Nonlinear $O(3)$ sigma model in discrete complex analysis

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Overview

- $X^d(X^{d+1}), M^d$: oriented Riemannian manifolds
- $\phi : X \to M$: a smooth map (field)
- dv_X: a volume form of X

Sigma model

$$E(\phi) := \int_X |d\phi|^2 dv_X$$

We study **discrete** energy and EL equation of **Nonlinear $O(3)$ sigma model** $\phi : \mathbb{C} \to S^2$ on **discrete Riemann surfaces** for Mercat, Bobenko, and Günther.

Background $d = 4$

Instantons in $\mathbb{R}^4$ (BPST, 1975).

For a principal $G := SU(2)$ bundle over $\mathbb{R}^4$,
its connection A and curvature $F := F_A$,

$$S := \frac{1}{2}\int_{\mathbb{R}^4} |F|^2 dv_{\mathbb{R}^4} = \|F \mp *F\|^2 \pm 8\pi^2 N \geq 8\pi^2 |N|$$

$N \in \pi_3(S^3)$: instanton number.

Satulation $\Leftrightarrow \mathrm{(A)SD} : F = \pm * F$

$\rightsquigarrow$ Instantons

Background $d=3$

Monopoles in $\mathbb{R}^3$('t Hooft, Polyakov, 1975, BPS)

For a principal $G := SU(2)$ bundle P over $\mathbb{R}^3$,
its connection A and curvature $F := F_A$,
ϕ: a section of the adjoint bundle $\mathrm{ad}\, P$,

$$\begin{aligned} E :&= \frac{1}{2}\int_{\mathbb{R}^3}\{|F|^2 + |\nabla\phi|^2\}dv_{\mathbb{R}^3} \\ &= \|F \mp *\nabla\phi\|^2 \pm 4\pi N \geq 4\pi|N| \end{aligned}$$

$N \in \pi_2(S^2)$: monopole charge
Satulation $\Leftrightarrow$ $\mathrm{(A)Bogomol'nyi} : F = \pm * \nabla\phi$
$\rightsquigarrow$ Monopoles

Nonlinear $O(3)$ sigma model in $\mathbb{C}$

- $\phi : \mathbb{C} \to S^2$:
  C^∞ map between Riemannian manifolds
- $N := \int_{\mathbb{C}} \phi^* dv_{S^2} \in \mathbb{Z} \cong \pi_2(S^2)$:
  topological quantum number, (mapping degree)
- $f := \mathrm{St} \circ \phi : \mathbb{C} \to S^2 \to \mathbb{C}$

Belavin-Polyakov, 1975

$$E = \frac{1}{2} \int_{\mathbb{C}} |d\phi|^2 dv_{\mathbb{C}} = \left\| \frac{df \mp i * df}{1 + |f|^2} \right\|^2 \pm 4\pi N \geq 4\pi |N|$$

Satulation $\Leftrightarrow$ $\mathrm{(A)holomorphic} : df = \pm i * df$
$\rightsquigarrow$ $O(3)$ Instantons

Discrete Complex Analysis

1847	Kirchhoff	(preholomorphic)
1941	Isaacs	preholomorphic
1944	Ferrand	preholomorphic
1956	Duffin	discrete analytic function
1981	Berg-Lüscher	topological quantumn number
2001	Smirnov	probability
2001	Mercat	**discrete Riemann surface**
2008	Bobenko-Suris	discrete differential geometry
2017	Bobenko-Günther	**discrete Riemann surface**

References:

1. Mercat: "*Discrete Riemann Surfaces*", Handbook of Teichmüller Theory I, 2007.
2. Smirnov: "*Discrete Complex Analysis and Probability*", Proceedings of ICM, 2010.

Discrete Riemann surface (Λ, ρ_Q, z_Q)

- Λ: a bipartite quadrilateral decomposition on $\Sigma \rightsquigarrow V(\Lambda), E(\Lambda), F(\Lambda) \ni Q \subset \mathbb{C}$
- $\diamondsuit := \Lambda^*$: the dual graph
- X: the medial graph $\leftsquigarrow$ midpoints of $E(\Lambda)$
- $\rho_Q := -i\dfrac{w_+ - w_-}{b_+ - b_-} \in \mathbb{C}$: discrete conformal structure

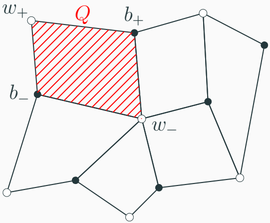


Figure 1: Λ

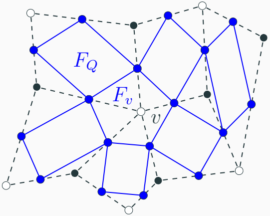


Figure 2: X

Discrete Cauchy-Riemann equation

discrete Cauchy-Riemann equation

$f : V(\Lambda) \to \mathbb{C}$: discrete holomorphic at $Q \in F(\Lambda)$

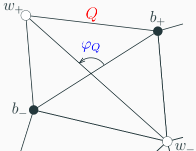
$$\overset{\text{def}}{\Longleftrightarrow} \frac{f(b_+) - f(b_-)}{b_+ - b_-} = \frac{f(w_+) - f(w_-)}{w_+ - w_-}$$

$$\Leftrightarrow f(w_+) - f(w_-) = i\rho_Q \left(f(b_+) - f(b_-)\right)$$

discrete differential operator ∂_Λ

$$\partial_\Lambda f(Q) := \lambda_Q \frac{f(b_+) - f(b_-)}{b_+ - b_-} + \overline{\lambda_Q} \frac{f(w_+) - f(w_-)}{w_+ - w_-},$$

where $2\lambda_Q := \exp\left(-i(\varphi_Q - \frac{\pi}{2})\right) / \sin(\varphi_Q)$



Discrete Cauchy-Riemann equation

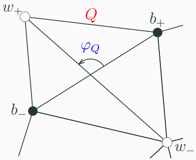
discrete differential operator ∂_Λ

$$\partial_\Lambda f(Q) := \lambda_Q \frac{f(b_+) - f(b_-)}{b_+ - b_-} + \overline{\lambda_Q} \frac{f(w_+) - f(w_-)}{w_+ - w_-},$$

where $2\lambda_Q := \exp\left(-i(\varphi_Q - \frac{\pi}{2})\right) / \sin(\varphi_Q)$

Discrete holomorphicity

f: discrete holomorphic at $Q \Leftrightarrow \overline{\partial_\Lambda} f(Q) = 0$



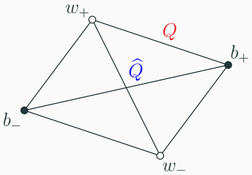
$\because$) $\overline{\partial_\Lambda} f(Q) \times 2i|w_+ - w_-||b_+ - b_-| \sin\varphi_Q =$
$(w_+ - w_-)(f(b_+) - f(b_-)) - (b_+ - b_-)(f(w_+) - f(w_-))$

Examples for discrete holomorphic functions

$\Lambda \subset \mathbb{C}$ (a planar bipartite quadrilateral decomposition)

1. $f(v) := v,\ v \in V(\Lambda)$: discrete holomorphic at $^\forall Q$.
   $\partial_\Lambda f(Q) = \lambda_Q + \overline{\lambda_Q} = 1,\ \overline{\partial_\Lambda} f(Q) = 0.$
2. $f(v) := v^2$: discrete holomorphic at Q
   $\Leftrightarrow b_+ + b_- = w_+ + w_- \Leftrightarrow Q$: parallelogram.
   In this case,

$$\partial_\Lambda f(Q) = \lambda_Q(b_+ + b_-) + \overline{\lambda_Q}(w_+ + w_-)$$
$$= (\lambda_Q + \overline{\lambda_Q})(b_+ + b_-) = 2\,\widehat{Q},\ \text{where } \widehat{Q} := \frac{b_+ + b_-}{2}$$



Discrete 1-form and discrete Hodge theory

- $\omega : E^{\mathrm{or}}(X) \to \mathbb{C}$: a discrete 1-form
  $\overset{\text{def}}{\iff} \omega(-e) = -\omega(e)$ for an oriented edge $e \in E^{\mathrm{or}}(X)$
- $\omega \underset{\text{loc}}{=} p\,dz + q\,d\bar{z}$ on e $\quad \left(\int_e dz = e \right)$

Hodge star operator $\star$ on a 1-form

$$\star\,\omega = \star(p\,dz + q\,d\bar{z}) := -ip\,dz + iq\,d\bar{z}$$

$$\rightsquigarrow \langle \omega, \omega' \rangle = \iint_{F(X)} \omega \wedge \star\overline{\omega'}\text{: Hermitian product}$$

Stokes' theorem

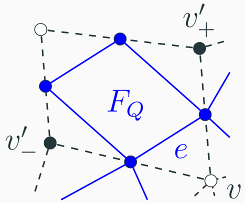
discrete differential operator d

$f : V(\Lambda) \to \mathbb{C}$

$\Rightarrow df := \partial_\Lambda f\, dz + \overline{\partial_\Lambda} f\, d\bar{z}$: discrete 1-form on X

Stokes' theorem

$$\int_e df = \frac{f(v) + f(v'_+)}{2} - \frac{f(v) + f(v'_-)}{2}$$



Period matrix

Theorem (Mercat, 2007)

Discrete period matrix $\to$ continuous one

We concentrate on the **nonliner $O(3)$ sigma model** for **discrete Riemann surfaces in $\mathbb{R}^2 \cong \mathbb{C}$**.

Energy $E^W_\diamondsuit(f)$ and Area $\mathcal{A}^W_\diamondsuit(f)$

- $\Lambda \subset \mathbb{C}$, $f : V(\Lambda) \to \mathbb{C}$, $h : V(\diamondsuit) \to \mathbb{C}$
- $W := |h(Q)|^2 \in \mathbb{R}_{>0}$: a weight (function)
- $E^W_\diamondsuit(f) := 2\langle hdf, hdf\rangle = 2\langle W df, df\rangle \geq 0$
- $\mathcal{A}^W_\diamondsuit(f) := -2i\langle hdf, h \star df\rangle = -2i\langle W df, \star df\rangle$
  The weighted discrete Dirichlet energy and area

$\|h(df \mp i \star df)\|^2 \geq 0$ gives us the formula

Belavin-Polyakov inequality

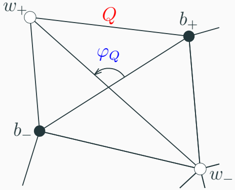
$$E^W_\diamondsuit(f) = \|h(df \mp i \star df)\|^2 \pm \mathcal{A}^W_\diamondsuit(f) \geq |\mathcal{A}^W_\diamondsuit(f)|$$

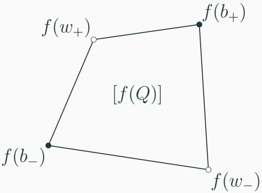
Saturation $\Leftrightarrow$ discrete (anti-)holomorphic $df = \pm i \star df$
$\rightsquigarrow$ instanton

A signed area

- $\text{area}(Q) = \frac{1}{2}\,\text{Im}\left((w_+ - w_-)\overline{(b_+ - b_-)}\right)$
  b_-, w_-, b_+, w_+: counterclockwise order $\Rightarrow \text{area}(Q) > 0$
- $[f(Q)]$: the quadrilateral consisting of the four ordered vertices $f(b_-)$, $f(w_-)$, $f(b_+)$, $f(w_+)$.

Rem f: discrete holomorphic $\Rightarrow \text{area}([f(Q)]) \geq 0$





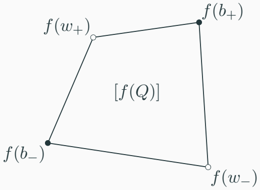
$E_\diamondsuit^W(f)$ and $\mathcal{A}_\diamondsuit^W(f)$ in detail ($\varphi_Q = \pi/2$)

$$E_\diamondsuit^W(f) = 2\langle hdf, hdf\rangle$$
$$= 2\sum_{Q\in V(\diamondsuit)} \iint_{F(X)} hdf \wedge \star\overline{hdf}$$
$$= 4\sum_{Q\in V(\diamondsuit)} \left(\int_e hdf \int_{e^*} \overline{hdf} - \int_{e^*} hdf \int_e \overline{hdf}\right)$$
$$= \cdots \qquad \left(\int_e \star\omega = \cot(\varphi_Q)\int_e \omega - \frac{|e|}{|e^*|\sin(\varphi_Q)}\int_{e^*}\omega\right)$$
$$= \sum_{Q\in V(\diamondsuit)} \frac{W}{\sin(\varphi_Q)}\Big(\rho_Q\big|f(b_+) - f(b_-)\big|^2 + \rho_Q^{-1}\big|f(w_+) - f(w_-)\big|^2$$
$$-2\cos(\varphi_Q)\,\mathrm{Re}\left(f(w_+) - f(w_-)\overline{f(b_+) - f(b_-)}\right)\Big) \geq 0.$$

$E^W_\diamondsuit(f)$ and $\mathcal{A}^W_\diamondsuit(f)$ in detail ($\varphi_Q = \pi/2$)

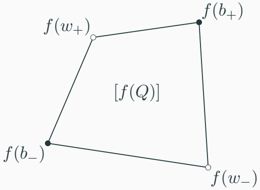
$$E^W_\diamondsuit(f) = \sum_{Q\in V(\diamondsuit)} \frac{W}{\sin(\varphi_Q)}\Big(\rho_Q\big|f(b_+)-f(b_-)\big|^2 + \rho_Q^{-1}\big|f(w_+)-f(w_-)\big|^2 - 2\cos(\varphi_Q)\,\mathrm{Re}\left(f(w_+)-f(w_-)\overline{f(b_+)-f(b_-)}\right)\Big) \geq 0.$$

$$\mathcal{A}^W_\diamondsuit(f) = 4\sum_{Q\in V(\diamondsuit)} W \underbrace{\frac{1}{2}\,\mathrm{Im}\left((f(w_+)-f(w_-))\overline{(f(b_+)-f(b_-))}\right)}_{=:\mathrm{area}([f(Q)])}$$



$E^W_\diamondsuit(f)$ and $\mathcal{A}^W_\diamondsuit(f)$ in detail ($\varphi_Q = \pi/2$)

$$\mathcal{A}^W_\diamondsuit(f) = 4 \sum_{Q \in V(\diamondsuit)} W \underbrace{\frac{1}{2} \operatorname{Im}\left((f(w_+) - f(w_-)) \overline{(f(b_+) - f(b_-))} \right)}_{=:\text{area}([f(Q)])}$$

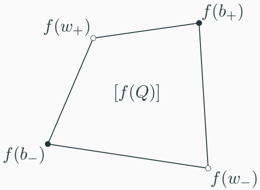


$E^W_\diamondsuit(f)$ and $\mathcal{A}^W_\diamondsuit(f)$ in detail ($\varphi_Q = \pi/2$)

$$\mathcal{A}^W_\diamondsuit(f)$$

$$= 4 \sum_{Q \in V(\diamondsuit)} W \underbrace{\frac{1}{2} \operatorname{Im}\Big((f(w_+) - f(w_-)) \overline{(f(b_+) - f(b_-))} \Big)}_{=:\mathrm{area}([f(Q)])}$$

$$= 4 \sum_{Q \in V(\diamondsuit)} W \, \mathrm{area}([f(Q)])$$



Discrete "area" $\mathcal{A}_{\diamondsuit}^{W}(f)$

- $f : V(\Lambda) \to \mathbb{C}$: a function
- $\mathrm{St} : S^2 \to \mathbb{C}$: Stereographic projection

Weight

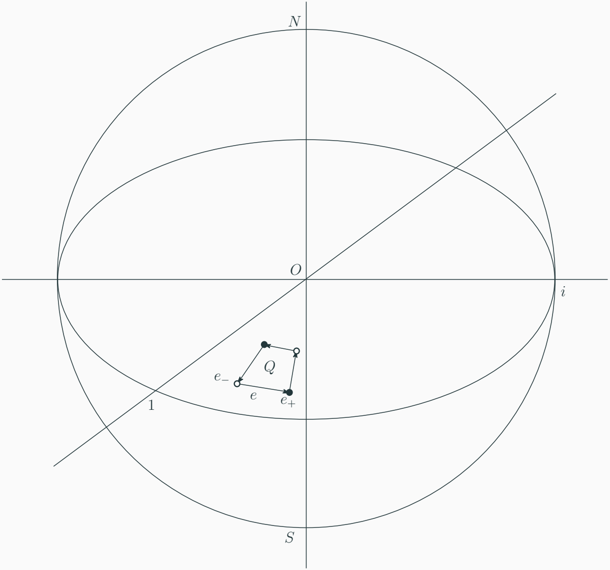
$$W := \frac{1}{4} \frac{\mathrm{area}(\mathrm{St}^{-1}([f(Q)]))}{\mathrm{area}([f(Q)])} \qquad (\mathrm{area}([f(Q)]) \neq 0)$$

Area on S^2

$f : V(\Lambda) \to \mathbb{C}$: discrete (anti-)holomorphic

$$\Rightarrow \mathcal{A}_{\diamondsuit}^{W}(f) = \sum_{Q \in V(\diamondsuit)} \mathrm{area}(\mathrm{St}^{-1}([f(Q)]))$$

Computation of $\mathrm{area}(\mathrm{St}^{-1}([f(Q)]))$



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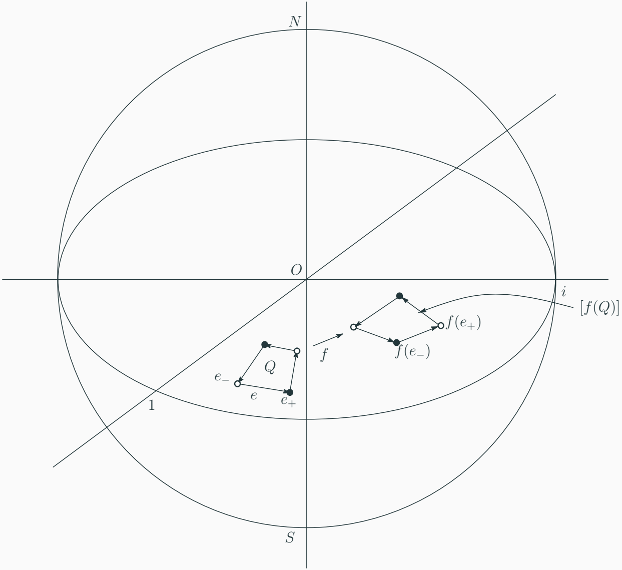
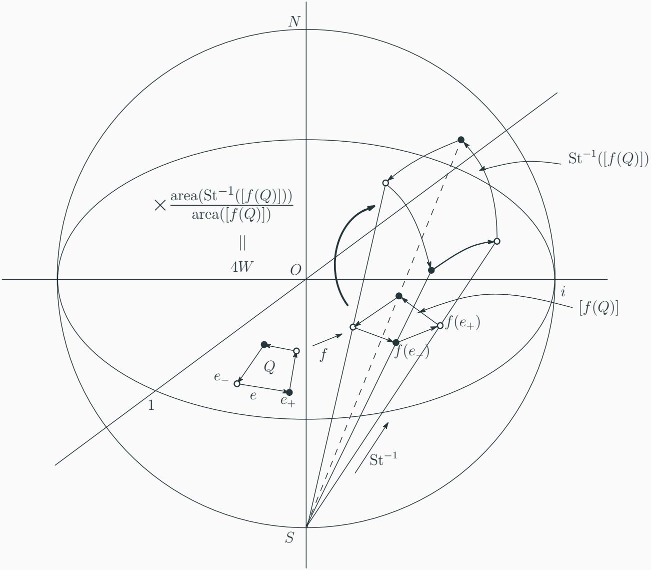
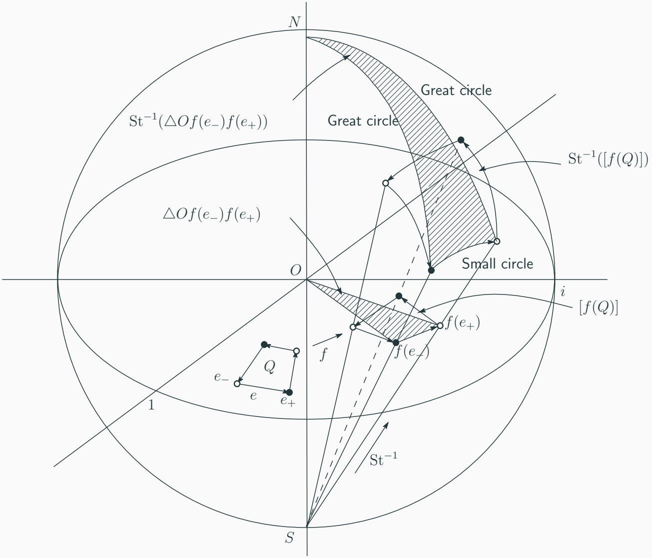


Computation of $\mathrm{area}(\mathrm{St}^{-1}([f(Q)]))$



Computation of $\mathrm{area}(\mathrm{St}^{-1}([f(Q)]))$



Area of a triangle (Cf. Berg and Lüscher, 1981)

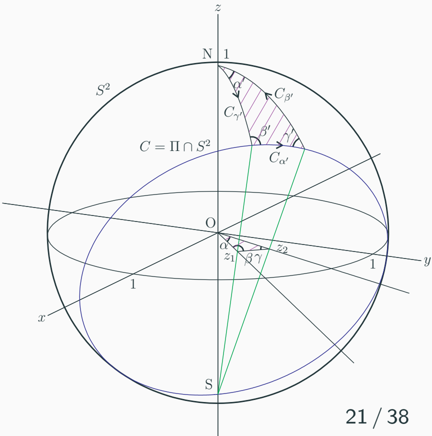
Gauss–Bonnet theorem

$$\int_M K\,dA + \int_{\partial M} \kappa_g\,ds = \iota_1 + \iota_2 + \iota_3 - \pi$$

- K: the Gaussian curvature of M
- κ_g : the geodesic curvature of ∂M
- $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \gamma$
  ($\because$ St: conformal)

$$\operatorname{area}(\mathrm{St}^{-1}(\triangle O z_1 z_2)) = \frac{2\,\mathrm{Im}(z_2\bar{z_1})}{L}\tan^{-1}\frac{L}{1+\mathrm{Re}(z_2\bar{z_1})},$$

where $L^2 = |z_2 - z_1|^2 + [\mathrm{Im}(z_2\bar{z_1})]^2$



Variation

- $f = f_1 + if_2 : V(\Lambda) \to \mathbb{C}$, $v_0 \in V(\Lambda)$: fixed
- $E_\diamondsuit^W := E_\diamondsuit^W(f_1, f_2) = E_\diamondsuit^W(f, \bar{f})$
- $\phi(v) := \delta_{vv_0}$: Kronecker's δ function

$$\frac{\partial E_\diamondsuit^W}{\partial f_1(v_0)} := \frac{d}{dt} E_\diamondsuit^W(f_1 + t\phi, f_2)\Big|_{t=0},$$
$$\frac{\partial E_\diamondsuit^W}{\partial f_2(v_0)} := \frac{d}{dt} E_\diamondsuit^W(f_1, f_2 + t\phi)\Big|_{t=0}$$

Discrete Euler-Lagrange equation

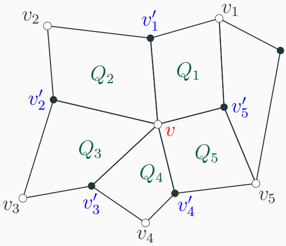
$$0 = -\frac{\partial E}{\partial \bar{f}} = \frac{1}{2}\left(\frac{\partial E}{\partial f_1} + i\frac{\partial E}{\partial f_2}\right)$$

Discrete Euler-Lagrange equation

Discrete EL equation $[\mathrm{EL}]^{\mathrm{disc.}}(v) = 0$ ~~$(\varphi_Q = \pi/2)$~~

$$
\begin{aligned}
[\mathrm{EL}]^{\mathrm{disc.}}(v) &:= -\frac{\partial E_{\diamond}^{W}}{\partial \bar{f}(v)}(f, \bar{f}) \\
&= \sum_{Q_s \sim v} \frac{e_{\diamond}([f(Q_s)])}{4\,\mathrm{area}([f(Q_s)])} \left(\frac{\partial F_s}{\partial \bar{f}(v)} - \frac{\partial F_{s-1}}{\partial \bar{f}(v)} \right) \\
&\quad + \sum_{Q_s \sim v} 2i \frac{W([f(Q_s)])\mathrm{area}(Q_s)}{\mathrm{area}([f(Q_s)])} \overline{(f(v'_s) - f(v'_{s-1}))}\, \partial_{\Lambda} f(Q_s)\, \overline{\partial_{\Lambda}} f(Q_s)
\end{aligned}
$$

- $Q_s \sim v$: incident quadrilaterals
- $e_{\diamond}([f(Q_s)]) :=$
  $\frac{1}{\sin(\varphi_Q)} \Big(\rho_s |f(v_s) - f(v)|^2 + \rho_s^{-1} |f(v'_s) - f(v'_{s-1})|^2$
  $-2\cos(\varphi_Q)\,\mathrm{Re}\left(f(w_+) - f(w_-) \overline{f(b_+) - f(b_-)} \right) \Big)$
- $F_s := \mathrm{area}(\mathrm{St}^{-1}(\triangle O f(v) f(v'_s)))$



Solutions of discrete EL equation

Solutions of discrete EL equation

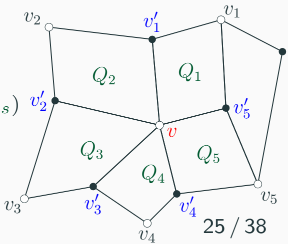
f: discrete (anti-)holomorphic $\Rightarrow$ $[\mathrm{EL}]^{\mathrm{disc.}}(v) = 0$

Outline of proof

- $\overline{\partial_\Lambda} f(Q_s) = 0$ or $\partial_\Lambda f(Q_s) = 0$
- $e_\diamondsuit([f(Q_s)]) = \pm 4\,\mathrm{area}([f(Q_s)])$

$$\rightsquigarrow [\mathrm{EL}]^{\mathrm{disc.}}(v) = \pm \sum_{Q_s \sim v} \left(\frac{\partial F_s}{\partial \bar{f}}(v) - \frac{\partial F_{s-1}}{\partial \bar{f}}(v) \right) = 0 \quad \square$$

$$[\mathrm{EL}]^{\mathrm{disc.}}(v) = \sum_{Q_s \sim v} \frac{e_\diamondsuit([f(Q_s)])}{4\,\mathrm{area}([f(Q_s)])} \left(\frac{\partial F_s}{\partial \bar{f}(v)} - \frac{\partial F_{s-1}}{\partial \bar{f}(v)} \right)$$
$$+ \sum_{Q_s \sim v} 2i \frac{W([f(Q_s)])\mathrm{area}(Q_s)}{\mathrm{area}([f(Q_s)])} \overline{(f(v_s') - f(v_{s-1}'))}\, \partial_\Lambda f(Q_s)\, \overline{\partial_\Lambda} f(Q_s)$$



Rhombic lattice $\Lambda_{2a,2b}$ in $\mathbb{C}$

$$(\Lambda, \rho) = (\Lambda_{2a,2b}, b/a)$$

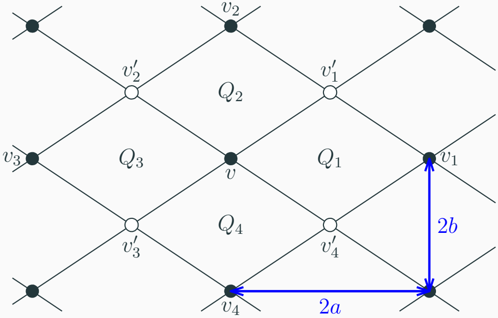


Figure 3: $\Lambda_{2a,2b}$

$$\mathrm{area}(Q_i) = 2ab$$

Discrete exponential function

Disc. exp. function $\exp(\lambda, z; z_0)$ on $\Lambda_{2a,2b}$ at $z_0 = 0$

$$\exp(\lambda, z; 0)$$

$$:= \left(\frac{1 + \frac{\lambda}{2}(a+bi)}{1 - \frac{\lambda}{2}(a+bi)}\right)^{\frac{x}{2a}+\frac{y}{2b}} \left(\frac{1 + \frac{\lambda}{2}(a-bi)}{1 - \frac{\lambda}{2}(a-bi)}\right)^{\frac{x}{2a}-\frac{y}{2b}}$$

$$\to e^{\lambda z} \quad \left(\sqrt{a^2+b^2} \to +0\right)$$

$$(z := x + iy \in V(\Lambda), \quad \lambda \neq \pm 2/(a \pm bi) \in \mathbb{C})$$

Discrete power function $z^{(N)}$

$\exp(\lambda, z; 0) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} z^{(N)}$ gives us

$$z^{(0)} = 1$$
$$z^{(1)} = z$$
$$z^{(2)} = z^2$$

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$$z^{(1)} = z$$

$$z^{(2)} = z^2$$

$$z^{(3)} = z^3 + (a^2 - b^2)z - \frac{1}{2}(a^2 + b^2)\bar{z},$$

Discrete power function $z^{(N)}$

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$$z^{(0)} = 1$$

$$z^{(1)} = z$$

$$z^{(2)} = z^2$$

$$z^{(3)} = z^3 + (a^2 - b^2)z - \frac{1}{2}(a^2 + b^2)\bar{z},$$

$$z^{(4)} = z^4 + 4(a^2 - b^2)z^2 - 2(a^2 + b^2)|z|^2.$$

$$z^{(5)} = z^5 + 10(a^2 - b^2)z^3 - 3(a^4 - b^4)\bar{z} + \tfrac{3}{2}(3a^4 - 10a^2b^2 + 3b^4)z - 5(a^2 + b^2)|z|^2 z.$$

Discrete power function $z^{(N)}$

$z^{(0)} = 1 \qquad \exp(\lambda, z; 0) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} z^{(N)}$

$z^{(1)} = z$

$z^{(2)} = z^2$

$$z^{(3)} = z^3 + (a^2 - b^2)z - \frac{1}{2}(a^2 + b^2)\bar{z},$$

$$z^{(4)} = z^4 + 4(a^2 - b^2)z^2 - 2(a^2 + b^2)|z|^2.$$

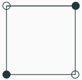
$$z^{(5)} = z^5 + 10(a^2 - b^2)z^3 - 3(a^4 - b^4)\bar{z} + \frac{3}{2}(3a^4 - 10a^2b^2 + 3b^4)z - 5(a^2 + b^2)|z|^2 z.$$

Facts

- $z^{(M)} z^{(N)} \neq z^{(M+N)}$
- $z^{(N)}(z)\big|_{a=b=1/\sqrt{2}} = e^{-i\frac{N\pi}{4}} z_F^{(N)}\left(e^{i\frac{\pi}{4}} z\right)$

$z_F^{(N)}$ is defined on $\mathbb{Z}^2$ (Ferrand, 1944)

$\mathbb{Z}^2$



Proof of $\mathcal{A}^W_\diamondsuit(z^{(N)}) = 4\pi N$ 1/3

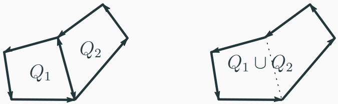
topological number

$$\mathcal{A}^W_\diamondsuit(z^{(N)}) = 4\pi N$$

Proof of $\mathcal{A}_{\diamondsuit}^{W}(z^{(N)}) = 4\pi N$

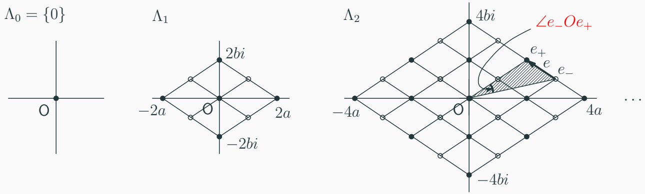
Assume Λ finite

$$\begin{aligned}
\mathcal{A}_{\diamondsuit}^{W}(f) &= \sum_{Q \in V(\diamondsuit)} \operatorname{area}(\mathrm{St}^{-1}([f(Q)])) \\
&= \sum_{Q \in V(\diamondsuit)} \sum_{e \in E(\partial Q)} \operatorname{area}(\mathrm{St}^{-1}(\triangle O f(e_-) f(e_+))) \\
&= \sum_{e \in E(\partial \Lambda)} \operatorname{area}(\mathrm{St}^{-1}(\triangle O f(e_-) f(e_+)))
\end{aligned}$$



Proof of $\mathcal{A}^W_\diamondsuit(z^{(N)}) = 4\pi N$ 2/3

a famliy of sublattices $\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_n \subset \cdots \subset \Lambda$



$$\Lambda_n := \left\{ z = x + iy \in \Lambda \,\middle|\, \left|\frac{x}{2a} + \frac{y}{2b}\right| \leq n,\ \left|\frac{x}{2a} - \frac{y}{2b}\right| \leq n \right\}$$

$z^{(N)} \sim z^N$ $(n \to \infty)$, $z \in V(\partial\Lambda_n) \rightsquigarrow$

$$\mathrm{area}(\mathrm{St}^{-1}(\triangle O z^{(N)}(e_-) z^{(N)}(e_+))) \xrightarrow{\text{Gauss–Bonnet}} 2\arg(e_+^N \overline{e_-^N})$$

$$= 2N \angle e_- O e_+$$

Proof of $\mathcal{A}^W_\diamondsuit(z^{(N)}) = 4\pi N$ 3/3

$$\begin{aligned}
\mathcal{A}^W_\diamondsuit(z^{(N)}) &= \lim_{n\to\infty} \mathcal{A}^W_{\diamondsuit_n}(z^{(N)}), \quad \diamondsuit_n := \Lambda_n^* \\
&= \lim_{n\to\infty} \sum_{e\in E(\partial\Lambda_n)} \operatorname{area}(\mathrm{St}^{-1}(\triangle O z^{(N)}(e_-) z^{(N)}(e_+))) \\
&= \lim_{n\to\infty} \sum_{e\in E(\partial\Lambda_n)} 2N\angle e_- O e_+ \\
&= 2N \cdot 2\pi = 4\pi N \quad \left(\sum_{e\in E(\partial\Lambda_n)} \angle e_- O e_+ = 2\pi \right)
\end{aligned}$$

“Continuous” EL equation

- $\phi : \mathbb{C} \to S^2$: a C^∞ map
- $\mathrm{St} : S^2 \to \mathbb{C}^2$: Stereographic projection
- $f := \mathrm{St} \circ \phi : \mathbb{C} \to S^2 \to \mathbb{C}$
- $E(\phi)$: the energy for nonlinear $O(3)$ sigma model

EL equation $[\mathrm{EL}]^{\mathrm{cont.}} = 0$ for $E(\phi)$

$$[\mathrm{EL}]^{\mathrm{cont.}} := -\frac{\partial E}{\partial \bar{f}}$$
$$= \frac{4}{(1+|f|^2)^2}\left(\partial_z \partial_{\bar{z}} f - \frac{2\bar{f}}{1+|f|^2}\partial_z f \cdot \partial_{\bar{z}} f\right)$$

For (anti-)holomorphic funcion f, $\Delta = 4\partial_z\partial_{\bar{z}}$ induces

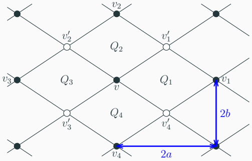
$$[\mathrm{EL}]^{\mathrm{cont.}} = 0 \quad \text{if} \quad \partial_{\bar{z}} f = 0 \quad (\partial_z f = 0)$$

Continuous limits of $E^W_\diamondsuit,\ \mathcal{A}^W_\diamondsuit,\ [\mathrm{EL}]^{\mathrm{disc.}}$

Continuous limits of our model

- $\Lambda = \Lambda_{2a,2b}$: A rhombic lattice, $v \in V(\Lambda)$
- $\rho = b/a$: fixed, $a \to +0$ $(b \to +0)$

1. $E^W_\diamondsuit(f) \to 2\int_{\mathbb{R}^2} \frac{|\partial_x f|^2 + |\partial_y f|^2}{(1+|f|^2)^2} dv_{\mathbb{C}} = \frac{1}{2}\int_{\mathbb{C}} |d\phi|^2 dv_{\mathbb{C}} = E$
2. $\mathcal{A}^W_\diamondsuit(f) \to 4\int_{\mathbb{C}} \frac{\mathrm{Im}(\partial_y f \cdot \partial_x \bar{f})}{(1+|f|^2)^2} dv_{\mathbb{C}} = \int_{\mathbb{C}} \phi^*(dv_{S^2}) = \mathcal{A}$,
3. $[\mathrm{EL}]^{\mathrm{disc.}}(v)/(2ab) \to [\mathrm{EL}]^{\mathrm{cont.}}(v)$



Summary

- Nonlinear $O(3)$ sigma model in discrete complex analysis
- Discrete energy, area and an inequality
- Their continuous limits for rhombic lattices
- Discrete power functions for rhombic lattices and their topological number

Question

What about $\Lambda \subset \Sigma_g$ $(g \geq 1)$?

Thank you very much!

Reference:

1. *Nonlinear $O(3)$ sigma model in discrete complex analysis*, arXiv:1602.08923v5? [hep-th], 2026