Einstein-like metrics on generalized flag manifolds

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Einstein-like metrics on generalized flag manifolds
based on joint works with
Andreas Arvanitoyeorgos and Marina Statha

- Introduction : a little history on Einstein-like metrics on Riemannian manifolds
  a characterization of Einstein-like metrics on a compact homogeneous space
  C. Peng and C. Qian (2016), C. Qian and A. Wu (2021),
  F. Li, H. Chen and Z. Chen (2023)
- invariant metrics on compact homogeneous space
  a characterization due to F. Li, H. Chen and Z. Chen (2023)
- generalized flag manifolds, a system of $\mathfrak{t}$-roots
- results for generalized flag manifolds
- outline of a proof

(M, g): Riemannian manifold,

- (M, g) is called Einstein if the Ricci tensor r_g of the metirc g satisfies $\mathrm{r}_g = cg$ for some constant c.
- In 1978, as a generalization of Einstein metrics, A. Gray introduced the notion of Einstein-like $\mathscr{A}$-metrics and $\mathscr{B}$-metrics on Riemannian manifolds.

  $\mathscr{X}(M)$: C^∞ vector fields on M
  For a Riemannian manifold (M, g),
- the metric g is called an $\mathscr{A}$-metric $\iff$

$$(\nabla_X \mathrm{r}_g)(X, X) = 0 \quad \text{for} \quad \text{all} \quad X \in \mathscr{X}(M)$$

- the metric g is called a $\mathscr{B}$-metric $\iff$

$$(\nabla_X \mathrm{r}_g)(Y, Z) = (\nabla_Y \mathrm{r}_g)(X, Z) \quad \text{for} \quad \text{all} \quad X, Y, Z \in \mathscr{X}(M)$$

- Note that, if a metric g has parallel Ricci tensor $\nabla \mathrm{r}_g = 0$, then g is an $\mathscr{A}$-metric and a $\mathscr{B}$-metric.
- Let R be the curvature tensor of (M, g).
  2^{nd} Bianchi identity:

  $$\nabla_X(R)(Y, Z) + \nabla_Y(R)(Z, X) + \nabla_Z(R)(X, Y) = 0.$$

  By contracting, we have

  $$\sum g(\nabla_{E_i}(R)(E_i, X)Y, Z) = \nabla_Y(r)(X, Z) - \nabla_Z(r)(X, Y).$$

  Thus, g is $\mathscr{B}$-metric iff the divergence of curvature tensor is zero, that is, harmonic curvature.
- Examples: If (M, g) is a conformally flat manifold with constant scalar curvature, then g is a $\mathscr{B}$-metric (cf. Besse, Einstein manifolds, 16. 3, page 435).

- Example of Gray (1978): For $n \geq 3$, let

$$M = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\} \quad \text{and} \quad g = {x_n}^{\alpha} g_0,$$

where g_0 is a flat metric and $\alpha = 4/(n-2)$. Then (M, g) is a conformally flat manifold with constant scalar curvature 0. We see the metric g is not Einstein, but it is a $\mathscr{B}$-metric.

- A. Derdziński has started to study $\mathscr{B}$-metrics (in case of $\dim M = 4$ in detail) around 1978. Most recent work is in (2025), J. Korean Math. 62.

- Open problem : Is there a homogeneous space G/K with an invariant Einstein-like metric which is not Einstein? (conjectured by Bourguignon due to Derdziński)

Known results for invariant Codazzi tensor fields on a homogeneous space G/K

- a symmetric 2-covariant tensor field β is called Codazzi $\iff$

$$(\nabla_X \beta)(Y, Z) = (\nabla_Y \beta)(X, Z) \quad \text{for all} \quad X, Y, Z \in \mathscr{X}(M)$$

- D'Atri (1985) studied left invariant Codazzi tensor fields β for left invariant metric on connected Lie group G.
- Marshall Reber and Terek (2024) extended D'Atri's results to a reductive homogeneous space G/K.
  For naturally reductive metrics on G/K, they proved that an invariant Codazzi tensor fields β are parallel, $\nabla \beta = 0$. In particular, a naturally reductive $\mathscr{B}$-metric g on G/K is $\nabla \mathrm{r}_g = 0$.
- Aberaouze and Boucett (2022) proved that invariant $\mathscr{B}$-metrics g on solvable Lie groups and Lie groups of dimension ≤ 6 are Ricci parallel.

- G : a compact semi-simple Lie group
  K : a closed subgroup of G
  G/K : homogeneous space
  $\mathfrak{m} = \mathfrak{k}^{\perp}$: the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ w.r.t. B ($= -$ Killing form of $\mathfrak{g}$). Then we have $\mathfrak{m} \cong T_o(G/K)$ and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad [\,\mathfrak{k}, \,\mathfrak{m}\,] \subset \mathfrak{m}$$

- Let g be a G-invariant metric on G/K.
  Then the Riemannian connection ∇ is given by

$$(\nabla_X Y)_o = -\frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y) \text{ for } X, Y \in \mathfrak{m},$$

where $[\ ,\ ]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component, and $U(X, Y) : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is defined by

$$2g(U(X, Y), Z) = \big(g([Z, X]_{\mathfrak{m}}, Y) + g(X, [Z, Y]_{\mathfrak{m}})\big).$$

A characterization of Einstein-like metrics on a compact homogeneous space G/K

- C. Peng and C. Qian (2016) obtained a characterization of Einstein-like invariant metrics on homogeneous space G/K :

Proposition 1

Let g be a G-invariant metric on G/K.

$$g \text{ is a } \mathscr{B}\text{-metric} \iff$$

$$(1/2)r_g([Z,X]_{\mathfrak{m}},Y) - (1/2)r_g([Y,X]_{\mathfrak{m}},Z) + r_g(X,[Z,Y]_{\mathfrak{m}}) \\ + r_g(U(Z,X),Y) - r_g(U(Y,X),Z) = 0,$$

for all X, Y, $Z \in \mathfrak{m}$.

- C. Peng and C. Qian (2016) classified homogeneous $\mathscr{A}$-metrics and $\mathscr{B}$-metrics on spheres and projective spaces.
- C. Qian and A. Wu (2021) classified Einstein-like invariant metrics on compact symmetric spaces M.

  That is, whenever there exists a closed proper subgroup G' of $G = \mathrm{Isom}_0(M)$ acting transitively on M, they found all the G'-invariant $\mathscr{A}$-metrics and $\mathscr{B}$-metrics on M.

- F. Li, H. Chen and Z. Chen (2023) obtained a characterization of Einstein-like invariant metrics on compact homogeneous space G/K and classified such invariant metrics on generalized Wallach spaces of exceptional type.
- In the above cases, $\mathscr{B}$-metrics on M and G/K are always Einstein. We can show that $\mathscr{B}$-metrics on generalized Wallach spaces are Einstein.

- We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, $[\,\mathfrak{k},\, \mathfrak{m}\,] \subset \mathfrak{m}$ and a decomposition of $\mathfrak{m}$ into irreducible Ad(K)-modules:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\nu.$$

- We assume that Ad(K)-modules $\mathfrak{m}_j$ $(j = 1, \cdots, \nu)$ are mutually non-equivalent.

  Then a G-invariant metric g on G/K can be written as

$$g = x_1 B|_{\mathfrak{m}_1} + \cdots + x_\nu B|_{\mathfrak{m}_\nu}, \tag{1}$$

  for positive real numbers $x_1, \cdots, x_\nu$.

- Note that the Ricci tensor r_g of a G-invariant Riemannian metric on G/K is of the same form as (1).

- Let $\{e_\alpha\}$ be a B-orthonormal basis adapted to the decomposition of $\mathfrak{m}$, i.e., $e_\alpha \in \mathfrak{m}_i$ for some i, and $\alpha < \beta$ if $i < j$ (with $e_\alpha \in \mathfrak{m}_i$ and $e_\beta \in \mathfrak{m}_j$).
- We put $C^\gamma_{\alpha\beta} = B([e_\alpha, e_\beta], e_\gamma)$, so that $[e_\alpha, e_\beta]_\mathfrak{m} = \sum_\gamma C^\gamma_{\alpha\beta} e_\gamma$.
- Set $A_{ijk} = \sum (C^\gamma_{\alpha\beta})^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i,\ e_\beta \in \mathfrak{m}_j,\ e_\gamma \in \mathfrak{m}_k$.
  Then, the non-negative number A_{ijk} is independent of the B-orthonormal bases chosen for $\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k$, and

$$A_{ijk} \ = \ A_{jik} \ = \ A_{jki}. \tag{2}$$

- Let $d_k = \dim \mathfrak{m}_k$. Then we have

Lemma 2

The components $r_1, \cdots, r_\nu$ *of Ricci tensor* r_g *of the metric* $g = x_1 B|_{\mathfrak{m}_1} + \cdots + x_\nu B|_{\mathfrak{m}_\nu}$ *on* G/K *are given by*

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} A_{kji} - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} A_{kji} \tag{3}$$

where the sum is taken over $i, j = 1, \cdots, \nu$.

- F.Li, H.Chen and Z.Chen obtained characterizations of invariant $\mathscr{B}$-metrics using Proposition 1 (2).

Theorem 3

Let g be a G-invariant metric on G/K of the form (1).
The metric g is a $\mathscr{B}$-metric if and only if

$$C^{\gamma}_{\alpha\beta}((x_j + x_i - x_k)r_j + (x_k + x_i - x_j)r_k - 2x_i r_i) = 0 \tag{4}$$

holds for any $i, j, k \in \{1, 2, \cdots, \nu\}$, $\alpha = 1, 2, \cdots, d_i$, $\beta = 1.2, \cdots, d_j$ and $\gamma = 1, 2, \cdots, d_k$.

Corollary 4

If $A_{iik} \neq 0$, then the Ricci components r_i, r_k of a $\mathscr{B}$-metric g of the form (1) satisfy $r_i = r_k$.

Following Corollaries are useful for us.

Corollary 5

If $A_{ijk} \neq 0$ and $r_j = r_k$, then the Ricci components of a $\mathscr{B}$-metric g of the form (1) satisfy $r_i = r_j = r_k$.

Corollary 6

If $A_{ijk} \neq 0$, for i, j, k mutually distinct, and

$$x_i{}^2 - 2x_i x_j - 2x_i x_k + x_j{}^2 - 2x_j x_k + x_k{}^2 \neq 0,$$

then the Ricci components of a $\mathscr{B}$-metric g of the form (1) satisfy $r_i = r_j = r_k$.

Generalized flag manifolds

We now consider invariant metrics on Generalized flag manifolds.

- A generalized flag manifold M is an adjoint orbit of a compact connected semi-simple Lie group G, and is a homogeneous space of the form $M = G/K$, where $K = C(S)$ is the centralizer of a torus S in G.
- Generalized flag manifolds exhaust compact simply connected homogeneous Kähler manifolds.
- A generalized flag manifold admits a finite number of G-invariant complex structures. For each G-invariant complex structure there is a compatible Kähler-Einstein metric.
- Generalized flag manifolds can be classified by use of painted Dynkin diagrams.
- Generalized flag manifolds are also referred to as Kähler C-spaces.

Theorem 7 (A. Arvanitoyeorgos, Y. S. and M. Statha)

For generalized flag manifolds G/K we have

1) if the second Betti number $b_2(G/K) = 1$, invariant $\mathscr{B}$-metrics on G/K are Einstein.

2) if the second Betti number $b_2(G/K) = 2$, invariant $\mathscr{B}$-metrics on G/K are Einstein.

- G : a compact semi-simple Lie group
  $\mathfrak{g}$: the Lie algebra of G, $\mathfrak{h}$: maximal abelian subalgebra of $\mathfrak{g}$.
  $\mathfrak{g}^{\mathbb{C}}$: the complexification of $\mathfrak{g}$, $\mathfrak{h}^{\mathbb{C}}$: the complexification of $\mathfrak{h}$
- Δ : the root system Δ of $\mathfrak{g}^{\mathbb{C}}$ relative to the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ ($\Delta \subset \mathfrak{h}_0 = \sqrt{-1}\mathfrak{h}$) by the duality defined by the Killing form of $\mathfrak{g}^{\mathbb{C}}$.
- $\Pi = \{\alpha_1, \cdots, \alpha_l\}$: a fundamental system of Δ
- $\{\Lambda_1, \cdots, \Lambda_l\}$ the fundamental weights of $\mathfrak{g}^{\mathbb{C}}$ corresponding to Π, that is

$$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \qquad (1 \leq i, j \leq \ell).$$

- For a subset Π_0 of Π, write $\Pi - \Pi_0 = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$ $(1 \leq \alpha_{i_1} < \cdots < \alpha_{i_r} \leq \ell)$. We put $[\Pi_0] = \Delta \cap \{\Pi_0\}_{\mathbb{Z}}$, where $\{\Pi_0\}_{\mathbb{Z}}$ denotes the subspace of $\mathfrak{h}_0$ generated by Π_0.

- $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in \Delta} \mathfrak{g}^{\mathbb{C}}_{\alpha}$: the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$
- $\mathfrak{u} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in [\Pi_0] \cup \Delta^+} \mathfrak{g}^{\mathbb{C}}_{\alpha}$: the parabolic subalgebra for a subset Π_0, where Δ^+ is the set of all positive roots relative to Π.
- Note that the nilradical $\mathfrak{n}$ of $\mathfrak{u}$ is given by

$$\mathfrak{n} = \sum_{\alpha \in \Delta^+ - [\Pi_0]} \mathfrak{g}^{\mathbb{C}}_{\alpha}.$$

We put $\Delta^+_{\mathfrak{m}} = \Delta^+ - [\Pi_0]$ and $\Delta^+_0 = [\Pi_0]^+$

- $G^{\mathbb{C}}$: a simply connected complex semi-simple Lie group whose Lie algebra is $\mathfrak{g}^{\mathbb{C}}$
  U : the parabolic subgroup of $G^{\mathbb{C}}$ generated by $\mathfrak{u}$
- Then $G^{\mathbb{C}}/U$ is a simply connected complex homogeneous manifold and compact Lie group G acts transitively on $G^{\mathbb{C}}/U$.
- $G^{\mathbb{C}}/U = G/K$ as C^{∞}-manifolds where $K = G \cap U$ is a connected closed subgroup of G
- $G^{\mathbb{C}}/U$ admits a G-invariant Einstein Kähler metric.
- We put $\mathfrak{t} = \left\{ H \in \mathfrak{h}_0 \mid (H,\ \Pi_0) = (0) \right\}$. Then $\{\Lambda_{i_1}, \cdots, \Lambda_{i_r}\}$ is a basis of $\mathfrak{t}$. Put $\mathfrak{s} = \sqrt{-1}\mathfrak{t}$. Then the Lie algebra $\mathfrak{k}$ is given by $\mathfrak{k} = \mathfrak{z}(\mathfrak{s})$ (the Lie algebra of centralizer of a torus S in G).
- It is known that the second Betti number $b_2(G/K) = r = \dim \mathfrak{t}$.

- We consider the restriction map

$$\kappa : \mathfrak{h}_0^* \to \mathfrak{t}^* \quad \alpha \mapsto \alpha|_{\mathfrak{t}}$$

and set $\Delta_{\mathfrak{t}} = \kappa(\Delta)$. An elements of $\Delta_{\mathfrak{t}}$ are called a **$\mathfrak{t}$-root** and $\Delta_{\mathfrak{t}}$ is called **a system of $\mathfrak{t}$-roots**.
(The notion of $\mathfrak{t}$-roots is introduced by Alekseevky and Perelomov around 1985 to study invariant Kähler-Einstein metrics of generalized flag manifolds.)

- There exists a 1-1 correspondence between $\mathfrak{t}$-roots ξ and irreducible submodules $\mathfrak{m}_\xi$ of the $\mathrm{Ad}_G(K)$-module $\mathfrak{m}^{\mathbb{C}}$ that is given by

$$\Delta_{\mathfrak{t}} \ni \xi \mapsto \mathfrak{m}_\xi = \sum_{\kappa(\alpha)=\xi} \mathfrak{g}_\alpha^{\mathbb{C}}.$$

- Thus we have a decomposition of the $\mathrm{Ad}_G(K)$-module $\mathfrak{m}^{\mathbb{C}}$:

$$\mathfrak{m}^{\mathbb{C}} = \sum_{\xi \in \Delta_{\mathfrak{t}}} \mathfrak{m}_\xi.$$

- Denote by $\Delta_{\mathfrak{t}}^{+}$ the set of all positive $\mathfrak{t}$-roots, that is, the restricton of the system Δ^{+}. Then $\mathfrak{n} = \sum_{\xi \in \Delta_{\mathfrak{t}}^{+}} \mathfrak{m}_{\xi}$.
- Denote by τ the complex conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$ (note that τ interchanges $\mathfrak{g}_{\alpha}^{\mathbb{C}}$ and $\mathfrak{g}_{-\alpha}^{\mathbb{C}}$) and by $\mathfrak{v}^{\tau}$ the set of fixed points of τ in a (complex) vector subspace $\mathfrak{v}$ of $\mathfrak{g}^{\mathbb{C}}$. Thus we have a decomposition of $\mathrm{Ad}_G(K)$-module $\mathfrak{m}$ into irreducible submodules:

$$\mathfrak{m} = \sum_{\xi \in \Delta_{\mathfrak{t}}^{+}} \left(\mathfrak{m}_{\xi} + \mathfrak{m}_{-\xi}\right)^{\tau}.$$

- For integers $j_1, \cdots, j_r$ with $(j_1, \cdots, j_r) \neq (0, \cdots, 0)$, we put $\Delta(j_1, \cdots, j_r) = \left\{ \sum_{j=1}^{\ell} m_j \alpha_j \in \Delta^+ \,\middle|\, m_{i_1} = j_1, \cdots, m_{i_r} = j_r \right\}$. There exists a natural 1-1 correspondence between $\Delta_{\mathfrak{t}}^+$ and the set $\{\Delta(j_1, \cdots, j_r) \neq \emptyset\}$
- For a generalized flag manifold G/K, we have a decomposition of $\mathfrak{m}$ into mutually non-equivalent irreducible $\mathrm{Ad}_G(K)$-modules:

$$\mathfrak{m} = \sum_{\xi \in \Delta_{\mathfrak{t}}^+} (\mathfrak{m}_\xi + \mathfrak{m}_{-\xi})^\tau = \sum_{j_1, \cdots, j_r} \mathfrak{m}(j_1, \cdots, j_r).$$

Thus a G-invariant metric g on G/K can be written as

$$g = \sum_{\xi \in \Delta_{\mathfrak{t}}^+} x_\xi B|_{(\mathfrak{m}_\xi + \mathfrak{m}_{-\xi})^\tau} = \sum_{j_1, \cdots, j_r} x_{j_1 \cdots j_r} B|_{\mathfrak{m}(j_1, \cdots, j_r)} \tag{5}$$

for positive real numbers x_ξ, $x_{j_1 \cdots j_r}$.

Put

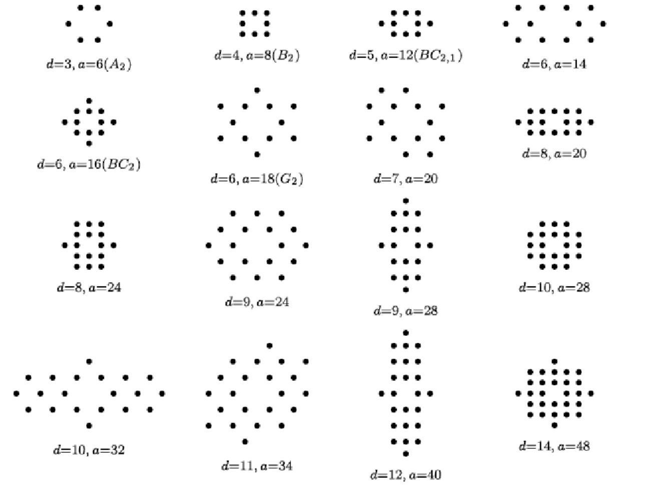
$$\delta = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha, \quad \delta_{\mathfrak{m}} = \frac{1}{2}\sum_{\alpha\in\Delta^+_{\mathfrak{m}}}\alpha, \quad \delta_{\mathfrak{k}} = \frac{1}{2}\sum_{\alpha\in\Delta^+_0}\alpha.$$

Note that $\delta_{\mathfrak{m}} = \delta - \delta_{\mathfrak{k}}$ and $\delta_{\mathfrak{m}} \in \sum_{\ell=1}^{r}\mathbb{Z}_+\Lambda_{j_\ell}$.
It is known that a Kähler Einstein metric g on $G^{\mathbb{C}}/U = G/K$ is given by

$$g = \sum_{\xi\in\Delta^+_{\mathfrak{t}}}(\delta_{\mathfrak{m}}, \xi)B|_{(\mathfrak{m}_\xi+\mathfrak{m}_{-\xi})^\tau}.$$

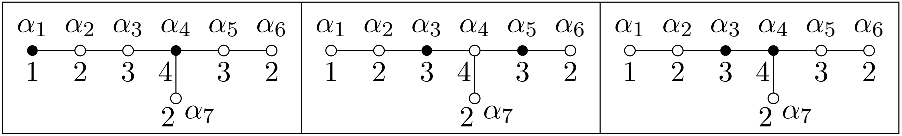
systems of $\mathfrak{t}$-roots for $b_2(G/K) = 2$

- For classical Lie groups, we have $\nu = 3$ and A_2-type, $\nu = 4$ and B_2-type, $\nu = 5$ and $BC_{2,1}$-type, $\nu = 6$ and BC_2-type.
- For exceptional Lie groups, we have 16 cases.



Consider $G/K = E_7/\,\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(3)\times\mathrm{U}(2)) = E_7/(T^2A_2A_2A_1)$.

Painted Dynkin diagram



$$\mathfrak{m} = \mathfrak{m}(1,0)\oplus\mathfrak{m}(0,1)\oplus\mathfrak{m}(1,1)\oplus\mathfrak{m}(0,2)\oplus\mathfrak{m}(1,2)\oplus\mathfrak{m}(0,3)\oplus\mathfrak{m}(1,3)\oplus\mathfrak{m}(1,4)$$

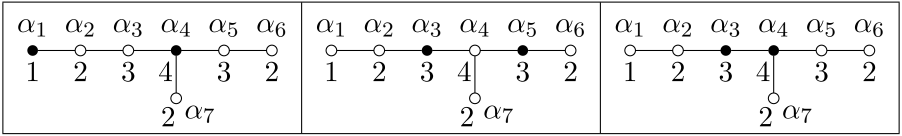
$$(\Delta_{\mathfrak{t}}^{+} = \{\,\xi_1=\kappa(\alpha_1), \xi_2=\kappa(\alpha_4),\ \xi_1+\xi_2=\kappa(\alpha_1+\alpha_2+\alpha_3+\alpha_4),$$
$$2\xi_2=\kappa(\alpha_3+2\alpha_4+\alpha_5+\alpha_7),\ \xi_1+2\xi_2=\kappa(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_7),$$
$$3\xi_2=\kappa(\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7),$$
$$\xi_1+3\xi_2=\kappa(\alpha_1+2\alpha_2+3\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7),\ \xi_1+4\xi_2\ \})$$

$$\mathfrak{m} \;=\; \mathfrak{l}(1,0)\oplus\mathfrak{l}(0,1)\oplus\mathfrak{l}(1,1)\oplus\mathfrak{l}(1,2)\oplus\mathfrak{l}(2,1)\oplus\mathfrak{l}(2,2)\oplus\mathfrak{l}(3,2)\oplus\mathfrak{l}(3,3)$$

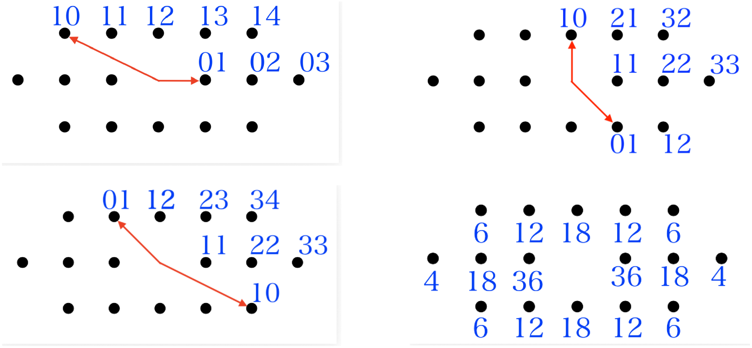
$$\mathfrak{m} \;=\; \mathfrak{n}(1,0)\oplus\mathfrak{n}(0,1)\oplus\mathfrak{n}(1,1)\oplus\mathfrak{n}(1,2)\oplus\mathfrak{n}(2,2)\oplus\mathfrak{n}(2,3)\oplus\mathfrak{n}(3,3)\oplus\mathfrak{n}(3,4)$$

$E_7/\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(3)\times\mathrm{U}(2))$

Painted Dynkin diagram



corresponding **system of T-roots**



Kähler Einstein metrics on $E_7/\,\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(3)\times\mathrm{U}(2))$

$$
\begin{aligned}
g = {} & x_{10}(-B)|_{\mathfrak{m}_{10}} + x_{01}(-B)|_{\mathfrak{m}_{01}} + x_{11}(-B)|_{\mathfrak{m}_{11}} + x_{02}(-B)|_{\mathfrak{m}_{02}} \\
& + x_{12}(-B)|_{\mathfrak{m}_{12}} + x_{03}(-B)|_{\mathfrak{m}_{03}} + x_{13}(-B)|_{\mathfrak{m}_{143}} + x_{14}(-B)|_{\mathfrak{m}_{14}} \qquad (6)
\end{aligned}
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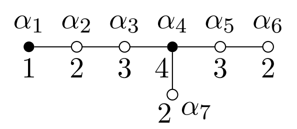
<td>
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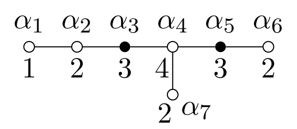
<td>
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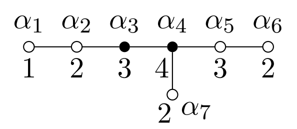
<td>
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<td>$2\delta_{\mathfrak{m}} = 4\Lambda_1 + 7\Lambda_4$</td>
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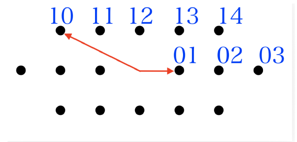
<td>
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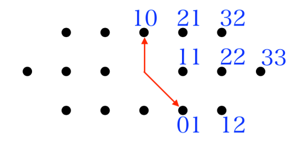
<td>
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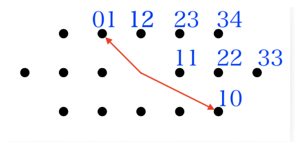
<td>
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<td>$x_{10}=4,\quad x_{01}=7,$
$x_{11}=11,\quad x_{02}=14,$
$x_{12}=18,\quad x_{13}=25,$
$x_{14}=32,\quad x_{03}=21$</td>
<td>$x_{10}=6,\quad x_{01}=11,$
$x_{11}=5,\quad x_{02}=22,$
$x_{12}=6,\quad x_{13}=17,$
$x_{14}=28,\quad x_{03}=33$</td>
<td>$x_{10}=4,\quad x_{01}=9,$
$x_{11}=5,\quad x_{02}=18,$
$x_{12}=14,\quad x_{13}=23,$
$x_{14}=32,\quad x_{03}=27$</td>
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For the decomposition

$$\mathfrak{m} = \mathfrak{m}(1,0) \oplus \mathfrak{m}(0,1) \oplus \mathfrak{m}(1,1) \oplus \mathfrak{m}(0,2) \oplus \mathfrak{m}(1,2) \oplus \mathfrak{m}(0,3) \oplus \mathfrak{m}(1,3) \oplus \mathfrak{m}(1,4),$$

we have

$$[\mathfrak{m}(1,0),\mathfrak{m}(0,1)] \subset \mathfrak{m}(1,1), \quad [\mathfrak{m}(0,1),\mathfrak{m}(0,1)] \subset \mathfrak{m}(0,2) \oplus \mathfrak{k},$$
$$[\mathfrak{m}(0,1),\mathfrak{m}(0,2)] \subset \mathfrak{m}(0,3) \oplus \mathfrak{m}(0,1), \quad [\mathfrak{m}(1,0),\mathfrak{m}(0,2)] \subset \mathfrak{m}(1,2),$$
$$[\mathfrak{m}(1,0),\mathfrak{m}(0,3)] \subset \mathfrak{m}(1,3), \quad [\mathfrak{m}(0,1),\mathfrak{m}(1,1)] \subset \mathfrak{m}(1,2) \oplus \mathfrak{m}(1,0),$$
$$[\mathfrak{m}(0,1),\mathfrak{m}(1,2)] \subset \mathfrak{m}(1,3) \oplus \mathfrak{m}(1,1), \quad [\mathfrak{m}(0,1),\mathfrak{m}(1,3)] \subset \mathfrak{m}(1,4) \oplus \mathfrak{m}(1,2),$$
$$[\mathfrak{m}(0,2),\mathfrak{m}(1,1)] \subset \mathfrak{m}(1,3), \quad [\mathfrak{m}(0,2),\mathfrak{m}(1,2)] \subset \mathfrak{m}(1,4) \oplus \mathfrak{m}(1,0),$$
$$[\mathfrak{m}(0,3),\mathfrak{m}(1,1)] \subset \mathfrak{m}(1,4).$$

By using Riemannian submersions and a Kähler-Einstein metric, we can compute Ricci components r_{ij} of a metric (6). That is, we see

$$A_{01\,13\,14} = 1, A_{01\,11\,12} = 2, A_{01\,01\,02} = 4, A_{11\,03\,14} = \frac{1}{3},$$
$$A_{01\,12\,13} = 2, A_{02\,12\,14} = 1, A_{01\,02\,03} = 1, A_{11\,02\,13} = 1,$$
$$A_{10\,01\,11} = 1, A_{10\,02\,12} = 1, A_{10\,03\,13} = \frac{1}{3}.$$

Assume that a metric of the form (6) is a $\mathscr{B}$-metric. Note that we have $r_{01} = r_{02}$ from $A_{01\,01\,02} \neq 0$ and Corollary 4. From $A_{01\,02\,03} \neq 0$ and Corollary 5, we see that $r_{01} = r_{02} = r_{03}$.
Put

$$q_1 = {x_{10}}^2 + {x_{01}}^2 + {x_{11}}^2 - 2x_{10}x_{01} - 2x_{10}x_{11} - 2x_{01}x_{11}.$$

Note that $A_{10\,01\,11} \neq 0$. If $q_1 \neq 0$, we have $r_{10} = r_{01} = r_{11}$ from Corollary 4. Then we see that $r_{01} = r_{11} = r_{12}$ from $A_{01\,11\,12} \neq 0$ and Corollary 4. From $A_{01\,12\,13} \neq 0$ and Corollary 3.4, we see that $r_{01} = r_{12} = r_{13}$, From $A_{01\,13\,14} \neq 0$ and Corollary 3.4, we see that $r_{01} = r_{13} = r_{14}$. Thus we see that the $\mathscr{B}$-metric g on $E_7/\,\mathrm{S}(\mathrm{U}(3)\times \mathrm{U}(3)\times \mathrm{U}(2))$ is Einstein, if $q_1 \neq 0$.
Put

$$q_2 = {x_{10}}^2 + {x_{02}}^2 + {x_{12}}^2 - 2x_{10}x_{02} - 2x_{10}x_{12} - 2x_{02}x_{12},$$
$$q_3 = {x_{10}}^2 + {x_{03}}^2 + {x_{13}}^2 - 2x_{10}x_{03} - 2x_{10}x_{13} - 2x_{02}x_{13},$$
$$q_4 = {x_{01}}^2 + {x_{11}}^2 + {x_{12}}^2 - 2x_{01}x_{11} - 2x_{01}x_{12} - 2x_{11}x_{12}.$$

Put

$$
\begin{aligned}
q_5 &= {x_{01}}^2 + {x_{12}}^2 + {x_{13}}^2 - 2x_{01}x_{12} - 2x_{01}x_{13} - 2x_{12}x_{13},\\
q_6 &= {x_{01}}^2 + {x_{13}}^2 + {x_{14}}^2 - 2x_{01}x_{13} - 2x_{01}x_{14} - 2x_{13}x_{14},\\
q_7 &= {x_{02}}^2 + {x_{11}}^2 + {x_{13}}^2 - 2x_{02}x_{11} - 2x_{02}x_{13} - 2x_{11}x_{13},\\
q_8 &= {x_{02}}^2 + {x_{12}}^2 + {x_{14}}^2 - 2x_{02}x_{12} - 2x_{02}x_{14} - 2x_{12}x_{14},\\
q_9 &= {x_{11}}^2 + {x_{03}}^2 + {x_{14}}^2 - 2x_{11}x_{03} - 2x_{11}x_{14} - 2x_{03}x_{14}.
\end{aligned}
$$

If $q_2 \neq 0$, we see that $r_{10} = r_{02} = r_{12}$ from $A_{01\,11\,12} \neq 0$ and Corollary 4, and thus we have $r_{10} = r_{02} = r_{01}$. Then we see that $r_{10} = r_{01} = r_{11}$ from $A_{10\,01\,11} \neq 0$ and Corollary 4. Thus we see that the $\mathscr{B}$-metric g on $E_7/\,\mathrm{S}(\mathrm{U}(3)\times \mathrm{U}(3)\times \mathrm{U}(2))$ is Einstein, if $q_2 \neq 0$.
Similarly, we see that, if one of the conditions $q_3 \neq 0$, $q_4 \neq 0$, $q_5 \neq 0$, $q_6 \neq 0$, $q_7 \neq 0$, $q_8 \neq 0$, $q_9 \neq 0$ holds, then the $\mathscr{B}$-metric g on $E_7/\,\mathrm{S}(\mathrm{U}(3)\times \mathrm{U}(3)\times \mathrm{U}(2))$ is Einstein.

Now we consider the cases when $q_1 = 0$, $q_2 = 0$, $q_3 = 0$, $q_4 = 0$, $q_5 = 0, q_6 = 0$, $q_7 = 0, q_8 = 0$ and $q_9 = 0$. Since $x_{10} > 0, x_{01} > 0, x_{11} > 0$, $x_{02} > 0, x_{03} > 0, x_{12} > 0, x_{13} > 0, x_{14} > 0$, we put $x_{10} = 1$ and $x_{01} = t^2$. Then, by solving $q_1 = 0$, we have that $x_{11} = (1-t)^2$ or $x_{11} = (1+t)^2$.

Lemma 1

Put $x_{10} = 1$ and $x_{01} = t^2$. Then the positive solutions $\{x_{11}, x_{02}, x_{03}, x_{12}, x_{13}, x_{14}\}$ of the system of equations

$$q_2 = 0, q_3 = 0, q_4 = 0, q_5 = 0, q_6 = 0, q_7 = 0, q_8 = 0, \textit{and } q_9 = 0 \tag{7}$$

are given by

$$\{x_{11} = (1-t)^2,\ x_{02} = 4(1-t)^2,\ x_{12} = (1-2t)^2,\ x_{03} = (2-t)^2,\ x_{13} = (1-t)^2,\ x_{14} = 1\}, \tag{8}$$

$$\{x_{11} = (1-t)^2,\ x_{02} = 4(1-t)^2,\ x_{12} = (1-2t)^2,\ x_{03} = t^2,\ x_{13} = (1-t)^2,\ x_{14} = 1\}, \tag{9}$$

$$\{x_{11}=(1-t)^2,\ x_{02}=4,\ x_{12}=1,\quad (10)$$
$$x_{03}=t^2,\ x_{13}=(1+t)^2,\ x_{14}=1\},$$
$$\{x_{11}=(1-t)^2,\ x_{02}=4t^2,\ x_{12}=(1-2t)^2,\quad (11)$$
$$x_{03}=9t^2,\ x_{13}=(1-3t)^2,\ x_{14}=(1-4t)^2\},$$
$$\{x_{11}=(1+t)^2,\ x_{02}=4(1+t)^2,\ x_{12}=(1+2t)^2,\quad (12)$$
$$x_{03}=(2+t)^2,\ x_{13}=(1+t)^2,\ x_{14}=1\},$$
$$\{x_{11}=(1+t)^2,\ x_{02}=4(1+t)^2,\ x_{12}=(1+2t)^2,\quad (13)$$
$$x_{03}=t^2,\ x_{13}=(1+t)^2,\ x_{14}=1\},$$
$$\{x_{11}=(1+t)^2,\ x_{02}=4,\ x_{12}=1,\quad (14)$$
$$x_{03}=t^2,\ x_{13}=(1-t)^2,\ x_{14}=1\},$$
$$\{x_{11}=(1+t)^2,\ x_{02}=4t^2,\ x_{12}=(1+2t)^2,\quad (15)$$
$$x_{03}=9t^2,\ x_{13}=(1+3t)^2,\ x_{14}=(1+4t)^2\}.$$

Moreover, we have more positive solutions, which are not included above, given by

$$\{t=\frac{2}{3},\ x_{10}=1,\ x_{01}=\frac{4}{9},\ x_{11}=\frac{1}{9},\ x_{02}=\frac{4}{9},\ x_{12}=\frac{1}{9},\ x_{03}=\frac{4}{9},\ x_{13}=\frac{1}{9},\ x_{14}=\frac{1}{9}\}, \tag{16}$$

$$\{t=2,\ x_{10}=1,\ x_{01}=4,\ x_{11}=1,\ x_{02}=4,\ x_{12}=1,\ x_{03}=4,\ x_{13}=1,\ x_{14}=9\}, \tag{17}$$

$$\{t=2,\ x_{10}=1,\ x_{01}=4,\ x_{11}=1,\ x_{02}=4,\ x_{12}=1,\ x_{03}=4,\ x_{13}=1,\ x_{14}=1\}, \tag{18}$$

$$\{t=-\frac{2}{3},\ x_{10}=1,\ x_{01}=\frac{4}{9},\ x_{11}=\frac{1}{9},\ x_{02}=\frac{4}{9},\ x_{12}=\frac{1}{9},\ x_{03}=\frac{4}{9},\ x_{13}=\frac{1}{9},\ x_{14}=\frac{1}{9}\}, \tag{19}$$

$$\{t=-2,\ x_{10}=1,\ x_{01}=4,\ x_{11}=1,\ x_{02}=4,\ x_{12}=1,\ x_{03}=4,\ x_{13}=1,\ x_{14}=9\}, \tag{20}$$

$$\{t=-2,\ x_{10}=1,\ x_{01}=4,\ x_{11}=1,\ x_{02}=4,\ x_{12}=1,\ x_{03}=4,\ x_{13}=1,\ x_{14}=1\}. \tag{21}$$

A proof of Theorem 7 for the case $E_7/\,\mathrm{S}(\mathrm{U}(3)\times \mathrm{U}(3)\times \mathrm{U}(2))$

Put $X(i,j,k) = \dfrac{x_i}{x_j x_k} - \dfrac{x_j}{x_i x_k} - \dfrac{x_k}{x_i x_j}$ for simplicity. Then we have Ricci components $\{\mathrm{r}_{01}, \mathrm{r}_{02}, \mathrm{r}_{03}\}$ for the metric g are given by

$$\mathrm{r}_{01} = \frac{1}{2x_{01}} + \frac{1}{72}X(01,10,11) + \frac{1}{72}X(01,02,03) + \frac{1}{36}X(01,11,12) + \frac{1}{36}X(01,12,13) + \frac{1}{72}X(01,13,14) - \frac{1}{18}\frac{x_{02}}{{x_{01}}^2},$$

$$\mathrm{r}_{02} = \frac{1}{2x_{02}} + \frac{1}{36}X(02,10,12) + \frac{1}{36}X(02,01,03) + \frac{1}{36}X(02,12,14) + \frac{1}{36}X(02,11,13) + \frac{1}{18}\Big(-\frac{2}{x_{02}} + \frac{x_{02}}{{x_{01}}^2}\Big),$$

$$\mathrm{r}_{03} = \frac{1}{2x_{03}} + \frac{1}{24}A_{10\,03\,13}X(03,10,13) + \frac{1}{8}X(03,01,02) + \frac{1}{24}X(03,11,14).$$

Now we substitute the values from (8) to (21) into r_{01}, r_{02}, r_{03} and see that $r_{01} = r_{02} = r_{03}$ can not occur, that is, we can see that these cases are not $\mathscr{B}$-metric.

For example, in case of (8), we see that

$$\begin{aligned}
&\{r_{01} - r_{02},\ r_{01} - r_{03}\} \\
&\quad = \Big\{ \frac{32t^7 - 198t^6 + 503t^5 - 916t^4 + 1282t^3 - 1044t^2 + 416t - 64}{72(t-2)(t-1)^2t^4(2t-1)}, \\
&\qquad\qquad \frac{16t^7 - 68t^6 + 78t^5 - 61t^4 + 170t^3 - 206t^2 + 96t - 16}{18(t-2)^2(t-1)t^4(2t-1)} \Big\}
\end{aligned}$$

Taking resultant Res of polynomials

$$\{32t^7 - 198t^6 + 503t^5 - 916t^4 + 1282t^3 - 1044t^2 + 416t - 64,$$
$$16t^7 - 68t^6 + 78t^5 - 61t^4 + 170t^3 - 206t^2 + 96t - 16\},$$

we see that $Res = 34391097085297007 0016$. Thus a system of equations $\{r_{01} - r_{02} = 0,\ r_{01} - r_{03} = 0\}$ does not have common root.

We have

$$\mathsf{r}_{11} = \frac{1}{2x_{11}} + \frac{1}{24}X(11,10,01) + \frac{1}{12}X(11,01,12) + \frac{1}{24}X(11,02,13) + \frac{1}{72}X(11,03,14),$$

$$\mathsf{r}_{12} = \frac{1}{2x_{12}} + \frac{1}{36}X(12,10,02) + \frac{1}{18}X(12,01,11) + \frac{1}{18}X(12,01,13) + \frac{1}{36}X(12,02,14),$$

$$\mathsf{r}_{13} = \frac{1}{2x_{13}} + \frac{1}{72}X(13,10,03) + \frac{1}{24}X(13,02,11) + \frac{1}{12}X(13,01,12) + \frac{1}{24}X(13,01,14),$$

$$\mathsf{r}_{14} = \frac{1}{2x_{14}} + \frac{1}{36}X(14,11,03) + \frac{1}{12}X(14,02,12) + \frac{1}{12}X(14,01,13),$$

$$\mathsf{r}_{10} = \frac{1}{2x_{12}} + \frac{1}{36}X(10,03,13) + \frac{1}{12}X(10,02,12) + \frac{1}{12}X(10,01,11).$$

In case of (8)

$$\{x_{11} = (1-t)^2,\ x_{02} = 4(1-t)^2,\ x_{12} = (1-2t)^2,$$
$$x_{03} = (2-t)^2,\ x_{13} = (1-t)^2,\ x_{14} = 1\},$$

we see that

$$r_{03}(x_{11}+x_{03}-x_{14})+r_{14}(x_{11}-x_{03}+x_{14})-2r_{11}x_{11} = -\frac{4t^2-14t+15}{18(t-2)}.$$

Since the equation $4t^2-14t+15=0$ does not have real roots, we see $r_{03}(x_{11}+x_{03}-x_{14})+r_{14}(x_{11}-x_{03}+x_{14})-2r_{11}x_{11} \neq 0$ for real value t. Thus we see that the case of (8) is not $\mathscr{B}$-metric.