The fundamental 2-form on a LCK solvmanifold

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Introduction

Definition 1

Let G be a simply-connected solvable Lie group (nilpotent Lie group), and Γ be a lattice, that is, a discrete co-compact subgroup of G. We call a compact manifold $\Gamma \backslash G$ a *solvmanifold (nilmanifold)*.

Theorem 1 (Hasegawa '06)

A Kähler solvmanifold is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus.

Definition 2

Let (M, g, J) be a Hermitian manifold, Ω be the fundamental 2-form $(\Omega(X, Y) := g(X, JY))$. (M, g, J) is called *locally conformal Kähler (LCK)*, if there exists a closed 1-form ω such that

$$d\Omega = \omega \wedge \Omega.$$

The closed 1-form ω is called *Lee form*.

Remark 1

If $\omega = df$, *then* $(M, e^{-f}g, J)$ *is Kähler.*

Definition 3

Let α be a closed 1-form on M.
We define the new differential operator from $A^p(M)$ to $A^{p+1}(M)$ defined by

$$d_\alpha \eta := \alpha \wedge \eta + d\eta$$

We easily see that $d_\alpha^2 = 0$, because α is closed.

Definition 4

- A p-form η is called *α-closed*, if $d_\alpha \eta = 0$.
- A p-form η is called *α-exact*, if $\eta = d_\alpha \theta$.

Let (M, g, J) be a LCK manifold with Lee form ω:

$$d\Omega = \omega \wedge \Omega \iff -\omega \wedge \Omega + d\Omega = 0 \iff d_{-\omega}\Omega = 0, \text{ that is., } \Omega \text{ is } -\omega\text{-closed.}$$

Examples of a LCK manifold

Example 1

- Hopf manifold (Vaisman, '79)
- Inoue surfaces (Tricerri, '82)
- Kodaira-Thurston manifold (Fernández etal, '86)
- O-T manifold (Oeljeklaus-Toma, '05)

Remark 2

- *Kodaira-Thurston manifold is a nilmanifold.*
- *Inoue surfaces and O-T manifold are solvmanifolds.*

Vaisman manifold

Definition 5

Let (M, g, J) be a LCK manifold with Lee form ω.
(M, g, J) is called a *Vaisman manifold*, if $\nabla\omega = 0$.

Example 1

- Hopf manifold and Kodaira-Thurston manifold are Vaisman manifolds.
- Inoue surfaces and O-T manifold are non-Vaisman LCK manifolds.

Theorem 2 (Tsukada, '94)

The fundamental 2-form Ω of a Vaisman manifold is given by

$$\Omega = d_{-\omega}(-\omega \circ J) = \omega \wedge \omega \circ J + d(-\omega \circ J).$$

LCK solvmanifold and its solvable Lie algebra

Let $(\Gamma\backslash G, g, J)$ be a LCK solvmanifold with Lee form ω and $\mathfrak{g}$ be the solvable Lie algebra of G. In this talk, we assume that

- J is left-invariant.
- There exits a left-invariant closed 1-form ω_0 such that $\omega_0 - \omega = df$.

Then, for ${}^\forall X, Y \in \mathfrak{g}$,

$$\langle X, Y \rangle := \int_{\Gamma\backslash G} e^f g(X, Y) d\mu,$$

where $d\mu$ is induced by a bi-invariant volume form on G.

Proposition 1 (Belgun, '00)

$(\langle\ ,\ \rangle, J)$ *is a left-invariant LCK structure with Lee form* ω_0.

We call $(\mathfrak{g}, \langle\ ,\ \rangle, J)$ a *LCK (Vaisman) solvable Lie algebra*.

Example 1

- Kodaira-Thurston manifold : $\mathbb{R} \times \mathfrak{h}(n)$,
  where $\mathfrak{h}(n)$ is a $(2n+1)$-dimensional Heisenberg Lie algebra (Vaisman).
- Inoue surface S^0 : $\mathbb{R} \ltimes \mathbb{R}^3 \to$ O-T manifold : $\mathbb{R}^n \ltimes \mathbb{R}^{n+2}$ (non-Vaisman).
- Inoue surface S^+ : $\mathbb{R} \ltimes \mathfrak{h}(1)$ (non-Vaisman).

Remark 3 (S. '07)

If a nilpotent Lie algebra $\mathfrak{g}$ has a LCK structure, then $\mathfrak{g}$ is $\mathbb{R} \times \mathfrak{h}(n)$ (Vaisman).

Proposition 2

Let $\mathfrak{g}$ be a non-nilpotent solvable Lie algebra such that $\mathfrak{g} = \mathbb{R}^n \ltimes \mathbb{R}^m$.

- *$\mathfrak{g}$ has no Vaisman structures (Kasuya, '13).*
- *If $\mathfrak{g}$ has a non-Vaisman LCK structure, then $\mathfrak{g} = \mathbb{R}^n \ltimes \mathbb{R}^{n+2}$ (Kanda).*

Case of a Vaisman structure

Kodaira-Thurston manifold

Let $\mathfrak{n}$ be the $(2n+2)$-dimensional nilpotent Lie algebra defined by

$$\mathfrak{n} = \text{ span } \{A, X_i, Y_i, Z\}$$
$$[X_i, Y_i] = Z$$

- Let J be a complex structure defined by $JA = Z, JX_i = Y_i$.
- Let $\{\alpha, x_i, y_i, z\}$ be the dual basis of $\{A, X_i, Y_i, Z\}$. Take the 2-form Ω_0 given by

$$\Omega_0 = -\alpha \wedge z - \sum_i x_i \wedge y_i.$$

Then, $(\Gamma\backslash N, \Omega_0, J)$ is a Vaisman nilmanifold with Lee form $\omega_0 = \alpha$. Moreover, we see that

$$\Omega_0 = -\alpha \wedge z + dz = d_{-\alpha} z = d_{-\omega_0}(-\omega_0 \circ J).$$

Let $(\mathfrak{g}, \langle\ ,\ \rangle, J)$ be a $2n$-dimensional LCK solvable Lie algebra with Lee form ω_0.

Theorem 3 (S. '12)

If $\Omega_0 = d_{-\omega_0}(-\omega_0 \circ J) = \omega_0 \wedge \omega_0 \circ J - d(\omega_0 \circ J)$, *then* $(\mathfrak{g}, \langle\ ,\ \rangle, J)$ *is Vaisman.*

Let $(\ ,\ )$ be an inner product on $\wedge^* \mathfrak{g}$ induced by $\langle\ ,\ \rangle$.

Theorem 4 (S. '15)

We may assume that $\|\omega_0\| = 1$.

$$(\Omega_0, d_{-\omega_0}(-\omega_0 \circ J))^2 \leq (\Omega_0, \Omega_0) \cdot (d_{-\omega_0}(-\omega_0 \circ J), d_{-\omega_0}(-\omega_0 \circ J))$$

$\iff 0 \leq \langle [A, JA], JA \rangle$, *where* $A = \omega_0^{\#}$.

Corollary 1

$(\mathfrak{g}, \langle\ ,\ \rangle, J)$ *is Vaisman* $\iff \langle [A, JA], JA \rangle = 0$.

Theorem (Andrada-Origlia ‘20, S. ‘21)

A solvable Lie algebra with a Vaisman structure is given by $\mathbb{R}^l \ltimes (\mathfrak{h}(n) \times \mathbb{R}^k)$, where $l + 2n + k + 1$ is even.

Moreover, we have

- A solvable Lie algebra with a Vaisman structure has no non-Vaisman LCK structures (S, ‘21).

Corollary 2 (cf. Belgun, ‘00, Kasuya, ‘13)

Inoue surfaces and O-T manifold have no Vaisman structures.

- A classification of a 6-dimensional Vaisman solvable Lie algebra (solvmanifold) (Andrada-Origlia, ‘20).
- A complex strcuture on a solvable Lie algebra with a Vaisman structure (Cortés-Hasegawa, ‘24).

Case of a non-Vaisman LCK structure (Main Result)

- We may assume that $\|\omega_0\| = 1$
- $p := \langle [A, JA], JA \rangle > 0$, where $A = \omega_0^{\#}$.
- Let $\mathfrak{n}$ be the maximal nilpotent ideal of $\mathfrak{g}$. Then, $\omega_0(\mathfrak{n}) = 0$ (S. '24).

Let $F_{d(\omega_0 \circ J)}$ be the linear map from $\mathfrak{g}$ to $\mathfrak{g}$ defined by

$$F_{d(\omega_0 \circ J)}(X) = (i(X)d(\omega_0 \circ J))^{\#}.$$

We have the orthogonal decomposition:

$$\mathfrak{g} = \mathrm{Im}\ F_{d(\omega_0 \circ J)} \oplus \mathrm{Ker}\ F_{d(\omega_0 \circ J)}$$

Lemma 1

- *Im $F_{d(\omega_0 \circ J)}$ is J-invariant.*
- *Ker $F_{d(\omega_0 \circ J)}$ is a J-invariant subalgebra.*

$\because$ Since ω_0 is closed, $d(\omega_0 \circ J)$ is J-invariant.

We decompose the fundamental 2-form Ω_0:

$$\Omega_0 = \Omega_0|_{\mathrm{Im}\ F_{d(\omega_0 \circ J)}} + \Omega_0|_{\mathrm{Ker}\ F_{d(\omega_0 \circ J)}} := \Omega_1 + \Omega_2.$$

Main result (S.)

$\Omega_1 = \dfrac{k}{n} d_{-\omega_0}(-\omega_0 \circ J)$, where $2k = \mathrm{rank} F_{d(\omega_0 \circ J)}$.

Corollary 3

$d\Omega_2 = \omega_0 \wedge \Omega_2$

∵ Since $d\Omega_0 = \omega_0 \wedge \Omega_0$ and $d\Omega_1 = \omega_0 \wedge \Omega_1$.

Corollary 4

(Ker $F_{d(\omega_0 \circ J)}, \Omega_2$) is Kähler and unimodular, that is., Ker $F_{d(\omega_0 \circ J)}$ is meta-abelian.

∵ Hano, ‘57.

Corollary 5

Im $F_{d(\omega_0 \circ J)}$ is a subalgebra.

Examples of Main Theorem

Inoue surface S^0

Let $\mathfrak{g}^0$ be the 4-dimensional solvable Lie algebra defined by

$$\mathfrak{g}^0 = \text{ span } \{A, X, Y_1, Y_2\}$$
$$[A, X] = 2X, [A, Y_1] = -Y_1 + cY_2, [A, Y_2] = -Y_2 - cY_1.$$

- Let J be a complex structure defined by $JA = X, JY_1 = Y_2$.
- Let $\{\alpha, x, y_i\}$ be the dual basis of $\{A, X, Y_i\}$. Take the 2-form Ω_0 given by

$$\Omega_0 = -4\alpha \wedge x - 4y_1 \wedge y_2 = \frac{1}{2} d_{-2\alpha}(-2\alpha \circ J) - 4y_1 \wedge y_2.$$

- Then, $(\Gamma \backslash G^0, \Omega_0, J)$ is a non-Vaisman LCK solvmanifold with Lee form $\omega_0 = 2\alpha$ such that $\|\omega_0\| = 1$.

6-dimensional O-T manifold

Let $\mathfrak{g}^{\text{O-T}}$ be the 6-dimensional solvable Lie algebra defined by

$$\mathfrak{g}^{\text{O-T}} = \text{ span } \{A_1, A_2, X_1, X_2, Y_1, Y_2\}$$
$$[A_i, X_1] = 2X_1, [A_i, Y_1] = -Y_1 + c_i Y_2, [A_i, Y_2] = -Y_2 - c_i Y_1 \ (i = 1, 2\ )$$

- Let J be a complex structure defined by $JA_i = X_i, JY_1 = Y_2$.
- Let $\{\alpha_i, x_i, y_i\}$ be the dual basis of $\{A_i, X_i, Y_i\}$. Take the 2-form Ω_0 given by

$$\begin{aligned}\Omega_0 &= -\frac{16}{3}(\alpha_1 \wedge x_1 + \alpha_2 \wedge x_2) - \frac{8}{3}(\alpha_1 \wedge x_2 + \alpha_2 \wedge x_1) - \frac{8}{3} y_1 \wedge y_2 \\ &= \frac{2}{3} d_{-2(\alpha_1+\alpha_2)}(-2(\alpha_1 + \alpha_2) \circ J) - \frac{8}{3} y_1 \wedge y_2.\end{aligned}$$

- Then, $(\Gamma \backslash G^{\text{O-T}}, \Omega_0, J)$ is a non-Vaisman LCK solvmanifold with Lee form $\omega_0 = 2(\alpha_1 + \alpha_2)$ such that $\|\omega_0\| = 1$.

Inoue surface S^+

Let $\mathfrak{g}^+$ be the 4-dimensional solvable Lie algebra defined by

$$\begin{gathered}\mathfrak{g}^+ = \text{ span }\{A, X, Y, Z\} \\ [A, X] = X, [A, Y] = -Y, [X, Y] = Z.\end{gathered}$$

- Let J be a complex structure defined by $JA = X, JZ = Y$.
- Let $\{\alpha, x, y, z\}$ be the dual basis of $\{A, X, Y, Z\}$. Take the 2-form Ω_0 given by

$$\begin{aligned}\Omega_0 &= -\alpha \wedge x - z \wedge y \\ &= \frac{1}{2} d_{-\alpha}(-\alpha \circ J) - z \wedge y.\end{aligned}$$

- Then, $(\Gamma \backslash G^+, \Omega_0, J)$ is a non-Vaisman LCK solvmanifold with Lee form $\omega_0 = \alpha$ such that $\|\omega_0\| = 1$.

Remark 4 (S. '24)

Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra. If $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{n}$ has a non-Vaisman LCK structure, then $\mathfrak{g}$ is $\mathfrak{g}^+$ as above.

Outline of Proof

Proposition 3

$[A, JA] \in$ span$\{JA\}$, that is., $[A, JA] = pJA$.

$\because$

- Sincce the linear map $J \circ \mathrm{ad}(JA)$ is symmetric with respect to $\langle\ ,\rangle$, we have the eigen-decomposition: $\mathfrak{g} = \sum \mathfrak{g}_\lambda$
- Let $\{A, JA, \boldsymbol{e}_i, J\boldsymbol{e}_i\}$ be an orthogonal basis of $(\langle\ ,\ \rangle, J)$. Then, $(n-1)JA = [A, JA] + \sum[\boldsymbol{e}_i, J\boldsymbol{e}_i]$, that is, $(n-1)A = J[JA, A] - \sum J[\boldsymbol{e}_i, J\boldsymbol{e}_i]$.
- For $X \in \mathfrak{g}_\lambda \cap (\mathrm{span}\{A\})^\perp$, we have

$$\begin{aligned} 0 = (n-1)\langle A, X\rangle &= \langle J[JA, A], X\rangle - \langle \sum J[\boldsymbol{e}_i, J\boldsymbol{e}_i], X\rangle \\ &= \langle A, J[JA, X]\rangle - \langle \sum J[\boldsymbol{e}_i, J\boldsymbol{e}_i], X\rangle \\ &= \langle A, \lambda X\rangle - \langle \sum J[\boldsymbol{e}_i, J\boldsymbol{e}_i], X\rangle = -\langle \sum J[\boldsymbol{e}_i, J\boldsymbol{e}_i], X\rangle. \end{aligned}$$

- Therefore, $[A, JA] \in$ span$\{JA\}$.

Corollary 6

$d(\omega_0 \circ J) = -p\omega_0 \wedge \omega_0 \circ J + \eta_0$, *where* $i(A)\eta_0 = i(JA)\eta_0 = 0$.

$\because$ Since $[A, JA] \in \mathrm{span}\{JA\}$ and $d\Omega_0 = \omega_0 \wedge \Omega_0$.

Corollary 7

$A, JA \in$ *Im* $F_{d(\omega_0 \circ J)}$.

Proposition 4

$p = \dfrac{n}{k} - 1$, *that is.,* $[A, JA] = (\dfrac{n}{k} - 1)JA$, *where* $2k =$ *rank*$F_{d(\omega_0 \circ J)}$.

$\because$ Since the unimodular condition, we see that

$$\begin{aligned} 0 \neq \Omega_0^{n-k} \wedge (d(\omega_0 \circ J))^k &= \Omega_0^{n-k} \wedge d(\omega_0 \circ J \wedge (d(\omega_0 \circ J))^{k-1}) \\ \Omega_2^{n-k} \wedge (d(\omega_0 \circ J))^k &= -d\Omega_0^{n-k} \wedge \omega_0 \circ J \wedge (d(\omega_0 \circ J))^{k-1} \\ &= -(n-k)\Omega_0^{n-k} \wedge \omega_0 \wedge \omega_0 \circ J \wedge (d(\omega_0 \circ J))^{k-1} \\ &= -(n-k)\Omega_2^{n-k} \wedge \omega_0 \wedge \omega_0 \circ J \wedge (d(\omega_0 \circ J))^{k-1}. \end{aligned}$$

We consider the following Schwarz's inequality:

$$\begin{aligned}
(\Omega_1, d_{-\omega_0}(-\omega_0 \circ J))^2 &\leq (\Omega_1, \Omega_1) \cdot (d_{-\omega_0}(-\omega_0 \circ J), d_{-\omega_0}(-\omega_0 \circ J)) \\
(\Omega_0, d_{-\omega_0}(\omega_0 \circ J))^2 &\leq (\Omega_1, \Omega_1) \cdot (d_{-\omega_0}(\omega_0 \circ J), d_{-\omega_0}(\omega_0 \circ J)) \\
n^2 &\leq k \cdot n(p+1) \ \ \text{(S. `15)} \\
&= k \cdot n(\frac{n}{k} - 1 + 1) = n^2.
\end{aligned}$$

Then the equation holds.

Moreover, we have $\Omega_1 = \dfrac{k}{n} d_{-\omega_0}(-\omega_0 \circ J)$ from $\Omega_0(JA, A) = 1$.

Future Work

$$\begin{array}{ccccc} \Omega_0 = \Omega_0|_{\text{Im } F_{d(\omega_0\circ J)}} + \Omega_0|_{\text{Ker } F_{d(\omega_0\circ J)}} & = & \dfrac{k}{n} d_{-\omega_0}(-\omega_0 \circ J) & + & \Omega_0|_{\text{Ker } F_{d(\omega_0\circ J)}} \\ & & \text{(Vaisman)} & + & \text{(Kähler)} \end{array}$$

- Case of rank $F_{d(\omega_0\circ J)} = 2$, we can prove that $\dim \mathfrak{g} = 4$, that is, S^0 or S^+.
- Case of $\mathfrak{g}$ is one-dimensional extension of $\mathfrak{n}$. We take the descending central series:

$$\mathfrak{n} \supset \mathfrak{n}^{(1)} = [\mathfrak{n}, \mathfrak{n}] \supset \mathfrak{n}^{(2)} = [\mathfrak{n}, \mathfrak{n}^{(1)}] \supset \cdots \supset \mathfrak{n}^{(r)} \supset \mathfrak{n}^{(r+1)} = \{0\},$$

where $\mathfrak{n}^{(i+1)} = [\mathfrak{n}, \mathfrak{n}^{(i)}]$ $(i \geq 1)$ and $\mathfrak{n}^{(r)} \neq 0$.
We can prove that $\mathfrak{n}^{(r)} \subset \text{Ker } F_{d(\omega_0\circ J)}$, $\mathfrak{n}^{(r)} \oplus J\mathfrak{n}^{(r)}$ is abelian (cf. **第 29 回研究会**).