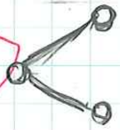
ニュートラル共形計量とヌル曲線にまつわる特異点

Neutral conformal metrics and singularities related to null curves

Numazu, 2015. 3. 9.　　Goo ISHIKAWA

A talk based on joint works with Y. Machida & M. Takahashi  D_3

$V = \mathbb{R}^{3,3}$　6-dim. vector space, signature $(3,3)$

$$Z = \{ (V_1, V_3^+, V_3^-) \mid \underset{\text{null}}{V_1}, \underset{\text{1-dim}}{V_3^+}, \underset{\text{3-dim}}{V_3^-} \subset V \quad V_1 \subset V_3^+,\ V_1 \subset V_3^- \quad \dim(V_3^+ \cap V_3^-) = 2 \}$$

flag manifold　$\dim Z = 6$　　$\cong \{ (V_1, V_2) \mid V_1 \subset \underset{\text{null}}{V_2} \subset V \}$

$M = \{ V_1 \subset V \text{ null lines} \} \subset P(V)$　$\dim M = 4$　　$M \cong Gr(2, \mathbb{R}^4)$

$N = \{ V_2 \subset V \text{ null 2-space} \}$　$\dim N = 5$

$Q = \{ V_3 \subset V \text{ null 3-space} \} = Q_+ \amalg Q_-$

$$Q_\pm \cong SO(3) \cong P^3 \cong Gr(1, \mathbb{R}^4)$$

$$\begin{tikzcd} & Z^6 \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & \\ N^5 \arrow[d, "\pi_+"'] \arrow[dr, "\pi_-"] & & M^4 \\ Q_+ & Q_- & \end{tikzcd}$$

contact (N^5)　　(2,2) metric (M^4)

$$V_1 \subset V_2 \begin{smallmatrix} \subset V_3^+ \\ \subset V_3^- \end{smallmatrix} \subset V_2^\perp \subset V_1^\perp \subset V$$

$E \subset TZ$　canonical distribution

$$E = \overset{E_1}{\ker \pi'_*} \oplus \overset{E_2}{\ker \pi_*} \qquad \operatorname{rank} E = 3$$

growth $(3, 5, 6)$

$E^{(2)} = E + [E, E]$　rank $= 5$

$E^{(2)}$ induces $D \subset TN$　contact structure　rank $D = 4$

$D = \ker \pi_{+*} \oplus \ker \pi_{-*}$　Lagrange pair

$D = D_+ \oplus D_-$

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$p = V_1 \in \underline{M}$ $\quad T_pM \cong V_1^{\perp}/V_1$ $\quad$ (2,2)-vector space

$\Gamma : I \longrightarrow Z$: E-integral curve $\quad f(t) = (V_1(t), V_3^+(t), V_3^-(t))$

$\pi \circ \Gamma = \gamma : \underset{\text{interval}}{I} \longrightarrow M$ null curve

$P(V_2(t)) \subset \underset{\text{normal}}{P(V_1^{\perp}(t)) \cap M}$ $\quad V_2(t)^{\perp} = V_3^+(t) + V_3^-(t)$

$V_1(t)$

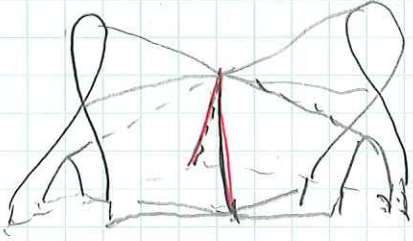
D_3 エボはスリーディー

envelope of $P(V_1^{\perp}(t)) \cap M$: $\boxed{D_3\text{-evolute}}$ of γ.

Lemma D_3-evolute of γ is given by $\bigcup_t P(V_3^+(t)) \cup P(V_3^-(t))$

It is a solution of Hamiton Jacobi equation

Ex.



$\times \mathbb{R}$

tri-umbrella "相合傘"

("Kazarjan's umbrella")

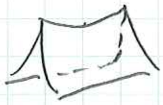
$\boxed{\text{tangent surface}}$ $\bigcup_t P(V_2(t)) \subseteq \bigcup_t P(V_3^+(t)) \cap \bigcup_t P(V_3^-(t))$

beak-to-beak $\quad$ 相合傘の下で beak-to-beak

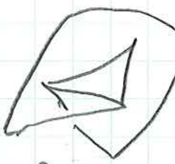
Th $\forall$ generic E-integral curve $\Gamma : I \to Z$

$\forall t_0 \in I$

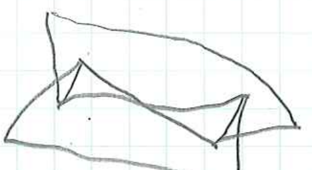
$\mathrm{Tan}(\pi \circ \Gamma)$,	$\mathrm{Tan}(\pi_+ \circ \pi' \circ \Gamma)$	$\mathrm{Tan}(\pi_- \circ \pi' \circ \Gamma)$
CE	CE	CE
OSW	M	M
CE	FU	SW
CE	SW	FU



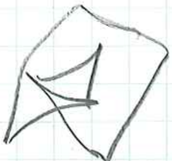
Cuspidal Edge



Open SwallowTail



Mond surface



SwallowTail

Folded Umbrella

M : semi-Riemannian 4-dim manifold of signature $(2,2)$.

$C \subset TM$ null cone field

$PC \doteq (C \setminus \text{zerosection})/\mathbb{R}^\times$ projectivization

$\pi : PC \to M$ $(S^1 \times S^1)/\mathbb{Z}_2 \cong S^1 \times S^1$ bundle

$E \subset T(PC)$ tautological subbundle

$(x, U_1) \in PC$ $E_{(x,U_1)} = \pi_*^{-1}(U_1)$, $\pi_* : T_{(x,U_1)}PC \to T_x M$, $U_1 \subset T_x M$

rank $E = 3$ growth $(3,5,6)$

<u>Lemma</u> $\exists$ intrinsic pseudo-product structure on E

$$E = E_1 \oplus E_2$$

rank $E_1 = 1$, rank $E_2 = 2$, E_1, E_2 integrable

E_1, E_2 is defined only by E using "singular curves"

<u>Rem.</u> E_1 : directions of null geodesics

E_2 : —— of π-fibres

$(x, U_1) \in PC$

$\exists 2$ null 2-spaces $U_2^+, U_2^- \subset C_x \subset T_x M$ ← (2,2) vector sp.

$$U_1 = U_2^+ \cap U_2^- \begin{array}{c}\subsetneq U_2^+ \subsetneq \\ \subsetneq U_2^- \subsetneq\end{array} U_1^\perp \subset T_x M$$

$$L^\pm_{(x,U_1)} := T_{(x,U_1)} P(U_2^\pm) \subset (E_2)_{(x,U_1)}$$

$$E_2 = L^+ \oplus L^-$$

Define $\tilde{D}^\pm := E_1 + L^\pm + [E_1, L^\pm] = (E_1 \oplus L^\pm)^{(2)}$

rank $\tilde{D}^\pm = 3$

$$\begin{tikzcd} & PC = Z \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & \\ N = PC/\mathcal{E}_1 & & PC/\mathcal{E}_2 = M \end{tikzcd}$$

$\exists D$: contact structure on N, $D^+, D^- \subset D$ Lag subbundle

$\pi'^{-1}_{*} D = E^{(2)}$, $\pi'^{-1}_{*} D^{\pm} = \tilde{D}^{\pm}$

$D = D^+ \oplus D^-$ Lagrangian pair.

Prop. The followings are equivalent:

(i) M is conformally flat

(ii) D^+, D^- integrable

(iii) $(N, D; D^+, D^-)$ standard

$$\begin{tikzcd} & PT^*P^3 \cong PT^*P^{3*} \arrow[dl] \arrow[dr] & \\ P^3 & & P^{3*} \end{tikzcd}$$

$\Gamma : I \to PC$: E-integral map

$\gamma : \pi \circ \Gamma : I \to M$ null-directed curve

$\Gamma(t) = (\gamma(t), [u(t)])$ $\quad \gamma'(t) = {}^{\exists}c(t)u(t) \quad u(t) \neq 0$

$u(t)^{\perp} \subset T_{\gamma(t)}M$ 3-space

$H_t = \mathrm{Exp}(u(t)^{\perp})$: 1-parameter family of hypersurfaces along γ

$S_t = \mathrm{Exp}(u(t)^{\perp} \cap C_{\gamma(t)})$: —— of Schubert variety of dim 2

$= S_t^+ \cup S_t^-$

Lemma Envelope of H_t is given by $\bigcup_t S_t = \left(\bigcup_t S_t^+\right) \cup \left(\bigcup_t S_t^-\right)$

(<u>D_3- evolute</u> of null directed curve γ)

Suppose M is <u>oriented</u> $(2,2)$-manifold.

twistor

$A \to M$ S^1-bundle of α-planes

$B \to M$ —— β-planes

$E^\alpha \subset TA$ tautological plane bundle, $\operatorname{rank} E^\alpha = 2$

$E^\beta \subset TB$ —— $\operatorname{rank} E^\beta = 2$

Th (Penrose) E^β: integrable $\Longleftrightarrow$ M : self-dual

E^α : 〃 $\Longleftrightarrow$ M : anti-self dual

$PC \to M$ is the fibre-product of

$$\begin{tikzcd} B \arrow[dr] & & A \arrow[dl] \\ & M & \end{tikzcd}$$

$$\begin{tikzcd} & PC \arrow[dl, "p_\beta"'] \arrow[dr, "p_\alpha"] & \\ B \arrow[dr, "\pi_\beta"'] & & A \arrow[dl, "\pi_\alpha"] \\ & M & \end{tikzcd}$$

Lemma $\hat{D}^+ = (p_\alpha)_*^{-1} E^\alpha$, $\hat{D}^- = (p_\beta)_*^{-1} E^\beta$

Lemma $\hat{D}^+$ integrable $\Longleftrightarrow$ E^α integrable

$\hat{D}^-$ 〃 $\Longleftrightarrow$ E^β 〃

Lemma $\hat{D}^\pm$ integrable $\Longleftrightarrow$ $D^\pm$ integrable

〈Proof of Prop〉

(i) $\Rightarrow$ (iii) $\Rightarrow$ (ii′) $\Longleftrightarrow$ E^α, E^β integrable $\Longleftrightarrow$ M : self-dual anti-self-dual

$\Longleftrightarrow$ M conformally flat

References

C. Lebrun, L.J. Mason : Non linear gravitons, null geodesics and holomorphic disks, Duke Math. J. 136-2 (2007) 205–273

松下泰雄, 鎌田博行, 中田文憲 : 4次元微分幾何学への招待 SGCライブラリー113 サイエンス社 (2014).

Global

contact 5-mfd (N, D) with Lagrangian pair $D = D^+ \oplus D^-$ $\overset{?}{\longleftrightarrow}$ 4-mfd M with $(2,2)$ neutral metric

?

$\forall$ maximal Legendre submfd of D^+ or D^- is S^2 or $\mathbb{R}P^2$

?

Zollfrei

$\forall$ maximal null geodesic is S^1