不定値計量と接触構造の双対性と特異性

Duality-singularity of indefinite metrics and contact structures

Goo Ishikawa

Department of Mathematics, Hokkaido University, Japan

第２４回沼津研究会 — 幾何，数理物理，そして量子論 —

沼津工業高等専門学校

２０１７年3月８日

A talk based on joint-works with

Y. Machida  and M. Takahashi 

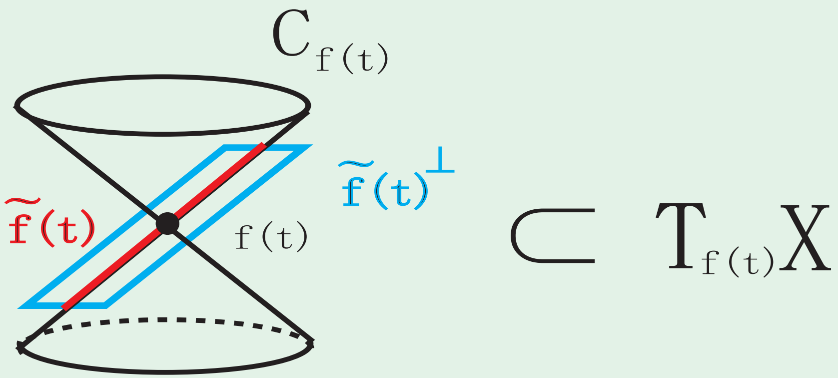
Let (X, g) be a C^∞ Lorentz 3-manifold (of signature $(1, 2)$).

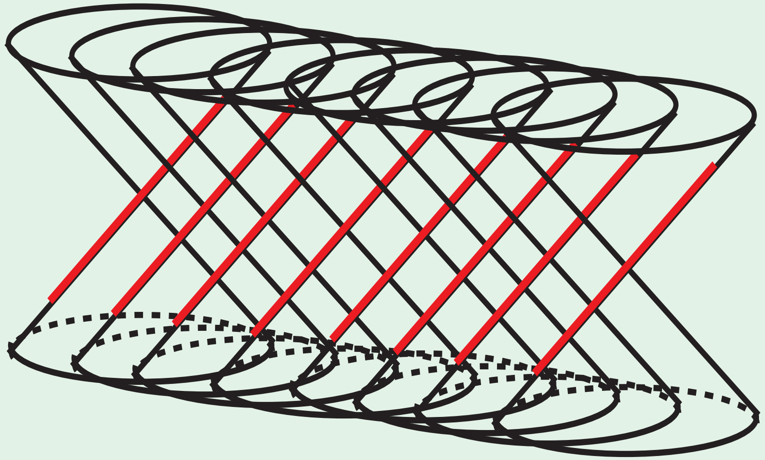
Example (Minkowskii space):
$X = \mathbf{R}^3, \quad g : ds^2 = dx_1^2 - dx_2^2 - dx_3^2.$

Definition. A C^∞ immersion $f : U(\subset \mathbf{R}^2) \to (X, g)$ is called null (or lightlike) if the pull-back metric f^*g is degenerate everywhere on U.

Let $C := \{v \in TX \mid g(v, v) = 0\}$ be the null quadratic cone field associated to the indefinite metric g.

It is easy to see that an immersion $f : U \to X$ is null if and only if $f_*(T_tU)$ is tangent to the cone $C_{f(t)}$ for any $t \in U$.
Then U is foliated by null curves.
(A curve is called null if its velocity vectors belong to $C \subset TX$.)





Let $Z := PC = \{(x, \ell) \mid x \in X, \ell$ is a null line in $T_x X\} \subset P(TX)$, the space of null directions, $\dim(Z) = 4$.
Denote by $\pi_X : Z \to X$ the natural projection, $\pi_X(x, \ell) = x$.

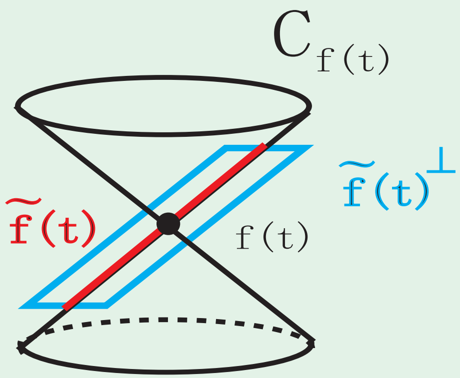
Lemma. Let $f : U \to X$ be a null immersion. Then there exists a unique C^∞ map $\widetilde{f} : U \to Z$ such that $\pi_X \widetilde{f} = f$ and that $f_*(T_t U) = \widetilde{f}(t)^\perp$, for any $t \in U$.

$$\widetilde{f}(t)^\perp := \{v \in T_{f(t)} X \ \mid \ g(v, u) = 0, \ \text{for any } u \in \widetilde{f}(t)\}.$$

$$\begin{array}{ccccc} & & Z & \hookrightarrow & PT^*X \\ & \widetilde{f} \nearrow & \downarrow \pi_X & \swarrow & \\ U & \xrightarrow{f} & X & & \end{array}$$

Definition. A (possibly non-immersive) C^∞ map-germ $f : (\mathbf{R}^2, p) \to X$ is called a null frontal surface or a lightlike frontal surface if there exists a C^∞ map-germ $\widetilde{f} : (\mathbf{R}^2, p) \to Z$ such that $\pi_X \widetilde{f} = f$ and that $f_*(T_t\mathbf{R}^2) \subset \widetilde{f}(t)^\perp$, for any $t \in \mathbf{R}^2$ nearby p.

$$\widetilde{f}(t)^\perp := \{v \in T_{f(t)}X \mid g(v, u) = 0, \text{ for any } u \in \widetilde{f}(t)\}.$$



$\subset T_{f(t)}X$

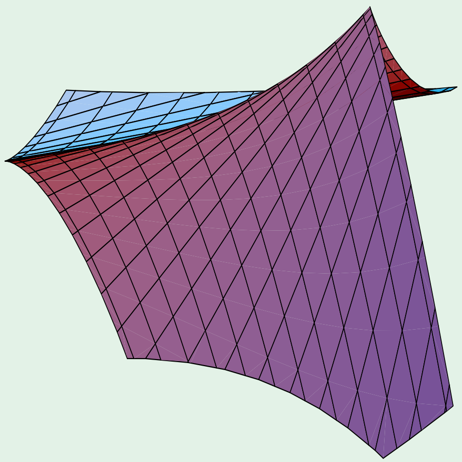
Problem. Classify "generic" singularities of null frontal surfaces up to local diffeomorphisms.

— Minkowskii case: Chino-Izumiya 2010 (lightlike developables).

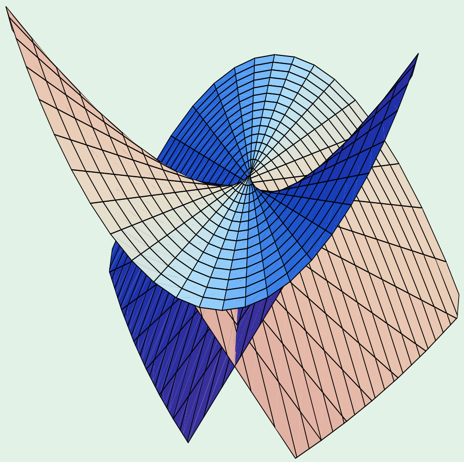
— $\mathrm{Sp}(\mathbf{R}^4)$-homogeneous case: Machida-Takahashi-I 2011 (tangent surfaces of directed null curves).

We show that any null frontal surface is a null tangent surface of a directed null curves (in a wider sense) and we give the generic classification of singularities of null frontal surfaces for general Lorentz 3-manifolds, moreover for arbitrary non-degenerate (strictly convex) cone fields on 3-manifolds.

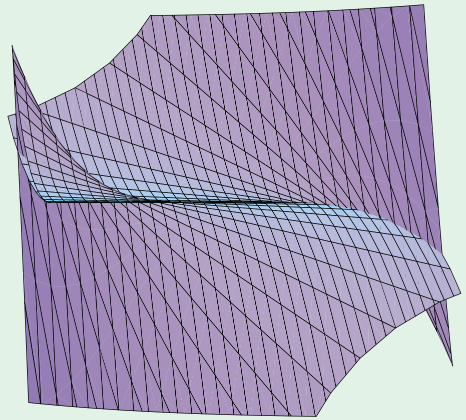
Solution. The list of generic singularities consists of three local diffeomorphism classes, cuspidal edge (CE), swallow-tail (SW) and Shcherbak surface (SB).



Cuspidal Edge



SWallowtail



ShcherBak surface

We observe the “stability” or, in other words, “robustness” of the classification of singularities.
Any partubation of the metric does not affect essentially on the classification of the singularities of null frontal surfaces.

Query. What are the geometric structures behind, which dominate the appearance of singularities of null frontals.

【 Pseudo-product Engel structure 】

Define a distribution

$$E = \bigcup_{(x,\ell)\in Z} (\pi_X)_*^{-1}(\ell) \subset TZ,$$

over Z, where $\pi_{X*} : TZ \to TX$ is the differential map of π_X.

Then E is of rank 2 with the growth $(2, 3, 4)$, i.e. $\mathcal{E}^2 := \mathcal{E} + [\mathcal{E}, \mathcal{E}]$ is generated by a subbunde E^2 of rank 3 and $\mathcal{E}^2 + [\mathcal{E}, \mathcal{E}^2] = \mathcal{TZ}$, in other words, E is an Engel structure on the 4-manifold Z.

Define other distributions

$$E_1 := \mathrm{Ker}(\pi_{X*}) \subset E \subset TZ, \;\; E_2 := \mathrm{ch}(E^2) \subset E \subset TZ,$$

Cauchy characteristic of E^2.

Then both E_1 and E_2 are of rank 1, and we have the decomposition

$$E = E_1 \oplus E_2,$$

a pseudo-product Engel structure (in the sense of Noboru Tanaka). Moreover we have

$$\mathcal{E} + [\mathcal{E}_1, \mathcal{E}] = \mathcal{E}^2, \quad \mathcal{E}^2 + [\mathcal{E}_1, \mathcal{E}^2] = \mathcal{TZ}, \quad [\mathcal{E}_2, \mathcal{E}^2] \subset \mathcal{E}^2.$$

Let Y denote the leaf space of E_2, which is regarded as the space of null geodesics.

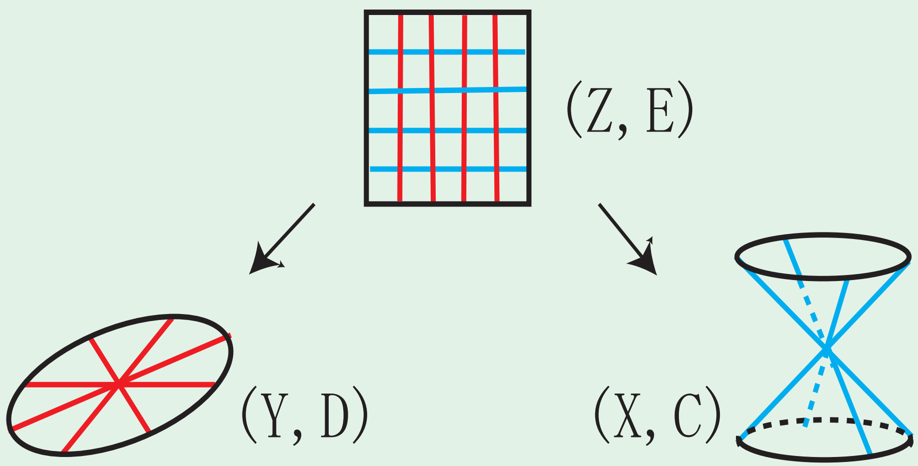
Denote by $\pi_Y : Z \to Y$ be the natural projection.

Locally we have the double fibration

$$Y^3 \xleftarrow{\pi_Y} Z^4 \xrightarrow{\pi_X} X^3.$$

The distribution E^2 on Z of rank 3 descends by π_Y to a contact structure D on Y:

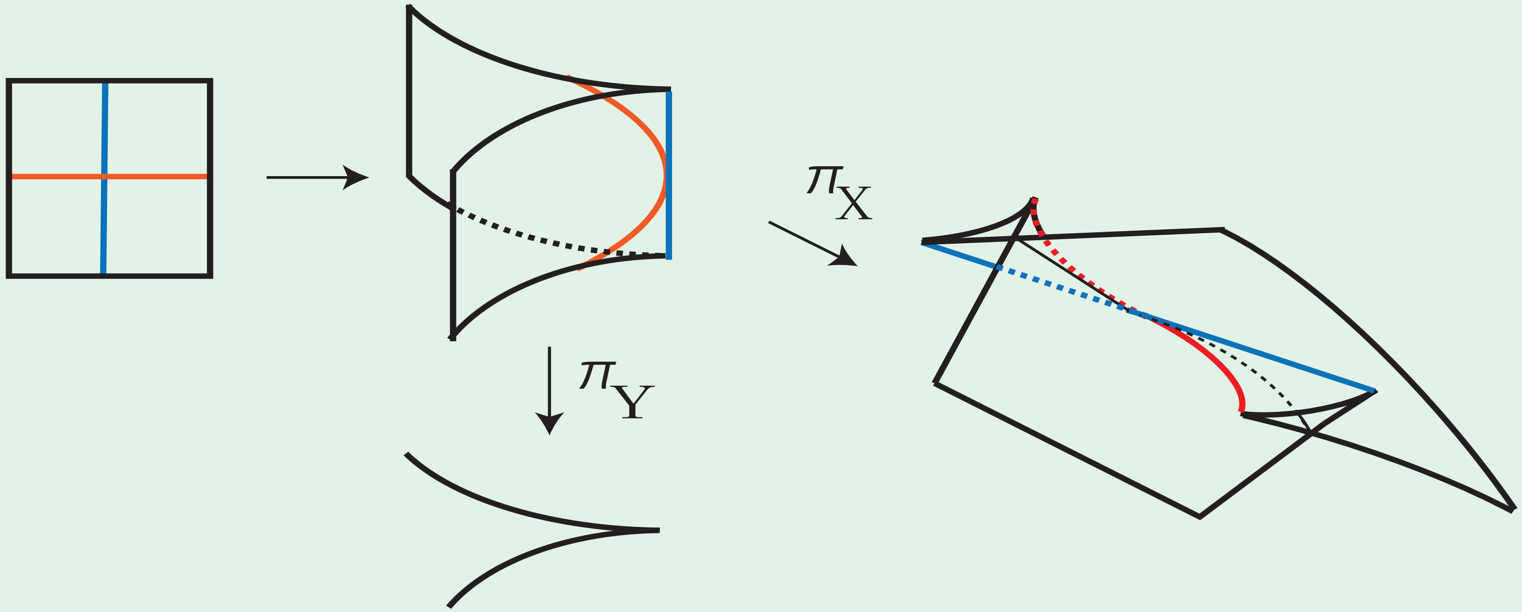
$$(\pi_{Y*})(E^2) = D, \quad E^2 = (\pi_{Y*})^{-1}(D).$$



【 Singularities of null frontal surfaces 】

Lemma. $f : (\mathbf{R}^2, p) \to X$ is a null frontal if and only if there exists an E^2-integral lift $\widetilde{f} : (\mathbf{R}^2, p) \to Z$ of f. $(\widetilde{f})_*(T_t\mathbf{R}^2) \subset (E^2)_{\widetilde{f}(t)}$, for any $t \in \mathbf{R}^2$ nearby p.
Then $\pi_Y \widetilde{f}$ is D-integral and, therefore, of rank ≤ 1.

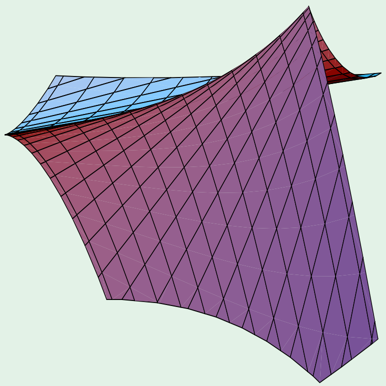
Thus $\widetilde{f}$ collapses by π_Y to a D-integral curve, in other words, to a "Legendre curve", and $\widetilde{f}$ is foliated by π_Y-fibres. Therefore f is ruled by a "Legendre family" of null geodesics.
The singular locus of $f = \pi_X \widetilde{f}$ consist of an E-integral curve γ and the π_Y-fibres of singular points of the Legendre curve.



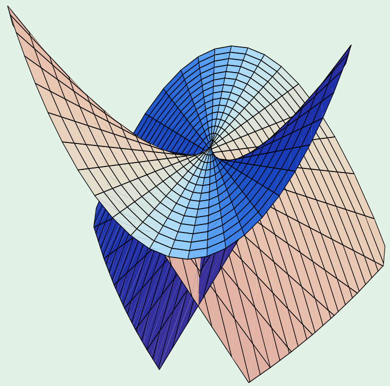
Thus the null frontal f is regarded as the tangent surface to a (directed) null curve $\pi_X \gamma$.

Avoiding very degenerate cases, we start with a germ of E-integral curve $\gamma : (\mathbf{R}, t_0) \to Z$ so that $\pi_Y^{-1}\pi_Y\gamma$ is right equivalent to $\widetilde{f}$, and f is right equivalent to $\pi_X\pi_Y^{-1}\pi_Y\gamma$.

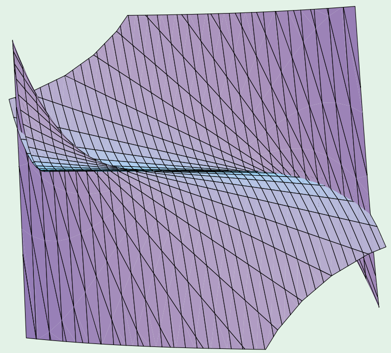
Theorem 1. (Machda-Takahashi-I)
For a generic E-integral curve $\gamma : I \to Z$, the induced null frontal $\pi_X \pi_Y^{-1} \pi_Y \gamma$ is diffeomorphic (right-left equivalent) along γ to cuspidal edge, swallowtail or Shcherbak surface.
The same classification result holds for arbitrary non-degenerate (strictly convex) cone structure $C \subset TX$.



Cuspidal Edge



SWallowtail



ShcherBak surface

$$(Z,\ E = E_1 \oplus E_2)$$

pseudo-product Engel structure

3rd order ODE ↙ ↘

$$(Y, D)$$

projective contact structure

$$(X, C)$$

non-degenerate (strictly convex) cone structure

Classification of geometric structures, contact geometry of 3rd order ODE, by E. Cartan, S.-S. Chern(1940), N. Tanaka, ...

"Wünschmann invariant" $= 0 \iff C$: metric cone.

Some details of the classification.

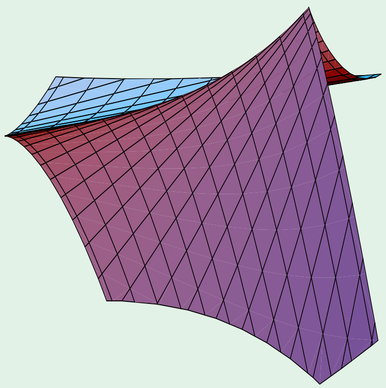
Let $C \subset TX$ be a non-degenerate cone structure of X.
Then $\exists$local.coord. x_1, x_2, x_3, θ of Z such that
$\pi_X : (x_1, x_2, x_3, \theta) \mapsto (x_1, x_2, x_3)$,
$E_1 = \left\langle \frac{\partial}{\partial\theta} \right\rangle, E_2 = \left\langle \frac{\partial}{\partial x_1} + \theta \frac{\partial}{\partial x_2} + a(x,\theta)\frac{\partial}{\partial x_3} + e(x,\theta)\frac{\partial}{\partial\theta} \right\rangle$,
(e is determined from a, C non-degenerate $\Leftrightarrow a_{\theta\theta} \neq 0$.
If $a = \frac{1}{2}\theta^2$, then $e = 0$ and C is flat.)

Theorem 2. Let $\gamma : (\mathbf{R}, t_0) \to Z$ be an E-integral curve, $f = \pi_X \pi_Y^{-1} \pi_Y \gamma : (\mathbf{R}^2, (t_0, 0)) \to X$ the null frontal generated by $\pi_Y\gamma$. Set $\varphi(t) := \theta'(t) - (e \circ \gamma)(t)x_1'(t)$. Then
$f \sim CE \Longleftrightarrow x_1'(t_0) \neq 0, \varphi(t_0) \neq 0$
$f \sim SW \Longleftrightarrow x_1'(t_0) = 0, \varphi(t_0) \neq 0, x_1''(t_0) \neq 0$
$f \sim SB \Longleftrightarrow x_1'(t_0) \neq 0, \varphi(t_0) = 0, \varphi'(t_0) \neq 0$

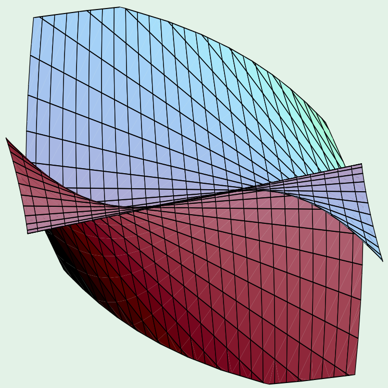
Here $\gamma(t) = (x_1(t), x_2(t), x_3(t), \theta(t))$.
Note that $\pi_X\gamma$ is a null geodesic $\Longleftrightarrow \varphi(t) \equiv 0$.

The dual objects to null frontals are given by tangent surfaces $\pi_Y \pi_X^{-1} \pi_X \gamma$ of Legendre curves $\pi_Y \gamma$ ruled by tangential "Legendre geodesics" (Legendre lines) $\pi_Y \pi_X^{-1}(x), (x \in X)$.

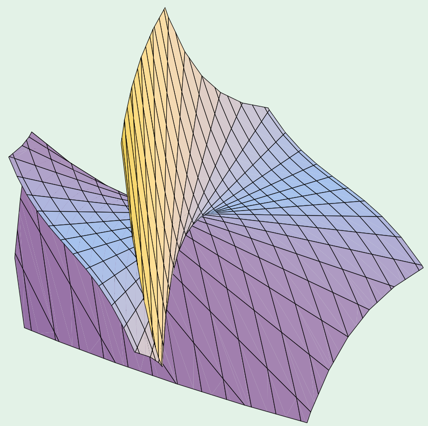
Theorem 3. For a generic E-integral curve $\gamma : I \to Z$, $\pi_Y \pi_X^{-1} \pi_X \gamma$ is diffeomorphic to cuspidal edge (CE), Mond surface (MD), or generic folded pleat (GFP).



cuspidal edge



Mond surface



generic folded pleat

∃local coordinates x, y, p, q of Z such that

$\pi_Y : (x, y, p, q) \mapsto (x, y, p)$

$$E_1 = \left\langle \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + f(x, y, p, q)\frac{\partial}{\partial q} \right\rangle, \quad E_2 = \left\langle \frac{\partial}{\partial q} \right\rangle.$$

For an E-integral curve $\gamma(t) = (x(t), y(t), p(t), q(t))$, we put $\psi(t) := q'(t) - (f \circ \gamma)(t)x'(t)$.

Theorem 4. $\pi_Y \pi_X^{-1} \pi_X \gamma$ is diffeomorphic at $(t_0, 0)$ to
cuspidal edge (CE) $\Longleftrightarrow x'(t_0) \neq 0, \psi(t_0) \neq 0$,
Mond surface (MD) $\Longleftrightarrow x'(t_0) \neq 0, \psi(t_0) = 0, \psi'(t_0) \neq 0$,
folded pleat (FP) $\Longleftrightarrow x'(t_0) = 0, \psi(t_0) \neq 0, x''(t_0) \neq 0$.

Note that $\pi_Y \gamma$ is a solution of 3rd order ODE $q' = f(x, y, p, q)$
$\Longleftrightarrow \psi(t) \equiv 0$.

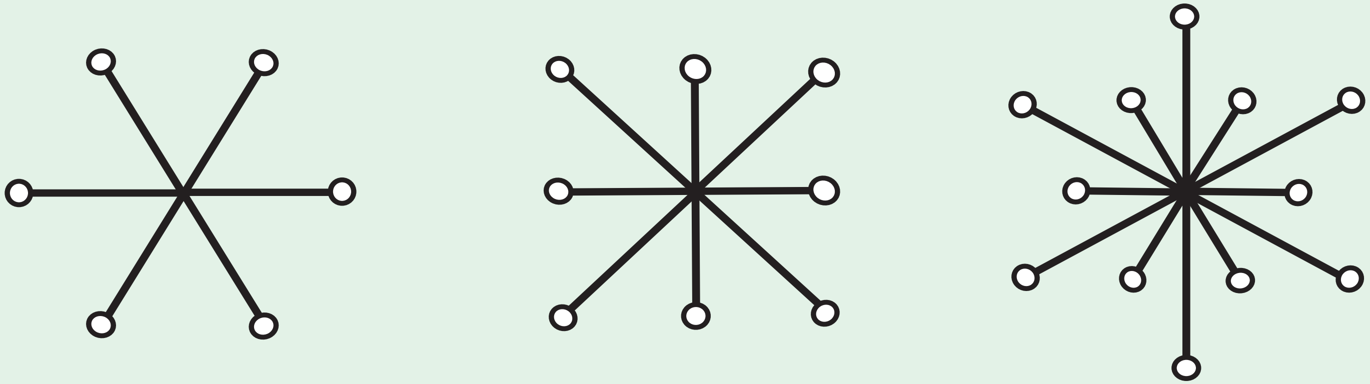
Thanks to Engel integral curves γ ("dancing on the heaven"), we have got the "asymmetric duality" of singularities of tangent surfaces.

Y	Z	X
$\pi_Y \pi_X^{-1} \pi_X \gamma$	γ	$\pi_X \pi_Y^{-1} \pi_Y \gamma$
CE	(I) non-tangent to E_1 non-tangent to E_2	CE
MD	(II) simply tangent to E_1 non-tangent to E_2	SW
FP	(III) non-tangent to E_1 simply tangent to E_2	SB

Some generalizations of the classification.

Simple Lie algebras of rank 2 :

A_2 $\quad$ $B_2 = C_2$ $\quad$ G_2

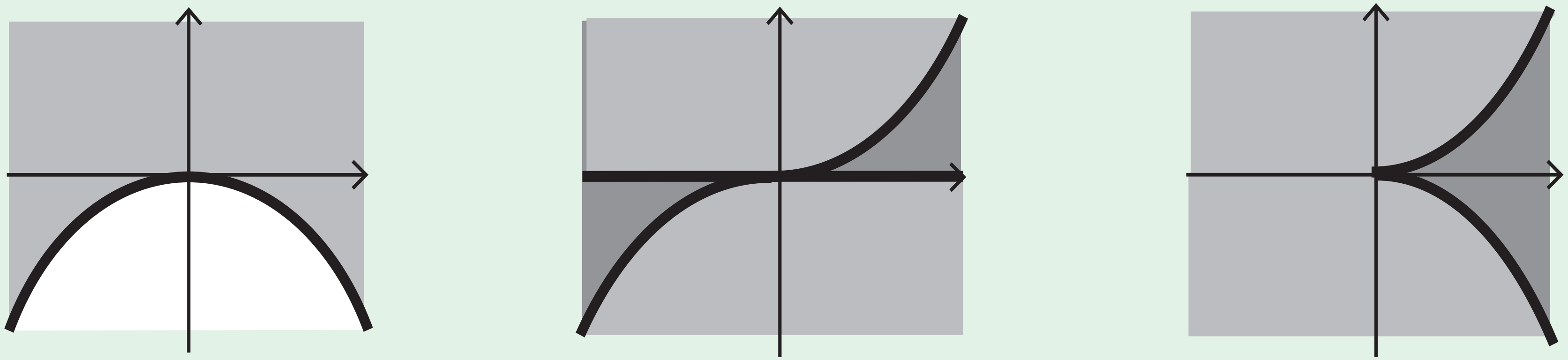


Root systems of types A_2, B_2 and G_2

— A_2 —

$G = \mathrm{PGL}(\mathbf{R}^3)$

For a plane curve $\gamma : I \to \mathbf{R}^2 \subset P(\mathbf{R}^3)$ and its projective dual $\gamma^* : I \to P(\mathbf{R}^{3*})$, we define their "tangent maps" $\mathrm{Tan}(\gamma) : I \times \mathbf{R} \to P(\mathbf{R}^3)$ and $\mathrm{Tan}(\gamma^*) : I \times \mathbf{R} \to P(\mathbf{R}^{3*})$, ruled by tangent lines.



convex point (fold map) | inflection point (beak-to-beak map) | cusp point (Whitney's cusp map)

We set the flag manifold (the incidence manifold):

$Z(A_2) := \{(V_1, V_2) \mid V_1 \subset V_2 \subset \mathbf{R}^3, \dim(V_1) = 1, \dim(V_2) = 2\}$,

Then we have the double fibrations:

$$Y(A_2) := P(\mathbf{R}^3) \xleftarrow{\pi_Y} Z(A_2) \xrightarrow{\pi_X} X(A_2) := P(\mathbf{R}^{3*}).$$

$$V_1 \longleftarrow (V_1, V_2) \longrightarrow V_2$$

Note that $\dim(Z) = 3, \ \dim(Y) = \dim(X) = 2$.

Moreover we obtain the contact structure

$$E := \mathrm{Ker}(\pi_{Y*}) \oplus \mathrm{Ker}(\pi_{X*}) \subset TZ$$

on Z from the double fibration.

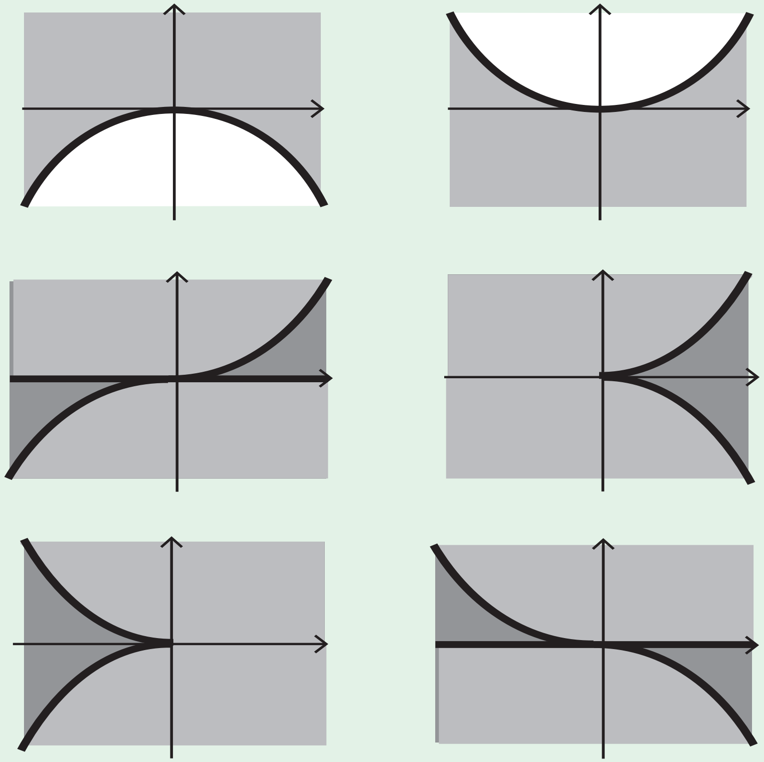
–Note that π_Y-fibers project by π_X to projective lines in $P(\mathbf{R}^{3*})$, and π_X-fibers project by π_Y to projective lines in $P(\mathbf{R}^3)$.

Theorem (A_2): For a generic Legendre curve $\gamma : I \to Z(A_2)$, the pair of diffeomorphism classes of tangent mappings to $\pi_Y\gamma, \pi_X\gamma$ at any point $t_0 \in I$ is given by one of

I : (fold, fold),

II : (beak-to-beak, Whitney's cusp),

III : (Whitney's cusp, beak-to-beak).



— B_2 —

Consider $G = \mathrm{O}(2,3)$ acting on $V = \mathbf{R}^{2,3}$.

$Z := \{V_1 \subset V_2 \subset \mathbf{R}^{2,3}\}$, the space of null flags, $\dim(Z) = 4$,

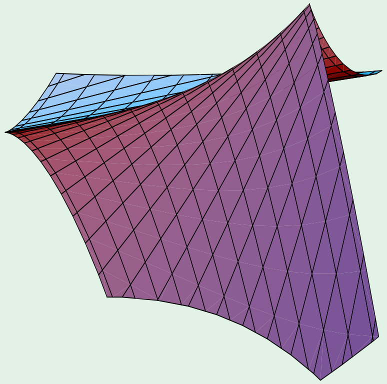
$X := \{V_1 \subset \mathbf{R}^{2,3}\}$, the space of null lines, $\dim(X) = 3$,

$Y := \{V_2 \subset \mathbf{R}^{2,3}\}$, the space of null planes, $\dim(Y) = 3$,

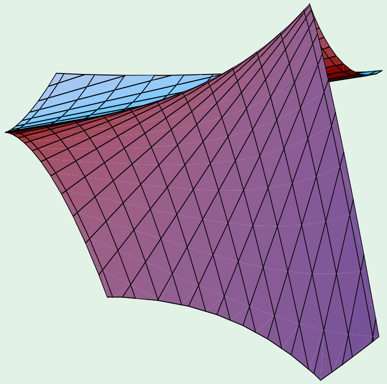
$$Y(B_2) \xleftarrow{\pi_Y} Z(B_2) \xrightarrow{\pi_X} X(B_2),$$

Just from the double fibration, we obtain an Engel structure $E := \mathrm{Ker}(\pi_{Y*}) \oplus \mathrm{Ker}(\pi_{X*}) \subset TZ$, a contact structure $D \subset TY$ and a quadratic cone field (conformally flat $(1,2)$-metric cone) $C \subset TX$.

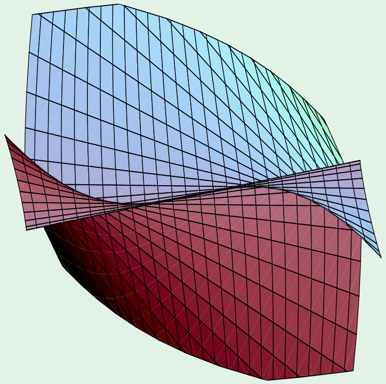
— π_Y-fibers project by π_X to "null lines" in X, and π_X-fibers project by π_Y to "Legendre lines" in Y.



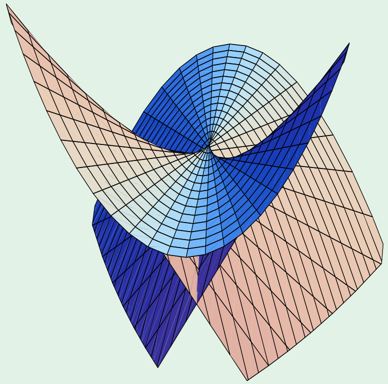
CE



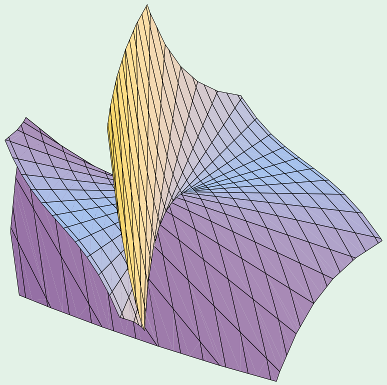
CE



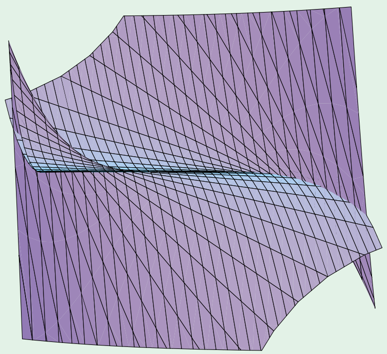
MD



SW



FP



SB

— G_2 —

$G = \mathrm{Aut}(\mathbf{O}')$, the split G_2. $\mathbf{O}'$ the split octonions.

Let $\mathbf{H} = \{a = x + yi + zj + wk \mid x, y, z, w \in \mathbf{R}\}$ be Hamilton's quarternion algebra and define the split octonions by $\mathbf{O}' = \mathbf{H} \oplus \mathbf{H}$ with the multiplication as

$$(a, b)(c, d) := (ac \mp \bar{d}b, da + b\bar{c}).$$

Note that $\mathbf{O}'$ is a non-associative algebra.

We set $V := \mathrm{Im}(\mathbf{O}')$, the imaginary part, $\dim(V) = 7$. Then $G = \mathrm{Aut}(\mathbf{O}')$ acts on V irreducibly.

Consider the split G_2 flag manifold

$Z(G_2) := \{(V_1, V_2) \mid V_1 \subset V_2 \subset V,\ V_1, V_2$ are null subalgebras$\}$,

where an $\mathbf{R}$-subspace $W \subset V$ is called a null subalgebra if $ww' = 0$, for any $w, w' \in W$.

Set $Y(G_2) := \{V_1 \mid V_1 \subset V,\ 1\text{-dimensional null subalgebra}\}$,

$X(G_2) := \{V_2 \mid V_2 \subset V,\ 2\text{-dimensional null subalgebra}\}$,

and consider the double fibrations:

$$Y(G_2) \xleftarrow{\pi_Y} Z(G_2) \xrightarrow{\pi_X} X(G_2),$$

$\dim(Z(G_2)) = 6,\ \dim(Y(G_2)) = \dim(X(G_2)) = 5.$

Set
$$E := \mathrm{Ker}(\Pi_{Y*}) \oplus \mathrm{Ker}(\Pi_{X*}) \subset TZ.$$

We called E the (split G_2) Engel distribution.

E is of rank 2 and with the small growth vector $(2, 3, 4, 5, 6)$ and the big growth vector $(2, 3, 4, 6)$.

Just from the split G_2 double fibrations, we obtain:

On the space Y, the Cartan structure $D \subset TY$ with the (small and big) growth vector $(2, 3, 5)$

On the space X, the Lagrange cubic cone field $C \subset D' \subset TX$ contained in a contact distribution D'. Such a structure is called a Monge structure.

π_Y-fibers project by π_X to "Monge lines" in X, which are tangent to the cubic cone C,

π_X-fibers project by π_Y to "Cartan lines" in Y, which are "abnormal" curves of the Cartan distibution D.

We have three types of double fibrations:

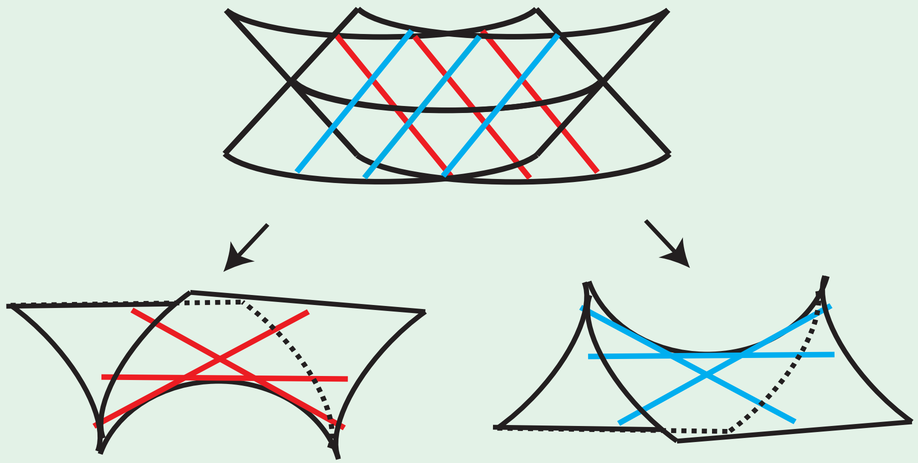
$$\begin{array}{ccc} & Z(A_2) & \\ \swarrow & & \searrow \\ Y(A_2) & & X(A_2) \end{array} \qquad \begin{array}{ccc} & Z(B_2) & \\ \swarrow & & \searrow \\ Y(B_2) & & X(B_2) \end{array} \qquad \begin{array}{ccc} & Z(G_2) & \\ \swarrow & & \searrow \\ Y(G_2) & & X(G_2) \end{array}$$

— $X(A_2)$ has the two-dimensional linear cone structure TX, $Y(A_2)$ has the two-dimensional distribution TY, and $Z(A_2)$ has the contact structure.

— $X(B_2)$ has the two-dimensional quadratic cone structure C, $Y(B_2)$ has the contact distribution D, and $Z(B_2)$ has the Engel structure.

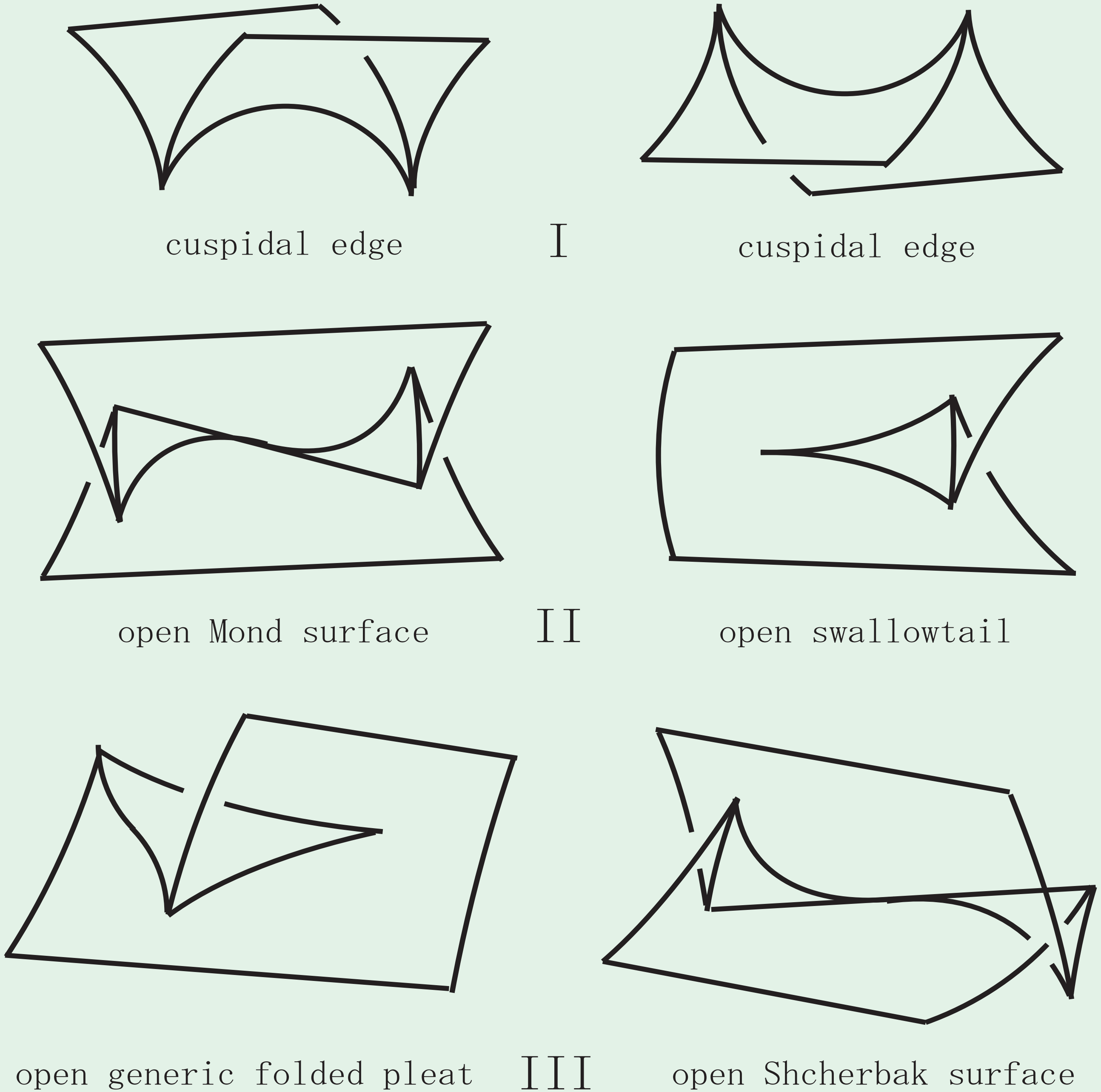
— $X(G_2)$ has the two-dimensional cubic cone structure C and the contact distribution $D' \supset C$, $Y(G_2)$ has the Cartan distribution D, and $Z(G_2)$ has the generalized Engel structure E.

A curve $\gamma : I \to (Z, E)$ from an open interval I is called a Engel integral curve if $\gamma_*(TI) \subset E(\subset TZ)$.
Then $\pi_X\gamma$ is a Monge integral curve in (X, C), and $\pi_Y\gamma$ is a Cartan integral curve in (Y, D).
The tangent surface $\pi_Y\pi_X^{-1}\pi_X\gamma$ of the Cartan integral curve $\pi_Y\gamma$ is ruled by tangent Cartan lines in Y. The tangent surface $\pi_X\pi_Y^{-1}\pi_Y\gamma$ of the Monge integral curve $\pi_X\gamma$ is ruled by tangent Monge lines in X.



Theorem (G_2) For a generic Engel integral curve $f : I \to (Z, E)$ in the split G_2 flag manifold Z, generic in C^∞ topology, the pair of diffeomorphism classes of tangent surfaces $\pi_Y \pi_X^{-1} \pi_X \gamma$ to Cartan curve $\pi_Y \circ f$ and tangent surfaces $\pi_X \pi_Y^{-1} \pi_Y \gamma$ to Monge curve $\pi_X \circ f$ at any point $t_0 \in I$ is given by one of

I : (cuspidal edge, cuspidal edge),
II : (open Mond surface, open swallowtail),
III : (generic open folded pleat, open Shcherbak surface).


cuspidal edge
I
cuspidal edge
open Mond surface
II
open swallowtail
open generic folded pleat
III
open Shcherbak surface

ご清聴ありがとうございます.

Thank you for your attention.