Rotating Wave Approximation of the Law's Effective Hamiltonian on the Dynamical Casimir Effect

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Abstract

In this paper we treat the Law's effective Hamiltonian of the Dynamical Casimir Effect in a cavity and construct an analytic approximate solution of the time–dependent Schrödinger equation under the general setting through a kind of rotating wave approximation (RWA). To the best of our knowledge this is the finest analytic approximate solution.

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1 Introduction

In this paper we revisit the so-called dynamical Casimir effect (DCE). It means the photon generation from vacuum due to the motion (change) of neutral boundaries, which corresponds to a kind of quantum fluctuation of the electro-magnetic field.

This phenomenon is a typical example of interactions between the microscopic and the macroscopic levels and is very fascinating from the point of view of not only (pure) Physics but also Mathematical Physics. As a general introduction to this topic, see for example [1].

In the paper we treat the effective Hamiltonian by Law [2] (which is time–dependent) and propose a kind of rotating wave approximation to the model and construct an analytic approximate solution under the general setting in terms of some infinite dimensional representation of the Lie algebra $su(1,1)$. This is the best solution known so far.

2 Model and RWA

First of all let us make a brief review of Law [2] within our necessity. See also [3] and [4]. In the following we set $\hbar = 1$ for simplicity.

As an effective Hamiltonian of the dynamical Casimir effect in a cavity (Cavity DCE for simplicity) we adopt the simplest one which is the special case of [2] (namely, $\epsilon(x,t) = \epsilon(t)$). However, this is fruitful enough as shown in the following. We set

$$H = \omega(t) a^\dagger a + i\chi(t) \left\{ (a^\dagger)^2 - a^2 \right\} \tag{1}$$

where $\omega(t)$ is a periodic function depending on the cavity form and $\chi(t)$ is given by

$$\chi(t) = \frac{1}{4\omega(t)} \frac{d\omega(t)}{dt} = \frac{1}{4} \frac{d}{dt} \log |\omega(t)|.$$

Here a and $a^\dagger$ are the cavity photon annihilation and creation operators respectively. Therefore, the physics that we are treating is two-photon generation processes from the vacuum state.

We would like to solve the Schrödinger equation

$$i\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle = H(t)|\Psi(t)\rangle \tag{2}$$

explicitly under the general setting.

If H is time–independent, then the general (formal) solution is given by

$$|\Psi(t)\rangle = e^{-itH}|\Psi(0)\rangle.$$

Of course, to calculate e^{-itH} exactly is another problem (which is in general very hard). However, in our case H is time–dependent, so solving (2) becomes increasingly difficult. We must make some approximation to the Hamiltonian.

We impose further restrictions on the model. Since $\omega(t)$ is periodic we take $\omega(t)$ as in [3]

$$\omega(t) = \omega_0(1 + \epsilon\sin(\eta t))$$

where ω_0, ϵ and η are real constants. We assume that $\omega_0 > 0$, $0 < \epsilon \ll 1$ and η is large enough. Under the restrictions we may set

$$\omega(t) \approx \omega_0 \quad \text{and} \quad \chi(t) = \frac{\epsilon\eta\cos(\eta t)}{4(1+\epsilon\sin(\eta t))} \approx \frac{\epsilon\eta}{4}\cos(\eta t).$$

Then the Hamiltonian (1) is approximated by

$$H = \omega_0 a^\dagger a + i\frac{\epsilon\eta}{4}\cos(\eta t)\left\{(a^\dagger)^2 - a^2\right\}. \tag{3}$$

We apply a kind of rotating wave approximation (RWA) to this model. Since

$$\cos(\eta t) = \frac{e^{i\eta t} + e^{-i\eta t}}{2} = \frac{1}{2}e^{i\eta t}\left(1 + e^{-2i\eta t}\right) = \frac{1}{2}e^{-i\eta t}\left(1 + e^{2i\eta t}\right)$$

we neglect the term $e^{-2i\eta t}$ or $e^{2i\eta t}$ like

$$\cos(\eta t)\left\{(a^\dagger)^2 - a^2\right\} = \cos(\eta t)(a^\dagger)^2 - \cos(\eta t)a^2 \approx \frac{1}{2}\left\{e^{-i\eta t}(a^\dagger)^2 - e^{i\eta t}a^2\right\}.$$

Note that the right hand side is anti–hermitian. See for example [5], [6] for more details on RWA.

As a result our Hamiltonian (3) changes to

$$\widetilde{H} = \omega_0 a^\dagger a + i\frac{\epsilon\eta}{8}\left\{e^{-i\eta t}(a^\dagger)^2 - e^{i\eta t}a^2\right\} = \omega_0 N + i\frac{\epsilon\eta}{8}\left\{e^{-i\eta t}(a^\dagger)^2 - e^{i\eta t}a^2\right\}, \tag{4}$$

which is hermitian. Our aim is to solve the modified Schrödinger equation

$$i\frac{d}{dt}|\Psi(t)\rangle = \widetilde{H}|\Psi(t)\rangle = \widetilde{H}(t)|\Psi(t)\rangle \tag{5}$$

explicitly under the general setting.

3 Method of Solution

In order to solve the equation (5) we use a well–known Lie algebraic method, see for example [7], [8]. If we set

$$K_+ = \frac{1}{2}(a^\dagger)^2, \quad K_- = \frac{1}{2}a^2, \quad K_3 = \frac{1}{2}\left(N + \frac{1}{2}\right) \tag{6}$$

it is not difficult to see both $K_+^\dagger = K_-,\ K_3^\dagger = K_3$ and the $su(1,1)$ relations

$$[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3 \tag{7}$$

by use of the relations (Heisenberg relations)

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a, a^\dagger] = 1 \ \ (\Longleftrightarrow [a^\dagger, a] = -1).$$

In terms of $\{K_+, K_-, K_3\}$ the Hamiltonian (4) can be written as

$$\widetilde{H} = -\frac{\omega_0}{2} + 2\omega_0 K_3 + \frac{i\epsilon\eta}{4}\left(e^{-i\eta t}K_+ - e^{i\eta t}K_-\right). \tag{8}$$

Next, we review the basic $su(1,1)$ relations. If we set $\{k_+, k_-, k_3\}$ as

$$k_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad k_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad k_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

it is easy to check the basic $su(1,1)$ relations

$$[k_3, k_+] = k_+, \quad [k_3, k_-] = -k_-, \quad [k_+, k_-] = -2k_3. \tag{10}$$

Note that $k_+^\dagger = -k_-$ ($\Leftrightarrow K_+^\dagger = K_-$). That is, $\{k_+, k_-, k_3\}$ are generators of the Lie algebra $su(1,1)$ of the non-compact group $SU(1,1)$.

Since $SU(1,1)$ is contained in the special linear group $SL(2;\mathbf{C})$, we **assume** that there exists an infinite dimensional unitary representation $\rho : SL(2;\mathbf{C}) \longrightarrow U(\mathcal{F})$ (group homomorphism[1]) satisfying

$$d\rho(k_+) = K_+, \quad d\rho(k_-) = K_-, \quad d\rho(k_3) = K_3 \tag{11}$$

where $d\rho$ is its differential representation and $\mathcal{F}$ is the Fock space generated by the generators $\{a^\dagger, a, N \equiv a^\dagger a\}$ stated above (see [8] for more details).

Then, since $d\rho$ is linear the main part of (8) becomes

$$\begin{aligned} 2\omega_0 K_3 + \frac{i\epsilon\eta}{4}\left(e^{-i\eta t}K_+ - e^{i\eta t}K_-\right) &= 2\omega_0 d\rho(k_3) + \frac{i\epsilon\eta}{4}\left(e^{-i\eta t}d\rho(k_+) - e^{i\eta t}d\rho(k_-)\right) \\ &= d\rho\left(2\omega_0 k_3 + \frac{i\epsilon\eta}{4}\left(e^{-i\eta t}k_+ - e^{i\eta t}k_-\right)\right) \end{aligned}$$

and we set

$$\widetilde{h} \equiv 2\omega_0 k_3 + \frac{i\epsilon\eta}{4}\left(e^{-i\eta t}k_+ - e^{i\eta t}k_-\right) = \begin{pmatrix} \omega_0 & \frac{i\epsilon\eta}{4}e^{-i\eta t} \\ \frac{i\epsilon\eta}{4}e^{i\eta t} & -\omega_0 \end{pmatrix}. \tag{12}$$

Note that $\widetilde{h}$ is not hermitian.

Here, we would like to solve the (small) Schrödinger–like equation

$$i\frac{d}{dt}|\psi(t)\rangle = \widetilde{h}|\psi(t)\rangle = \widetilde{h}(t)|\psi(t)\rangle. \tag{13}$$

For the purpose we decompose $\widetilde{h} = \widetilde{h}(t)$ into

$$\begin{pmatrix} \omega_0 & \frac{i\epsilon\eta}{4}e^{-i\eta t} \\ \frac{i\epsilon\eta}{4}e^{i\eta t} & -\omega_0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\eta}{2}t} & \\ & e^{i\frac{\eta}{2}t} \end{pmatrix} \begin{pmatrix} \omega_0 & \frac{i\epsilon\eta}{4} \\ \frac{i\epsilon\eta}{4} & -\omega_0 \end{pmatrix} \begin{pmatrix} e^{i\frac{\eta}{2}t} & \\ & e^{-i\frac{\eta}{2}t} \end{pmatrix}$$

and set

$$|\phi(t)\rangle = \begin{pmatrix} e^{i\frac{\eta}{2}t} & \\ & e^{-i\frac{\eta}{2}t} \end{pmatrix}|\psi(t)\rangle \Longleftrightarrow |\psi(t)\rangle = \begin{pmatrix} e^{-i\frac{\eta}{2}t} & \\ & e^{i\frac{\eta}{2}t} \end{pmatrix}|\phi(t)\rangle.$$

[1] In general, to construct a (Lie) group homomorphism is very hard. See some textbook of Lie groups.

Then, it is easy to see

$$i\frac{d}{dt}|\phi(t)\rangle = \begin{pmatrix} \omega_0 - \frac{\eta}{2} & \frac{i\epsilon\eta}{4} \\ \frac{i\epsilon\eta}{4} & -\left(\omega_0 - \frac{\eta}{2}\right) \end{pmatrix} |\phi(t)\rangle \equiv A|\phi(t)\rangle. \tag{14}$$

Since A is time–independent the general solution is given by

$$|\phi(t)\rangle = e^{-itA}|\phi(0)\rangle,$$

and to calculate the term e^{-itA} is relatively easy.

For simplicity we set

$$A = \begin{pmatrix} \omega_0 - \frac{\eta}{2} & \frac{i\epsilon\eta}{4} \\ \frac{i\epsilon\eta}{4} & -\left(\omega_0 - \frac{\eta}{2}\right) \end{pmatrix} \equiv \begin{pmatrix} c & id \\ id & -c \end{pmatrix}, \quad c = \omega_0 - \frac{\eta}{2},\ d = \frac{\epsilon\eta}{4}.$$

Noting

$$A^2 = (c^2 - d^2)1_2, \quad c^2 - d^2 = \left(\omega_0 - \frac{\eta}{2}\right)^2 - \left(\frac{\epsilon\eta}{4}\right)^2$$

(we assume that $c > d$) we have

$$\begin{aligned} e^{-itA} &= \sum_{n=0}^{\infty} \frac{(-it)^{2n}}{(2n)!} A^{2n} + \sum_{n=0}^{\infty} \frac{(-it)^{2n+1}}{(2n+1)!} A^{2n+1} \\ &\equiv \begin{pmatrix} \bar{\alpha} & \beta \\ \beta & \alpha \end{pmatrix} \end{aligned} \tag{15}$$

where

$$\alpha = \alpha(t) = \cos\left(t\sqrt{c^2 - d^2}\right) + i\frac{\sin\left(t\sqrt{c^2 - d^2}\right)}{\sqrt{c^2 - d^2}}c, \quad \beta = \beta(t) = \frac{\sin\left(t\sqrt{c^2 - d^2}\right)}{\sqrt{c^2 - d^2}}d$$

and

$$\frac{\beta}{\alpha} = \frac{\frac{\sin\left(t\sqrt{c^2-d^2}\right)}{\sqrt{c^2-d^2}}d}{\cos\left(t\sqrt{c^2 - d^2}\right) + i\frac{\sin\left(t\sqrt{c^2-d^2}\right)}{\sqrt{c^2-d^2}}c}.$$

Going back to $|\psi(t)\rangle$ from $|\phi(t)\rangle$ we obtain

$$|\psi(t)\rangle = \begin{pmatrix} e^{-i\frac{\eta}{2}t} & \\ & e^{i\frac{\eta}{2}t} \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \beta \\ \beta & \alpha \end{pmatrix} |\psi(0)\rangle = \begin{pmatrix} \bar{\alpha}e^{-i\frac{\eta}{2}t} & \beta e^{-i\frac{\eta}{2}t} \\ \beta e^{i\frac{\eta}{2}t} & \alpha e^{i\frac{\eta}{2}t} \end{pmatrix} |\psi(0)\rangle \tag{16}$$

with (15). Here, note that

$$\begin{vmatrix} \bar{\alpha}e^{-i\frac{\eta}{2}t} & \beta e^{-i\frac{\eta}{2}t} \\ \beta e^{i\frac{\eta}{2}t} & \alpha e^{i\frac{\eta}{2}t} \end{vmatrix} = |\alpha|^2 - \beta^2 = 1 \Longrightarrow \begin{pmatrix} \bar{\alpha}e^{-i\frac{\eta}{2}t} & \beta e^{-i\frac{\eta}{2}t} \\ \beta e^{i\frac{\eta}{2}t} & \alpha e^{i\frac{\eta}{2}t} \end{pmatrix} \in SL(2;\mathbf{C}).$$

Next, let us recall the (well–known) Gauss decomposition of elements in $SL(2;\mathbf{C})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{d} & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{pmatrix} \quad (d \neq 0,\ ad - bc = 1)$$

because

$$\frac{1}{d} + \frac{bc}{d} = \frac{1+bc}{d} = \frac{ad}{d} = a.$$

This formula gives

$$\begin{aligned} \begin{pmatrix} \bar{\alpha}e^{-i\frac{\eta}{2}t} & \beta e^{-i\frac{\eta}{2}t} \\ \beta e^{i\frac{\eta}{2}t} & \alpha e^{i\frac{\eta}{2}t} \end{pmatrix} &= \begin{pmatrix} 1 & \frac{\beta}{\alpha}e^{-i\eta t} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha}e^{-i\frac{\eta}{2}t} & 0 \\ 0 & \alpha e^{i\frac{\eta}{2}t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\beta}{\alpha} & 1 \end{pmatrix} \\ &= \exp\begin{pmatrix} 0 & \frac{\beta}{\alpha}e^{-i\eta t} \\ 0 & 0 \end{pmatrix} \times \\ &\quad \exp\begin{pmatrix} -\log\alpha - i\frac{\eta}{2}t & 0 \\ 0 & \log\alpha + i\frac{\eta}{2}t \end{pmatrix} \times \\ &\quad \exp\begin{pmatrix} 0 & 0 \\ \frac{\beta}{\alpha} & 0 \end{pmatrix} \\ &= \exp\left(\frac{\beta}{\alpha}e^{-i\eta t}k_+\right) \exp\left(-2\left(\log\alpha + i\frac{\eta}{2}t\right)k_3\right) \exp\left(-\frac{\beta}{\alpha}k_-\right) \end{aligned}$$

by (9).

Here, we apply the (group) homomorphism ρ to this case. Since ρ is a group homomorphism (namely, $\rho(ABC) = \rho(A)\rho(B)\rho(C)$) we have

$$
\begin{aligned}
&\rho\left(\left(\begin{array}{cc} \bar{\alpha}e^{-i\frac{\eta}{2}t} & \beta e^{-i\frac{\eta}{2}t} \\ \beta e^{i\frac{\eta}{2}t} & \alpha e^{i\frac{\eta}{2}t} \end{array}\right)\right) \\
= &\ \rho\left(\exp\left(\frac{\beta}{\alpha}e^{-i\eta t}k_+\right)\right)\rho\left(\exp\left(-2\left(\log\alpha + i\frac{\eta}{2}t\right)k_3\right)\right)\rho\left(\exp\left(-\frac{\beta}{\alpha}k_-\right)\right) \\
= &\ \exp\left(\frac{\beta}{\alpha}e^{-i\eta t}d\rho(k_+)\right)\exp\left(-2\left(\log\alpha + i\frac{\eta}{2}t\right)d\rho(k_3)\right)\exp\left(-\frac{\beta}{\alpha}d\rho(k_-)\right) \quad (\Downarrow \text{ by definition}) \\
= &\ \exp\left(\frac{\beta}{\alpha}e^{-i\eta t}K_+\right)\exp\left(-2\left(\log\alpha + i\frac{\eta}{2}t\right)K_3\right)\exp\left(-\frac{\beta}{\alpha}K_-\right) \quad (\Downarrow \text{ by (11)}).
\end{aligned}
$$

This is just the disentangling formula that we are looking for. Note that our derivation is heuristic (not logical) because we have assumed the existence of the Lie group homomorphism ρ, which is not established.

We can **conjecture** the general solution of the equation (5) with (8) as

$$
|\Psi(t)\rangle = e^{i\frac{\omega_0}{2}t}\exp\left(\frac{\beta}{\alpha}e^{-i\eta t}K_+\right)\exp\left(-2\left(\log\alpha + i\frac{\eta}{2}t\right)K_3\right)\exp\left(-\frac{\beta}{\alpha}K_-\right)|\Psi(0)\rangle. \quad (17)
$$

Let us prove this conjecture. For the purpose we set

$$
U(t) = e^{i\frac{\omega_0}{2}t}e^{f(t)K_+}e^{g(t)K_3}e^{h(t)K_-}, \quad U(0) = 1 \text{ (identity)} \quad (18)
$$

with unknown functions $f(t), g(t), h(t)$ $(f(0) = g(0) = h(0) = 0)$. By use of the method developed in [8] or [4] we obtain

$$
i\frac{d}{dt}U(t) = \left\{-\frac{\omega_0}{2} + i(\dot{f} - \dot{g}f + \dot{h}e^{-g}f^2)K_+ + i(\dot{g} - 2\dot{h}e^{-g}f)K_3 + i\dot{h}e^{-g}K_-\right\}U(t) \quad (19)
$$

where we have used the notations like $\frac{df}{dt} = \dot{f}$, etc for simplicity.

If we set

$$
\begin{aligned}
f(t) &= \frac{\beta}{\alpha}e^{-i\eta t} = \frac{\frac{\sin\left(t\sqrt{c^2-d^2}\right)}{\sqrt{c^2-d^2}}d}{\cos\left(t\sqrt{c^2-d^2}\right) + i\frac{\sin\left(t\sqrt{c^2-d^2}\right)}{\sqrt{c^2-d^2}}c}e^{-i\eta t}, \\
g(t) &= -2\left(\log\alpha + i\frac{\eta}{2}t\right) = -2\left\{\log\left(\cos\left(t\sqrt{c^2-d^2}\right) + i\frac{\sin\left(t\sqrt{c^2-d^2}\right)}{\sqrt{c^2-d^2}}c\right) + i\frac{\eta}{2}t\right\}, \\
h(t) &= -\frac{\beta}{\alpha} = -\frac{\frac{\sin\left(t\sqrt{c^2-d^2}\right)}{\sqrt{c^2-d^2}}d}{\cos\left(t\sqrt{c^2-d^2}\right) + i\frac{\sin\left(t\sqrt{c^2-d^2}\right)}{\sqrt{c^2-d^2}}c} \quad (20)
\end{aligned}
$$

with $c = \omega_0 - \frac{\eta}{2}$ and $d = \frac{\epsilon\eta}{4}$, then a long but straightforward calculation shows

$$\begin{aligned}
& -\frac{\omega_0}{2} + i(\dot{f} - \dot{g}f + \dot{h}e^{-g}f^2)K_+ + i(\dot{g} - 2\dot{h}e^{-g}f)K_3 + i\dot{h}e^{-g}K_- \\
= & -\frac{\omega_0}{2} + \frac{i\epsilon\eta}{4}e^{-i\eta t}K_+ + 2\omega_0 K_3 - \frac{i\epsilon\eta}{4}e^{i\eta t}K_- \\
= & -\frac{\omega_0}{2} + 2\omega_0 K_3 + \frac{i\epsilon\eta}{4}\left(e^{-i\eta t}K_+ - e^{i\eta t}K_-\right) \\
= & \widetilde{H}.
\end{aligned}$$

As a result the general solution is given by

$$U(t)|\Psi(0)\rangle = |\Psi(t)\rangle. \tag{21}$$

This concludes the proof.

For example, when $|\Psi(0)\rangle = |0\rangle$ the vacuum state ($a|0\rangle = 0$), some calculation by use of (6) gives

$$\begin{aligned}
|\Psi(t)\rangle &= e^{i\frac{\omega_0}{2}t}\exp\left(\frac{\beta}{\alpha}e^{-i\eta t}K_+\right)\exp\left(-2\left(\log\alpha + i\frac{\eta}{2}t\right)K_3\right)\exp\left(-\frac{\beta}{\alpha}K_-\right)|0\rangle \\
&= \frac{e^{i\frac{c}{2}t}}{\sqrt{\alpha}}\sum_{n=0}^{\infty}\sqrt{{}_{2n}C_n}\left(\frac{\beta}{2\alpha}e^{-i\eta t}\right)^n|2n\rangle
\end{aligned} \tag{22}$$

with $\alpha = \alpha(t), \beta = \beta(t)$ and $c = \omega_0 - \frac{\eta}{2}$ in (15).

4 Reconfirmation

In this section let us reconfirm our method. Our real target is the following "Hamiltonian"

$$h = \begin{pmatrix} \omega_0 & \frac{i\epsilon\eta}{2}\cos(\eta t) \\ \frac{i\epsilon\eta}{2}\cos(\eta t) & -\omega_0 \end{pmatrix}$$

(h is not hermitian) and to solve the (small) Schrödinger–like equation

$$i\frac{d}{dt}|\psi(t)\rangle = h|\psi(t)\rangle = h(t)|\psi(t)\rangle.$$

However, it is very difficult to solve at the present time.

Since

$$\begin{pmatrix} e^{i\frac{\eta}{2}t} & \\ & e^{-i\frac{\eta}{2}t} \end{pmatrix} h \begin{pmatrix} e^{-i\frac{\eta}{2}t} & \\ & e^{i\frac{\eta}{2}t} \end{pmatrix} = \begin{pmatrix} \omega_0 & \frac{i\epsilon\eta}{4} \\ \frac{i\epsilon\eta}{4} & -\omega_0 \end{pmatrix} + \begin{pmatrix} & \frac{i\epsilon\eta}{4}e^{2i\eta t} \\ \frac{i\epsilon\eta}{4}e^{-2i\eta t} & \end{pmatrix}$$

we can neglect the last term by the rotating wave approximation when η is large enough. Therefore, we changed h into $\widetilde{h}$

$$\widetilde{h} = \begin{pmatrix} e^{-i\frac{\eta}{2}t} & \\ & e^{i\frac{\eta}{2}t} \end{pmatrix} \begin{pmatrix} \omega_0 & \frac{i\epsilon\eta}{4} \\ \frac{i\epsilon\eta}{4} & -\omega_0 \end{pmatrix} \begin{pmatrix} e^{i\frac{\eta}{2}t} & \\ & e^{-i\frac{\eta}{2}t} \end{pmatrix} = \begin{pmatrix} \omega_0 & \frac{i\epsilon\eta}{4}e^{-i\eta t} \\ \frac{i\epsilon\eta}{4}e^{i\eta t} & -\omega_0 \end{pmatrix}$$

($\widetilde{h}$ is still not hermitian) and solved the equation

$$i\frac{d}{dt}|\psi(t)\rangle = \widetilde{h}|\psi(t)\rangle = \widetilde{h}(t)|\psi(t)\rangle$$

explicitly.

5 Concluding Remarks

In this paper we treated the Law's effective Hamiltonian of the Dynamical Casimir Effect in a cavity, and considered a kind of rotating wave approximation for the model, and constructed an analytic approximate solution under any initial condition. Our construction is based on some infinite dimensional representation of the Lie algebra $su(1,1)$. We believe that the method is simple, powerful and beautiful. As for related topics see [9], [10] and [11], [12] (the last two are highly recommended).

Our approximate Hamiltonian is still time–dependent. In general, to solve the Schrödinger equation with a time–dependent Hamiltonian explicitly is very hard. As stated in the abstract this is the best analytic approximate solution as far as we know.

What we did in the paper is of course nothing but an intermediate stage because we must detect photon generated by the vacuum state. How do we detect ? How do we construct a unified model containing the detection ? Such a model has been presented by [3] and [4]. In a forthcoming paper (papers) we will apply the result in this paper to the model.

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