D4幾何における三対性への特異点論の試み　　Numazu 2013.3 (修正稿)

Goo ISHIKAWA

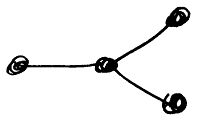
①

Trial to Triality　　町田, 高橋との共同研究

Joint with Machida, Takahashi

特異点論から triality に trial

$\boxed{D_4\text{の三対五徳八元旗}}$

Dynkin 図式  D_4

E. Cartan (1925)

$$Out(D_4) = Aut(D_4)/Inn(D_4) \cong \mathfrak{S}_3$$

Geometric triality　E. Study (1913)

$V = \mathbb{R}^{4,4}$ ($= \mathbb{O}'$ split 8元数)

metric $(x|x) = x_0x_7 + x_1x_6 + x_2x_5 + x_3x_4$

$Q_0 = \{V_1 \subset V$ null line$\}$ quadric in $P(V)$

$\dim Q_0 = 6$

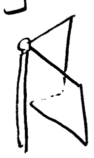
$M^9 = \{V_2 \subset V$ null plane$\} = \{$lines in $Q_0\}$

null 4-spaces

$V_4 \sim V_4' \overset{def}{\iff} \dim(V_4 \cap V_4')$ is even

$Q_+^6 \sqcup Q_-^6$ (disj. union)

$Z^{12} = \{(V_1, V_2, V_3) \mid V_1 \subset V_2 \subset V_3 \subset V,\ V_i:$ null$,\ \dim V_i = i\}$　flag mfd



$$V_1 \subset V_2 \subset V_3 \begin{smallmatrix} \subset \exists^1 V_4^+ \\ \subset V_4^- \end{smallmatrix}$$ "double flag"

$$\begin{tikzcd} & Z^{12} \arrow[rr, "\pi_1"] \arrow[dl, "\pi_0"'] \arrow[d, "\pi_+"] \arrow[dr, "\pi_-"] & & M^9 \\ Q_0^6 & Q_+^6 & Q_-^6 & \end{tikzcd}$$

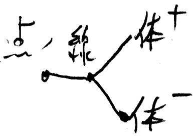
$Z \cong \{(V_1, V_4^+, V_4^-) \mid V_1 \subset V_2 \begin{smallmatrix} \subset V_4^+ \\ \subset V_4^- \end{smallmatrix}$ incident$\}$

$$V_1 \subset V_2 \begin{smallmatrix} \subset V_4^+ \\ \subset V_4^- \end{smallmatrix}$$

V_4^+, V_4^- incident

$\iff \dim(V_4^+ \cap V_4^-) = 3$

《五徳の話》



の3対性

Q_0, Q_+, Q_- : conformal structure

differential system $E = (\ker \pi_{0*} \cap \ker \pi_{+*} \cap \ker \pi_{-*}) \oplus \ker \pi_{2*} \subset TZ$ rank 4.

$f : I \xrightarrow{\subset \text{interval}} Z$ E-integral curve $\overset{\text{def.}}{\Longleftrightarrow}$ $f_* TI \subset E$

("non-dege" null curve in Q_0 lifts to E-int curve )

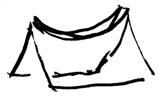
$\gamma_0 = \pi_0 \circ f : I \to Q_0$ null curve

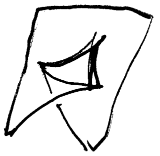
$\mathrm{Tan}(\gamma_0) : I \times \mathbb{R} \to Q_0$ tangent surface, null surface.

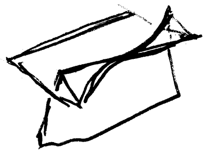
$\boxed{\text{singularity}}$ $\qquad \pi_0 \pi_1^{-1} \pi_1 f$

Th For a generic E-integral curve $f : I \to Z$

$\mathrm{Tan}(\gamma_0)$, $\mathrm{Tan}(\gamma_+)$, $\mathrm{Tan}(\gamma_-)$ have singularites

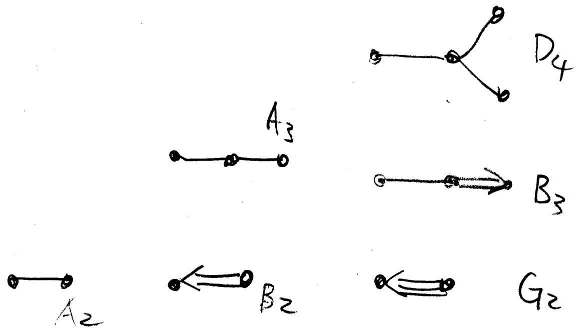
cuspidal edge 

open swallowtail 

open Mond 

	$\mathrm{Tan}(\gamma_0)$	$\mathrm{Tan}(\gamma_+)$	$\mathrm{Tan}(\gamma_-)$
5通り	CE	CE	CE
	OSW	CE	CE
	CE	OSW	CE
	CE	CE	OSW
	OM	OM	OM

5徳.



Numazu 2013.3.

Trial to triality in D_4-geometry from singularity theory
D_4幾何における三対性への特異点論の試み

Goo ISHIKAWA

講演原稿（初稿）

へび
がま かえる
なめくじ

Triality の話（三対性，三角関係，三すくみ，トロイカ，三人娘）

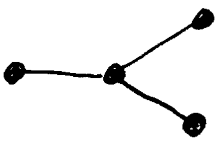
Joint with Machida, Takahashi

美空ひばり
江利ちえみ
雪村いづみ

中尾ミエ
園まり
伊東ゆかり

山口百恵
桜田淳子
森昌子

$\boxed{D_4\text{の三対五徳八元旗}}$



D_4

E. Cartan (1925) (cf. E. Cartan 1869–1951)

$$\mathrm{Out}(D_4) = \mathrm{Aut}(D_4)/\mathrm{Inn}(D_4) \cong \mathfrak{S}_3$$

Algebraic triality

Chevalley, Freudenthal, Springer, Jacobson

Geometric triality ("Kinematic triality")

E. Study (1913) (cf. Eduard Study 1862–1930)

Tits (1956), Kostant (1988)

(see I. Porteous, Topological Geometry 2nd ed. (1981).)
"quadratic triality"

3

$V = \mathbb{R}^{4,4}$ 8-dim. vector space of type (4,4)

metric $x_0x_7 + x_1x_6 + x_2x_5 + x_3x_4$ ($= \mathbb{O}'$ the split octonians)

$Q_0^6 = \{V_1 \subset V \text{ null lines}\} \subset P(V)$ quadric

$M^9 = \{V_2 \subset V \text{ null plane}\} = \{\text{lines on } Q_0\}$

null 4-spaces are devided into 2 kinds :

$$V_4 \sim V_4' \iff \dim(V_4 \cap V_4') : \text{even}$$

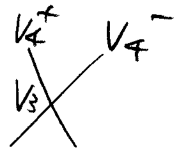
$\{V_4 \subset V \text{ null 4-space}\} = Q_+^6 \cup Q_-^6$

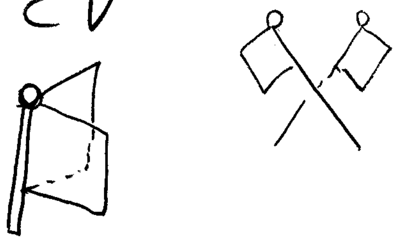
V_4^+, V_4^- incident $\iff \dim(V_4^+ \cap V_4^-) = 3$.

$$Z^{12} = \{(V_1, V_2, V_3) \mid V_1 \subset V_2 \subset V_3 \subset V \ \dim V_i = i, \ V_i : \text{null}\}$$

$$V_1 \subset V_2 \subset V_3 \begin{array}{l} \overset{\exists 1}{\subset} V_4^+ \\ \underset{\exists 1}{\subset} V_4^- \end{array} \overset{\text{incident}}{\begin{array}{l} \subset \\ \subset \end{array}} V_3^\perp \subset V_2^\perp \subset V_1^\perp \subset V$$

"double flag"





$$\begin{tikzcd} & Z^{12} \arrow[r, "\pi_1"] \arrow[dl, "\pi_0"'] \arrow[d, "\pi_+"] \arrow[dr, "\pi_-"] & M^9 \\ Q_0^6 & Q_+^6 & Q_-^6 \end{tikzcd}$$

V_2

$V_1 \qquad V_4^+ \qquad V_4^-$

（ここで五徳の話；昔，犬の足は3本だった 不便だった．一方「五徳」は動かないのに足が4本あって四徳とよばれていた．神様が犬をふびんに思い，四徳に足を1本ゆずるようにたのんだ．

<u>lines</u> on	Q_0	Q_+	Q_-
	$\pi_0\pi_1^{-1}$	$\pi_+\pi_1^{-1}$	$\pi_-\pi_1^{-1}$

犬はもらった足が大切なので用をたすとき片足あげる
四徳は徳があるということで五徳とよばれた．
犬の名前は「ハチ」いまも元気．）

Duality

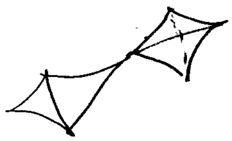
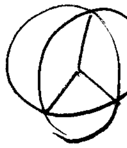
Ex. A_3  duality

$V_1 \subset V_2 \subset V_3 \subset \mathbb{R}^4$

$$\begin{tikzcd} & Z^6 \arrow[r, "\pi_2"] \arrow[dl, "\pi_1"'] \arrow[d, "\pi_3"] & Gr(2,\mathbb{R}^4) \\ \mathbb{R}P^3 & \mathbb{R}P^{3*} & \end{tikzcd}$$

lines $\pi_1 \pi_2^{-1}$; $\pi_3 \pi_2^{-1}$

- Lagrange, Legendre singularity theory (duality の singularity)

D_4 caustic  wavefront 

(deformation of $\ast$)

Thom, Arnold, Brieskorn, …

Triality (上のきめ方は Q_0 できめれば $Q_\pm$ 上の 3-sp についても きまる？)

pt on Q_0 $\longleftrightarrow$ 3-space on Q_+ of type $-$ $\longleftrightarrow$ 3-space on Q_- of type $+$

3-sp on Q_0 of type $+$ $\longleftrightarrow$ pt on Q_+ $\longleftrightarrow$ 3-sp on Q_- of type $-$

3-sp on Q_0 of type $-$ $\longleftrightarrow$ 3-sp on Q_+ of type $+$ $\longleftrightarrow$ pt on Q_-

line on Q_0 $\longleftrightarrow$ line on Q_+ $\longleftrightarrow$ line on Q_-

---

Embedding G_2 flag to D_4 flag $\boxed{a+b\varepsilon}$ G $\mathbb{O}'$ split 8元代数 $\varepsilon^2=1$

$a(d\varepsilon)=(da)\varepsilon$ $(b\varepsilon)c=(b\bar{c})\varepsilon$ $(b\varepsilon)(d\varepsilon)=\bar{d}b$

$$Z(G_2)^6 = \{(V_1,V_2) \mid V_1 \subset V_2 \subset \overset{3,4}{\mathrm{Im}\,\mathbb{O}'} \text{ null subalg},\ \dim V_i = i\} \hookrightarrow Z^{11}$$

$$\begin{tikzcd} & Z(G_2) \arrow[dl, "\pi_Y"'] \arrow[d, "\pi_X"] \\ Y^5 & X^5 \end{tikzcd}$$

V_2 : null subspace となる

$E(G_2) = \mathrm{Ker}\,\pi_{Y*} \oplus \mathrm{Ker}\,\pi_{X*} \subset TZ(G_2)$

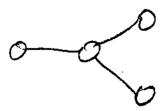
rank 2

$$E = (\ker \pi_{0*} \cap \ker \pi_{+*} \cap \ker \pi_{-*}) \oplus \ker \pi_{2*} \subset TZ \quad \text{rank } 4$$

Th (D_4) generic E-integral curve $f: I \to Z$,
$\forall t \in I$, singularity of tangent surfaces to projections on Q_0, Q_+, Q_-, M respectively are classifed up to diffeomorphisms:

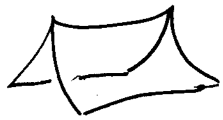
$\pi_0 \pi_1^{-1} \pi_1 f$, $\pi_+ \pi_1^{-1} \pi_1 f$ $\pi_- \pi_1^{-1} \pi_1 f$

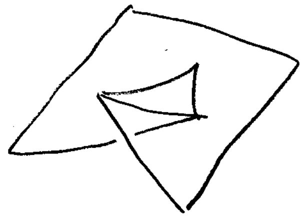
CE	CE	CE	CE
OSW	CE	CE	CE
CE	OSW	CE	CE
CE	CE	OSW	CE
OM	OM	OM	OSW

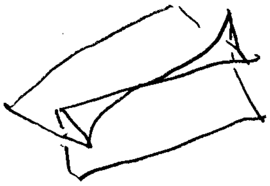


$\pi_0 \pi_+^{-1} \pi_+ f$
$\pi_0 \pi_-^{-1} \pi_- f$

ただし

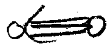
CE  cuspidal edge (1 2 3 ⋯)

OSW  open swallowtail (2 3 4 5 ⋯)

OM  open Mond (1 3 4 5 ⋯)

Th(G_2) generic $E(G_2)$-integral curve $f: I \to Z(G_2)$
$\forall t \in I$ singularity of tangent surfaces to projections on Y, X respectively are classified up to diffeom.:

5 CE	CE 5
OM	OSW
OGFP	OSh

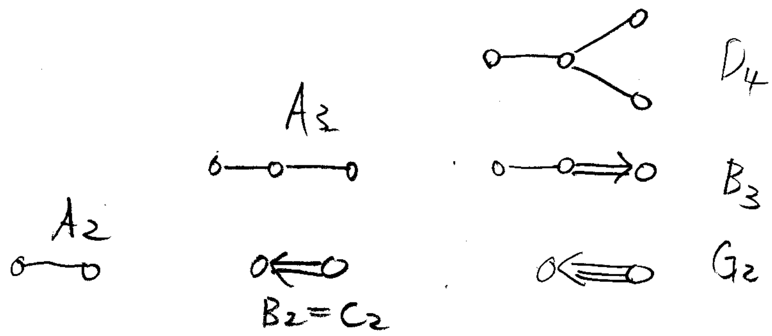
 G_2

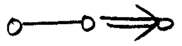
generic (2,3.5,7,8)
OGFP (Open generic folded pleat).
OSh ← (open Shcherbak surface) (1 3 5 7 8)

6

Witt-Dynkin diagram folding $\leftrightarrow$ fibration tree $\curvearrowright$ embedding (flag)

removing $\leftrightarrow$ —— $\curvearrowright$ local projection





$\underline{\underline{Th}}$ (B_3) $\mathbb{R}^{4,3}$

5	6	7
CE	CE	CE
OSW	CE	CE
UFU	OSW	CE
OM	OM	OSW

UFU = unfurled folded umbrella (1, 2, 4, 6, 7)

Idea of proof of Th.

≪ Triality of weights ≫

Q_0	Q_+	Q_-
w_{10}	w_{42}	w_{32}
$w_{21}+w_{10}$	$w_{42}+w_{21}$	$w_{32}+w_{21}$
$w_{32}+w_{21}+w_{10}$	$w_{42}+w_{21}+w_{10}$	$w_{32}+w_{21}+w_{10}$
$w_{42}+w_{21}+w_{10}$	$w_{42}+w_{32}+w_{21}$	$w_{42}+w_{32}+w_{21}$
$w_{42}+w_{32}+w_{21}+w_{10}$	$w_{42}+w_{32}+w_{21}+w_{10}$	$w_{42}+w_{32}+w_{21}+w_{10}$
$w_{42}+w_{32}+2w_{21}+w_{10}$	$w_{42}+w_{32}+2w_{21}+w_{10}$	$w_{42}+w_{32}+2w_{21}+w_{10}$

$$
\begin{tikzcd}[arrows=dash]
 & & w_{42} \\
w_{10} \arrow[r] & w_{21} \arrow[ur] \arrow[dr] & \\
 & & w_{32}
\end{tikzcd}
$$

M^9

w_{21}

$w_{21}+w_{10}$

$w_{21}+w_{32}$

$w_{21}+w_{42}$

$w_{32}+w_{21}+w_{10}$

$w_{42}+w_{21}+w_{10}$

$w_{42}+w_{32}+w_{21}$

$w_{42}+w_{32}+w_{21}+w_{10}$

$w_{42}+w_{32}+2w_{21}+w_{10}$

formula on "negative" roots
$\parallel$
positive roots

$\rightsquigarrow$ order of flag coordinates of generic E-integral curve.

$\rightsquigarrow$ normal form of tangent surface

7 add

Th(A_3)	3	3	4
	CE	CE	CE
	SW	FU	CE
	FU	SW	CE
	M	M	OSW

(1 2 4)
FU = folded umbrella

M = Mond
(1 3 4)

Th(C_3)	5	6	7
	CE	CE	CE
	OSW	CE	CE
	OFU	OSW	CE
	UM	OM	OSW

(1 2 4 5 ⋯)
OFU = open folded umbrella

UM = unfurled Mond surface
(1 3 4 6 7)
(Ishikawa 2012 Th.7.3)

Th(C_2) (MIT 2011.)	3	3
	CE	CE
	M	SW
	GFP	Sh

target surface tr
{OM (1, 3, 4, 5, ⋯)
UM} (1, 3, 4, 6, ⋯)

B_2

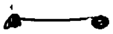
||

C_2

generic
(2 3 5)
GFP = <u>generic</u> folded pleat

Sh = Shcherbak surface
(1 3 5)

Th(A_2)	2	2
	(1 2) fold	fold (1 2)
	(1 3) beak-to-beak	Whitney Cusp (2, 3)
	(2 3) Whitney Cusp	beak-to beak (1 3)



Singularies in geometric triality

(8) add.

D_4 $\quad V=\mathbb{R}^{4,4}$ $\quad Q_0$: null lines

Q : null 4-planes $\quad Q=Q_+\cup Q_-$ disjoint

Q_+, Q_- のとり方は任意. Fix a null 4-planes $V_4^0 \in Q$

$Q_+=\{V_4\in Q \mid \dim V_4\cap V_4^0 \text{ even}\} = \{(-1)^{\dim V_4\cap V_4^0}=\pm 1\}$
$Q_-=\{V_4\in Q \mid \qquad\qquad \text{odd}\}$

V_4^+, V_4^- incident $\overset{\text{def}}{\Longleftrightarrow}$ $\dim(V_4^+\cap V_4^-)=3$

$V_1\in Q_0$, V_4^+, V_4^- : genein

$l=\{V_2 \mid V_1\subset V_2 \subset V_4^+,\ V_2\subset V_4^-\}\subset M^9$ $\qquad \pi_1(\pi_0^{-1}\cap\pi_+^{-1}\cap\pi_-^{-1})$ line in M

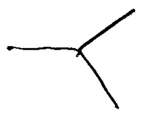
$$\begin{tikzcd}
 & Z^{12} \arrow[dl, "\pi_0"'] \arrow[d, "\pi_+"] \arrow[dr, "\pi_-"] \arrow[rr, bend left, "\pi_1"] & & M^9 \\
Q_0 & Q_+ & Q_- &
\end{tikzcd}$$

$\pi_0\circ\pi_1^{-1}$ lines in Q_0
$\pi_+\circ\pi_1^{-1}$ lines in Q_+
$\pi_-\circ\pi_1^{-1}$ lines in Q_-

$\pi_0\circ\pi_+^{-1}$ 3-spaces of one kind
$\pi_0\circ\pi_-^{-1}$ 3-spaces of another kind
$\pi_+\circ\pi_0^{-1}$
$\pi_+\circ\pi_-^{-1}$
$\pi_-\circ\pi_0^{-1}$
$\pi_-\circ\pi_+^{-1}$

In Q_0 $\qquad \pi_0\circ\pi_1^{-1}$, $\pi_0\circ\pi_+^{-1}$, $\pi_0\circ\pi_-^{-1}$

$TZ\supset E=\underbrace{(\ker\pi_{0*}\cap\ker\pi_{+*}\cap\ker\pi_{-*})}_{1次元}\oplus\underbrace{\ker\pi_{2*}}_{3次元}$



$f: I\to Z$ $\quad E$-integral curve

$\pi_0\pi_1^{-1}\pi_1 f$ $\quad \pi_+\pi_1^{-1}\pi_1 f$ $\quad \pi_-\pi_1^{-1}\pi_1 f$ $\quad$ tangent surfaces

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$\pi_0\pi_+^{-1}\pi_+ f$, $\pi_0\pi_1^{-1}\pi_- f$ $\quad$ $+4$-fold
-4-fold