_{-3})\},$$

There exists just the canonical line sub-bundle $L \subset E(\subset TZ)$ (from geometric control theory or from the theory of graded Lie algebras).

【 G_2-geometry — The cone fields 】

The original $(2,3,5)$-distribution D is obtained as the linear hull of the cone field (“bowtie”) induced from K:

$$D_y = \text{linear hull}\left(\bigcup_{z \in \pi_Y^{-1}(y)} \pi_{Y*}(K_z) \subset T_y Y \right).$$

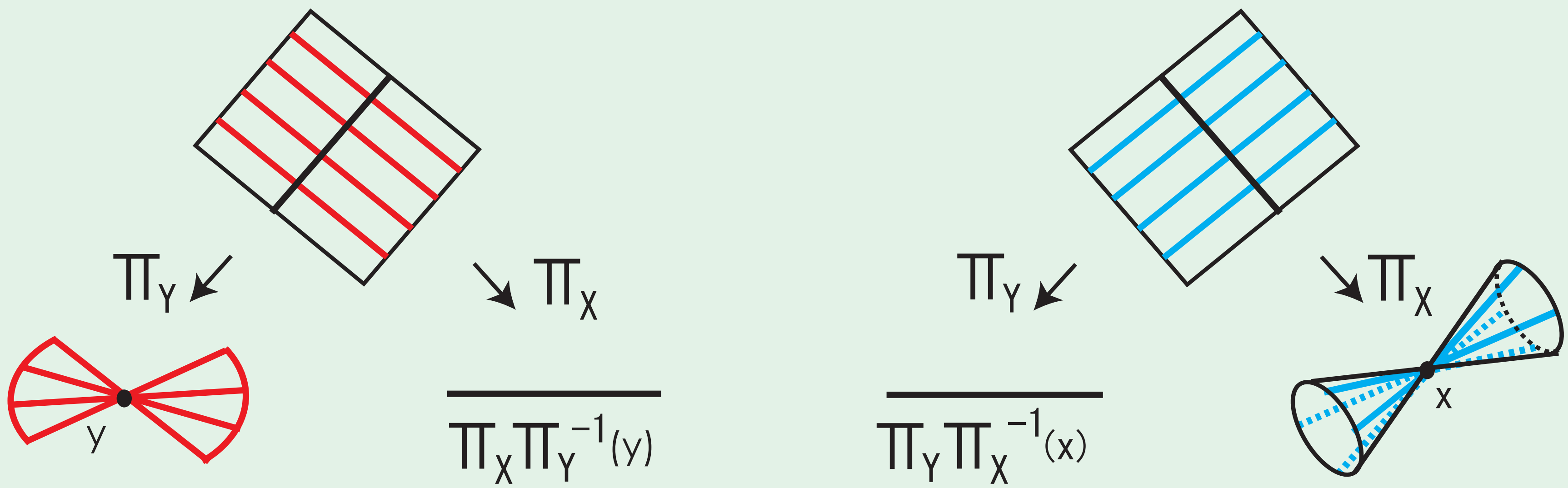
Also, the $(2,3,5)$-distribution D is obtained as the reduction of ∂E by Cauchy characteristic $L = \mathrm{Ker}(\pi_{Y*})$.

Moreover we have the cone field $C \subset TX$ on X by setting, for each $x \in X$,

$$C_x := \bigcup_{z \in \pi_X^{-1}(x)} \pi_{X*}(L_z) \subset T_x X.$$

【 The cone fields (continued) 】

Then the linear full $D' \subset TX$ of the cone field C turns to be a contact structure on X induced from $\partial^{(3)}E$ via π_X. Thus (X, C) is a "Lagrangian cone structure".



We have sequences of cones on Y, Z, X respectively:

$(\underline{2}, 2, 3, \underline{5}, 5)$ on Y $\longleftarrow$ $(2, 3, 4, 5, 6)$ on Z $\longrightarrow$ $(\underline{2}, \underline{3}, \underline{4}, 4, 5)$ on X.

So far, we have distributions D, E, L, K and the cone filed C from the double fibration $Y \xleftarrow{\pi_Y} Z \xrightarrow{\pi_X} X$:

$$\begin{array}{ccccc}
TY & \xleftarrow{\pi_{Y*}} & TZ & \xrightarrow{\pi_{X*}} & TX \\
\cup & & \cup & & \cup \\
D & \xleftarrow{\pi_{Y*}|_E} & E = K \oplus L & \xrightarrow{\pi_{X*}|_E} & C \\
\downarrow & & \downarrow & & \downarrow \\
Y & \xleftarrow{\pi_Y} & Z & \xrightarrow{\pi_X} & X
\end{array}$$

Now, we regard the cone field as a control system over X:

$$\mathbb{C} : L \xrightarrow{\pi_{X*}|_L} TX \to X.$$

Then we have the following main theorem:

【 G_2-Duality 】

Theorem (Duality [IKY]).
Singular paths of the control system

$$\mathbb{C} : L \xrightarrow{\pi_{X*}|_L} TX \to X$$

are given by π_X-images of π_Y-fibres.
Therefore, for any $x \in X$ and for any direction $\ell \subset C_x$, there exists uniquely a singular $\mathbb{C}$-paths passing through x with the direction ℓ at x.
The original space Y is identified with the space of singular paths for (X, C), while X is the space of singular paths for (Y, D).

【 A remaining problem on G_2-duality 】

The description of the duality on $(2, 3, 5)$-distributions (Y, D) and Lagrangian cone structures (X, C) via (Z, E) should be completed by answering the question:

> What kinds of Lagrangian cone structures do they correspond to $(2, 3, 5)$-distributions ?

【 Cone structures 】

Let $C \subset TX$ be a two dimensional cone field ($= \mathbf{R}^\times$-invariant, locally trivial subset of TX) on a 5-dimensional manifold X. Suppose that $C_x \subset T_xX$ has singularity just at 0, for any $x \in X$. Then $Z = PC := (C \setminus \{0\})/\mathbf{R}^\times$ is a 6-dimensional manifold and $\pi_X : Z \to X$ is a C^∞-fibration with non-singular projective curves $PC_x \subset P(T_xX) \cong P^4$ as fibres.

As a non-degeneracy condition, we assume that the first, second and third derivatives are linearly independent everywhere on PC_x, for any $x \in X$.

【 The cone structure as a control system 】

Let $L \subset (\pi_X)^{-1}(TX)$ be the "tautological line bundle" over $Z = PC$: $L := \{((x, \ell), v) \in Z \times TX \mid v \in \ell \subset T_xX\}$, which is regarded also as a fibration over X by $\pi_L : L \to X$, $((x, \ell), v) \mapsto x$, with 2-dimensional fibres.

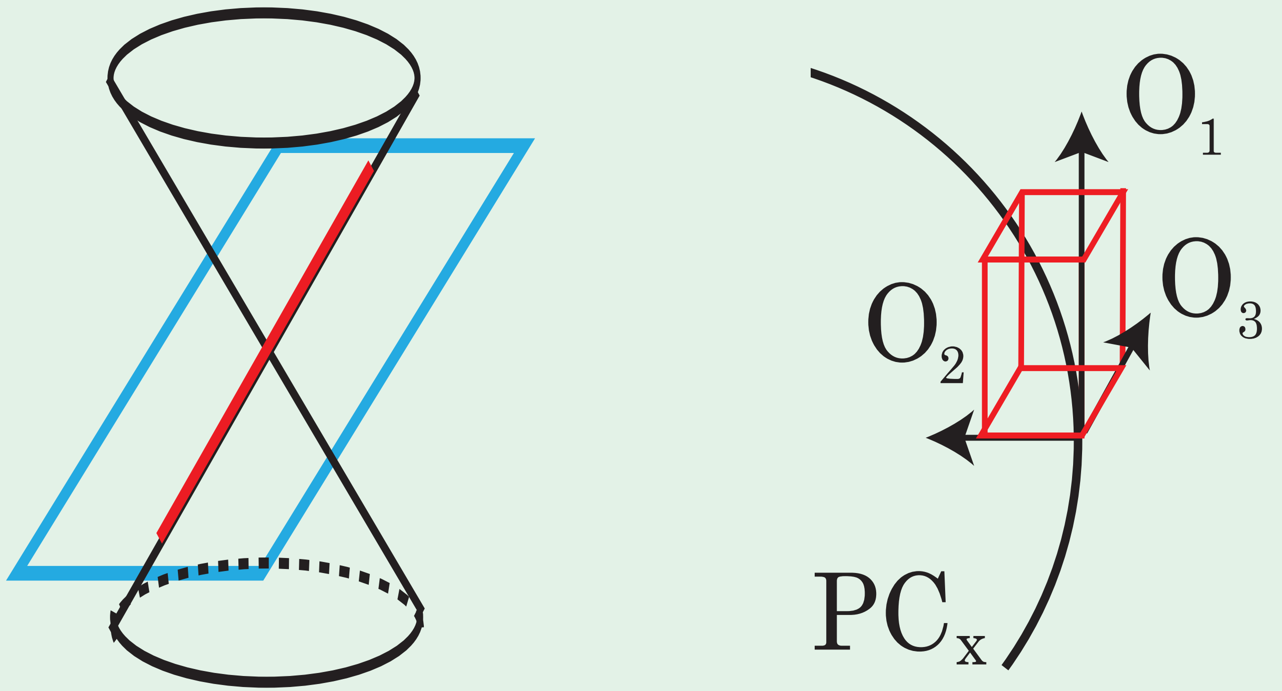
Then we regard the cone structure C in X as the control system over X:

$$\mathbb{C} : L \to TX \to X, \quad L \ni ((x, \ell), v) \mapsto (x, v) \mapsto x,$$

with 2-control parameters.

Each section $s : X \to L \setminus \{0\}$ of the fibration $\pi_L : L \to X$ defines a direction field in C over X and the control-linear approximation T_sC of C along s, which is a subbundle of TX of rank 2.

Moreover, for each section $s : X \to L \setminus \{0\}$, we define osculating bundles $O_s^{(2)}C \subset TX$ of rank 3 and $O_s^{(3)}C \subset TX$ of rank 4, generated by osculating planes O_2 and 3-dimensional osculating spaces O_3 to PC_x with direction s:



【 non-degenerate Lagrangian cone structure 】

Definition.

A cone field $C \subset TX$ is called a non-degenerate Lagrangian cone structure on a 5-dimensional manifold X if

(i) $D' := O_s^{(3)} C \subset TX$ is independent of choice of direction field $s : X \to L \setminus \{0\}$ and is a contact structure on X,

(ii) $T_s C$ is a Lagrangian sub-bundle of D', i.e. the derived system $\partial(T_s C) \subset D'$, for any direction field s.

【 The complete description of the duality 】

Theorem. There exist natural bijective correspondences:

$$\{(2,3,5)\text{-distributions } (Y, D)\}/\cong \;\longleftrightarrow$$

$$\left\{\begin{array}{c}\text{pseudo-product structures of } G_2\text{-type (Z, E):}\\ (2,3,4,5,6)\text{-distributions } E \text{ with a decomposition}\\ E = K \oplus L,\ \operatorname{rank}(K) = \operatorname{rank}(L) = 1,\\ [\mathcal{K}, \mathcal{L}] = \partial\mathcal{E}\ (:= [\mathcal{E}, \mathcal{E}] = \mathcal{E} + [\mathcal{E}, \mathcal{E}]),\\ [\,\mathcal{K}, \partial\mathcal{E}\,] = \partial^{(2)}\mathcal{E},\ [\,\mathcal{L}, \partial\mathcal{E}\,] = \partial\mathcal{E},\\ [\,\mathcal{K}, \partial^{(2)}\mathcal{E}\,] = \partial^{(3)}\mathcal{E},\ [\,\mathcal{L}, \partial^{(2)}\mathcal{E}\,] = \partial^{(2)}\mathcal{E},\\ [\,\mathcal{K}, \partial^{(3)}\mathcal{E}\,] = \partial^{(3)}\mathcal{E},\ [\,\mathcal{L}, \partial^{(3)}\mathcal{E}\,] = \partial^{(4)}\mathcal{E}.\end{array}\right\}/\cong$$

$$\longleftrightarrow \left\{\begin{array}{c}\text{non-degenerate Lagrangian cone structures } (X, C)\\ \text{on 5-dimensional manifolds } X \text{ with the condition}\\ \partial(T_sC) \subset O_s^{(2)}C, \text{ for any section } s : X \to L \setminus \{0\}.\end{array}\right\}/\cong$$

【 The complete description of the duality 】

pseudo-product structures of type G_2

$\nearrow\swarrow$ $\searrow\nwarrow$

(2, 3, 5)-distributions

non-degenerate Lagrangian cone structures satisfying $\partial(T_s C) \subset O_s^{(2)} C$, for any s

【 From Lagrangian cone to pseudo-product of G_2-type 】

Suppose (X, C) is a non-degenerate Lagrangian cone structure. Then we define a subbundle $E \subset TZ$ of rank 2 by setting

$$E_{(x,\ell)} := (\pi_X)_*^{-1}(\ell).$$

Then we have that E has weak growth $(2, 3, 4, 5, 6)$.

Set $K = \mathrm{Ker}(\pi_X)_* \subset E$.

Moreover the tautological line-bundle L is embedded in E as the Cauchy characteristic of the derived system ∂E, and we have the decomposition $E = K \oplus L$. Moreover it is a pseudo-product structure of G_2-type if and only if the condition $\partial(T_s C) \subset O_s^{(2)} C$ is fulfilled, for any $s : X \to L \setminus \{0\}$.

This completes the explanation on duality.

【 Example of G_2-flat case [MIT] 】

$G = \mathrm{Aut}(\mathbf{O}')$, the split G_2. $\mathbf{O}'$ the split octonions.

Let $\mathbf{H} = \{a = x + yi + zj + wk \mid x, y, z, w \in \mathbf{R}\}$ be Hamilton's quarternion algebra and define the split octonions by $\mathbf{O}' = \mathbf{H} \oplus \mathbf{H}$ with the multiplication as

$$(a, b)(c, d) := (ac + \overline{d}b, da + b\overline{c}).$$

Note that $\mathbf{O}'$ is a non-associative algebra.

We set $V := \mathrm{Im}(\mathbf{O}')$, the imaginary part, $\dim(V) = 7$.
Then $G = \mathrm{Aut}(\mathbf{O}')$ acts on V irreducibly.

Consider the split G_2 flag manifold

$$Z := \{(V_1, V_2) \mid V_1 \subset V_2 \subset V, \ V_1, V_2 \text{ are oriented null subalgebras}\},$$

where an $\mathbf{R}$-subspace $W \subset V$ is called a null subalgebra if $ww' = 0$, for any $w, w' \in W$.

Set

$Y := \{V_1 \mid V_1 \subset V, \ \text{1-dimensional oriented null subalgebra}\}$,
$X := \{V_2 \mid V_2 \subset V, \ \text{2-dimensional oriented null subalgebra}\}$.
Then $Z \cong S^3 \times S^3, Y \cong S^3 \times S^2, X \cong S^3 \times S^2$.

Consider the double fibrations: $Y \xleftarrow{\Pi_Y} Z \xrightarrow{\Pi_X} X$,

Set

$$E := \mathrm{Ker}(\Pi_{Y*}) \oplus \mathrm{Ker}(\Pi_{X*}) \subset TZ,$$

which is called the (split G_2) Engel distribution,
which is of rank 2 and with weak growth $(2, 3, 4, 5, 6)$.

Just from the split G_2 double fibrations, we can obtain:

On the projective space Y of null vectors, a $(2, 4, 5)$-distribution $D \subset TY$, called a Cartan structure.

On the Grassmannian X of null subalgebras, the Lagrange cubic non-degenerate Lagrangian cone field $C \subset D' \subset TX$ contained in a contact distribution D'. Such a structure is called a Monge structure.

The double fibration $Y \xleftarrow{\Pi_Y} Z \xrightarrow{\Pi_X} X$ is described via some local coordinates λ, x, y, z, u, v of the split G_2 flag manifold Z explicitly by

$\Pi_Y(\lambda, x, y, z, u, v) = \left(\lambda,\ y + \lambda z,\ x + \lambda y,\ v + \lambda x,\ u + \lambda(y^2 - xz)\right)$,

and $\Pi_X(\lambda, x, y, z, u, v) = (x, y, z, u, v)$.

The (2, 3, 4, 5, 6)-G_2-Engel structure E on Z is given by

$$\alpha_1 := dy + \lambda dz = 0, \quad \alpha_2 := dx - \lambda^2 dz = 0,$$

$$\alpha_3 := dv + \lambda^3 dz = 0, \quad \alpha_4 := du - (\lambda^3 z + 2\lambda^2 y + \lambda x)dz = 0.$$

A local frame (ξ_1, ξ_2) of E is given by

$$\xi_1 = \frac{\partial}{\partial \lambda}, \quad \xi_2 = \frac{\partial}{\partial z} - \lambda \frac{\partial}{\partial y} + \lambda^2 \frac{\partial}{\partial x} - \lambda^3 \frac{\partial}{\partial v} + (\lambda^3 z + 2\lambda^2 y + \lambda x)\frac{\partial}{\partial u}.$$

The (2, 3, 5) Cartan structure $D \subset TY$ is given,
in terms of local coordinates $(\lambda, \nu, \mu, \tau, \sigma)$, by

$$\begin{aligned}
\beta_1 &:= -\nu d\lambda + \lambda d\nu + d\mu = 0, \\
\beta_2 &:= (\lambda\nu - \mu)d\lambda - \lambda^2 d\nu + d\tau = 0, \\
\beta_3 &:= -\nu^2 d\lambda + (\lambda\nu + \mu)d\nu + d\sigma = 0.
\end{aligned}$$

The local frame of D is given by

$$\begin{aligned}\eta_1 &= \frac{\partial}{\partial\lambda} + \nu\frac{\partial}{\partial\mu} - (\lambda\nu - \mu)\frac{\partial}{\partial\tau} + \nu^2\frac{\partial}{\partial\sigma},\\ \eta_2 &= \frac{\partial}{\partial\nu} - \lambda\frac{\partial}{\partial\mu} + \lambda^2\frac{\partial}{\partial\tau} - (\lambda\nu + \mu)\frac{\partial}{\partial\sigma}.\end{aligned}$$

The Monge structure $C \subset D' \subset TX$ on the Grassmannian X is given in terms of local coordinates (x, y, z, u, v) of X

$$\begin{aligned}C \quad &: \quad dxdy - dzdv = 0,\ dxdz - (dy)^2 = 0,\ (dx)^2 - dydv = 0,\\ &\quad\ du - 2ydx + xdy + zdv = 0,\\ D' \quad &: \quad du - 2ydx + xdy + zdv = 0.\end{aligned}$$

【 The Lagrangian cone structure in G_2-flat case 】

Consider the cone structure C on $(\mathbf{R}^5, 0)$,

$$F(x; r, \theta) = r\left(\frac{\partial}{\partial x_1} + \theta\frac{\partial}{\partial x_2} + \theta^2\frac{\partial}{\partial x_3} + \theta^3\frac{\partial}{\partial x_4} + (x_3\theta - 2x_2\theta^2 + x_1\theta^3)\frac{\partial}{\partial x_5}\right),$$

with control parameter r, θ. Then C is a non-degenerate Lagrangian cone structure for the contact structure D' : $dx_5 - x_3dx_2 + 2x_2dx_3 - x_1dx_4 = 0$. Moreover C satisfies the condition $\partial(T_sC) \subset O_s^{(2)}C$ for any $s : X \to L \setminus \{0\}$ and it corresponds to the G_2-flat $(2, 3, 5)$-distribution.

【 Cartan geometries (parabolic geometries) 】

pseudo-product structures of type G_2

$\swarrow \nearrow$

$(2, 3, 5)$-distributions

G_2-contact structures

A G_2-contact structure on a 5-dimensional manifold X is a contact structure $D' \subset TX$ with a "cubic non-degenerate Lagrangian cone structure" $C \subset TX$ parametrised by a vector bundle F of rank 2 over X such that the Levi bracket $\mathfrak{L} : D' \times D' \to TX/D'$ is $\mathfrak{gl}(F)$-invariant.

Theorem. Any $(2,3,5)$-distribution (Y, D) which corresponds to a cubic cone structure (X, C) must be flat.

This fact is suggested by Professor Hajime Sato from the calculations of curvatures of psuedo-product G_2-structure and of G_2-contact structures. Here we provide alternative proof.

Proof. For each $x \in X$, the cone $C_x \subset D'_x(\subset T_xX)$ gives the (reduced) "Jacobi curve" in the sense of Agrachev and Zelenko. Then it is known that "Cartan tensor" of D is recovered by a projective invariant, the fundamental invariant, a kind of cross ratio, of $P(C_x)$ point-wise. All non-degenerate cubic cones are projectively equivalent. If D corresponds to a cubic cone structure, then Cartan tensor D vanishes, and therefore D is flat. $\square$

【 Fundamental invariants 】

For the cone $C_x \subset D_x \cong \mathbf{R}^4$, there is associated a curve $\Lambda(\theta)$ in Grassmannian $\mathrm{Gr}(2, \mathbf{R}^4)$.

For $\Delta, \Gamma, \Lambda \in \mathrm{Gr}(2, \mathbf{R}^4)$ such that Δ and Γ are transverse to Λ, we consider 2×2 matrix $\langle \Delta, \Gamma, \Lambda \rangle$ such that

$$\Gamma = \{v + \langle \Delta, \Gamma, \Lambda \rangle v \mid v \in \Delta\}.$$

Then we set

$$\mathrm{trace}\left(\tfrac{d}{ds} \langle \Lambda(\theta_1), \Lambda(s), \Lambda(\theta_0) \rangle|_{s=\theta_1} \circ \tfrac{d}{ds} \langle \Lambda(\theta_0), \Lambda(s), \Lambda(\theta_1) \rangle|_{s=\theta_0} \right) = -\tfrac{4}{(\theta_0 - \theta_1)^2} - g_\Lambda(\theta_0, \theta_1).$$

and define the fundamental form by

$$\mathcal{A}(\theta) := \frac{1}{2} \frac{\partial^2 g_\Lambda^2}{\partial \theta_1^2}(\theta_0, \theta_1)\Big|_{\theta_0 = \theta_1 = \theta} (d\theta)^4.$$

【 Examples of cubic Lagrangian cone structures not corresponding to (2, 3, 5)-distributions 】

Consider the cubic cone structure C on $(\mathbf{R}^5, 0)$, near $\theta = 0$,

$$F(x; r, \theta) = r\left(\frac{\partial}{\partial x_1} + \theta\frac{\partial}{\partial x_2} + (\theta^2 + a)\frac{\partial}{\partial x_3} + (\theta^3 - 3\theta a)\frac{\partial}{\partial x_4} + \{x_3\theta - 2x_2(\theta^2 + a) + x_1(\theta^3 - 3\theta a)\}\frac{\partial}{\partial x_5}\right),$$

defined by a C^∞ function $a(x_1)$ with $a(0) = 0$.

Then C is a non-degenerate Lagrangian cone structure for the contact structure $D' : dx_5 - x_3dx_2 + 2x_2dx_3 - x_1dx_4 = 0$. Moreover C satisfies the condition $\partial(T_sC) \subset O_s^{(2)}C$ for any $s : X \to L \setminus \{0\}$, to correspond to a $(2, 3, 5)$-distribution, if and only if $a \not\equiv 0$. (The case $a \equiv 0$ corresponds to the G_2-homogeneous case.)

【 Examples of non-cubic Lagrangian cone structures corresponding to $(2,3,5)$-distributions 】

Consider a cone field on $(\mathbf{R}^5, 0)$.

$$F(x; r, \theta) = r\left(\frac{\partial}{\partial x_1} + \theta\frac{\partial}{\partial x_2} + (\theta^2 + b)\frac{\partial}{\partial x_3} + (\theta^3 + c)\frac{\partial}{\partial x_4} + \{x_3\theta - 2x_2(\theta^2 + b) + x_1(\theta^3 + c)\}\frac{\partial}{\partial x_5}\right),$$

where $b = b(\theta), c = c(\theta)$, $\mathrm{ord}(b) \geq 3, \mathrm{ord}(c) \geq 4$ at $\theta = 0$.

Then F is a non-degenerate Lagrangian cone structure satisfying the condition $\partial(T_sC) \subset O_s^{(2)}C$ for any $s : X \to L\setminus\{0\}$ (and therefore it corresponds to a $(2,3,5)$-distribution), if and only if $c_\theta = 3\theta b_\theta - 3b$. (Then $D' = \{dx_5 - x_3dx_2 + 2x_2dx_3 - x_1dx_4 = 0\}$.)

If $b_{\theta\theta\theta\theta} \neq 0$, then F is not cubic.

Thank you for your attention.