F_4 型 $(8,15)$ 分布の特異曲線による延長

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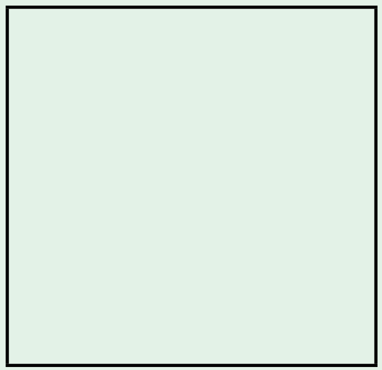
対称性と特異性 (Symmetry and Singularity).

相反する概念？

きれいな花にはトゲがある.

トゲがあるから魅力的？

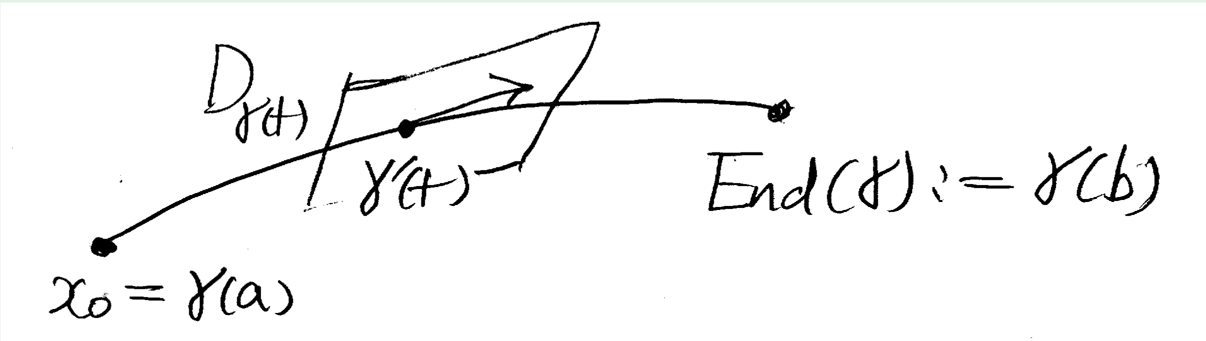
特異性から対称性を推し量ることはできないか？



例：四つの角は四角形がもつ対称性を調べる手がかりになる.

【 Singular curves of a distribution 】

Let $D \subset TM$ be a distribution, $x_0 \in M$ and $I = [a, b]$ an interval. Let Ω be the set of curves $\gamma : I \to M$ with $\dot\gamma(t) \in D_{\gamma(t)}$ for almost all $t \in I$ (D-integral) and $\gamma(a) = x_0$. Then the endpoint mapping $\mathrm{End} : \Omega \to M$ is defined by $\mathrm{End}(\gamma) := \gamma(b)$. A curve $\gamma \in \Omega$ is called a D-singular curve if γ is a critical point of the endpoint map End, i.e. the differential map $d_\gamma\mathrm{End} : T_\gamma\Omega \to T_{\gamma(b)}M$ is not surjective, for an appropriate manifold structure of Ω (and M).



【 Local characterisation of singular curves 】

Let $D \subset TM$ be a distribution of rank r generated by $\xi_1, \xi_2, \dots, \xi_r$ on a manifold M.

Lemma. A curve $\gamma : I \to M$ is D-singular if and only if there exists a lift $\Gamma : I \to T^*M \times \mathbb{R}^r$ of γ such that $\Gamma(t) = (x(t), p(t), u(t))$ satisfies the following Hamiltonian equation

$$(\star) \begin{cases} \dot{x} = u_1\xi_1(x) + u_2\xi_2(x) + \cdots + u_r\xi_r(x), \\ \dot{p} = -\left(u_1\dfrac{\partial H_{\xi_1}}{\partial x} + u_2\dfrac{\partial H_{\xi_2}}{\partial x} + \cdots + u_r\dfrac{\partial H_{\xi_r}}{\partial x}\right), \end{cases}$$

with constraints

$$H_{\xi_1} = 0, H_{\xi_2} = 0, \cdots, H_{\xi_r} = 0 \ \text{ and } \ \boxed{p \neq 0},$$

where $H_{\xi_i}(x, p) := \langle p, \xi_i(x) \rangle$.

Lemma. For a distribution D generated by $\xi_1, \dots, \xi_r$, along the solution $(x(t), p(t), u(t))$ of the constrained Hamiltonian equation $(\star)$ we have

$$\frac{d}{dt} H_{\xi_i}(t) = \sum_{j=1}^{r} u_j(t) H_{[\xi_i, \xi_j]}(t), \quad (1 \leq i \leq r).$$

Proof: We put $p = \sum_{j=1}^{r} p_j dx_j$ and $\xi_i = \sum_{k=1}^{r} \xi_{ik} \frac{\partial}{\partial x_k}$. Then we have $H_{\xi_i} = \sum_{j=1}^{r} p_j \xi_{ij}(x)$. Setting $H(x, p, u) = \sum_{1 \leq i,j \leq r} u_i p_j \xi_{ij}(x)$, we have by the Hamiltonian equation, for $1 \leq i \leq r$:

$$
\begin{aligned}
\frac{d}{dt} H_{\xi_i}(t) &= \textstyle\sum_{j=1}^{r} (p_j' \xi_{ij} + p_j \xi_{ij}') \\
&= \textstyle\sum_{j=1}^{r} \left(p_j' \xi_{ij} + \sum_{\ell=1}^{r} p_j \frac{\partial \xi_{ij}}{\partial x_\ell} x_\ell' \right) \\
&= \textstyle\sum_{j=1}^{r} \left(-\frac{\partial H}{\partial x_j} \xi_{ij} + \sum_{\ell=1}^{r} p_j \frac{\partial \xi_{ij}}{\partial x_\ell} \frac{\partial H}{\partial p_\ell} \right) \\
&= -\textstyle\sum_{k\ell j} u_k p_\ell \frac{\partial \xi_{k\ell}}{\partial x_j} \xi_{ij} + \sum_{j\ell k} p_j \frac{\partial \xi_{ij}}{\partial x_\ell} u_k \xi_{k\ell} \\
&= -\textstyle\sum_{k\ell j} u_k p_\ell \frac{\partial \xi_{k\ell}}{\partial x_j} \xi_{ij} + \sum_{\ell jk} p_\ell \frac{\partial \xi_{i\ell}}{\partial x_j} u_k \xi_{kj} \\
&= \textstyle\sum_{k\ell} u_k p_\ell \left(\sum_{j=1}^{r} \left(\xi_{ij} \frac{\partial \xi_{k\ell}}{\partial x_j} - \xi_{kj} \frac{\partial \xi_{i\ell}}{\partial x_j} \right) \right) \\
&= \textstyle\sum_{k=1}^{r} u_k \langle p, [\xi_i, \xi_k] \rangle \\
&= \textstyle\sum_{j=1}^{r} u_j H_{[\xi_i, \xi_j]}.
\end{aligned}
$$

by straightforward calculations. $\square$

【 Characteristic matrix 】

We introduce the characteristic matrix of D for the system of generators $\xi_1, \dots, \xi_r$ of D by the skew-symmetric matrix

$$A := \begin{pmatrix} H_{[\xi_1,\xi_1]} & H_{[\xi_1,\xi_2]} & \cdots & H_{[\xi_1,\xi_r]} \\ H_{[\xi_2,\xi_1]} & H_{[\xi_2,\xi_2]} & \cdots & H_{[\xi_2,\xi_r]} \\ \vdots & \vdots & \ddots & \vdots \\ H_{[\xi_r,\xi_1]} & H_{[\xi_r,\xi_2]} & \cdots & H_{[\xi_r,\xi_r]} \end{pmatrix},$$

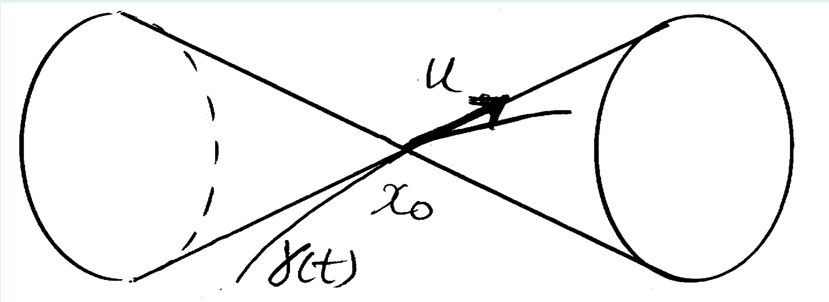
whose components are regarded, by the restriction to $D^{\perp}(\subset T^*M)$, as elements in $(D^{\perp})^*$.

Note that, if $\dot{x}(t) = \sum_{i=1}^{r} u_i(t)\xi_i(x(t))$ is the velocity vector of a singular curve $\gamma(t) = x(t)$, then necessarily $A\boldsymbol{u} = \mathbf{0}$.

【 Singular velocity cone 】

Definition. (Singular Velocity Cone)

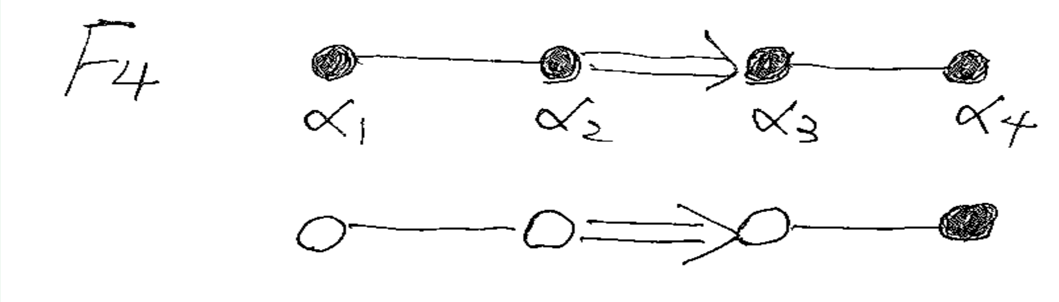
$\mathrm{SVC}(D) := \{(x_0, u) \in TM \mid$ There exists a D-singular curve
$\gamma : (\mathbb{R}, 0) \to M,$ satisfying that
$\gamma(0) = x_0,$ and $\gamma'(0) = u\},$

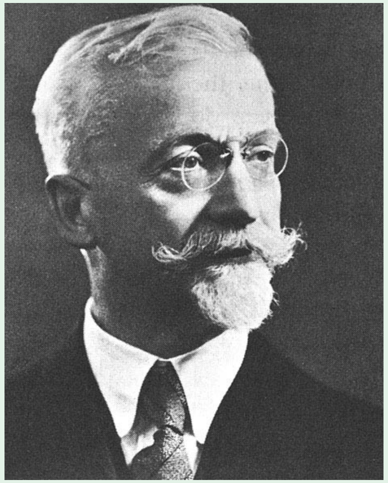


Note that the singular velocity cone $\mathrm{SVC}(D) \subset TM$ is a field of tangential cones (i.e., $\mathbb{R}^\times$-invariant).

【 Cartan's (8, 15)-distribution 】

In his paper "Über die einfachen Transformationsgruppen (1893)", E. Cartan realises the simple Lie algebra F_4 as the infinitesimal symmetry algebra of an $(8, 15)$-distribution.




Elie Cartan (1869–1951)

The distribution he gives is given concretely as $D \subset T\mathbb{R}^{15}$ on $\mathbb{R}^{15}$ by

$$\left\{\begin{array}{lcl}
X_1 & = & \frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z} - x_2\frac{\partial}{\partial x_{12}} - x_3\frac{\partial}{\partial x_{13}} - x_4\frac{\partial}{\partial x_{14}}, \\
X_2 & = & \frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial z} + x_1\frac{\partial}{\partial x_{12}} - x_3\frac{\partial}{\partial x_{23}} - x_4\frac{\partial}{\partial x_{24}}, \\
X_3 & = & \frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial z} + x_1\frac{\partial}{\partial x_{12}} + x_2\frac{\partial}{\partial x_{23}} - x_4\frac{\partial}{\partial x_{34}}, \\
X_4 & = & \frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial z} + x_1\frac{\partial}{\partial x_{12}} - x_2\frac{\partial}{\partial x_{13}} + x_3\frac{\partial}{\partial x_{34}}, \\
Y_1 & = & \frac{\partial}{\partial y_1} - y_4\frac{\partial}{\partial x_{23}} + y_3\frac{\partial}{\partial x_{24}} - y_2\frac{\partial}{\partial x_{34}}, \\
Y_2 & = & \frac{\partial}{\partial y_2} + y_4\frac{\partial}{\partial x_{13}} - y_3\frac{\partial}{\partial x_{14}} + y_1\frac{\partial}{\partial x_{34}}, \\
Y_3 & = & \frac{\partial}{\partial y_3} - y_4\frac{\partial}{\partial x_{12}} + y_2\frac{\partial}{\partial x_{14}} - y_1\frac{\partial}{\partial x_{24}}, \\
Y_4 & = & \frac{\partial}{\partial y_4} + y_3\frac{\partial}{\partial x_{12}} - y_2\frac{\partial}{\partial x_{13}} + y_1\frac{\partial}{\partial x_{23}},
\end{array}\right.$$

with the coordinates $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z$ and x_{ij} $(1 \leq i < j \leq 4)$ of $\mathbb{R}^{15}$. $(8 + 1 + 6 = 15)$.

Then, by setting $Z = \frac{\partial}{\partial z}$ and $X_{ij} = \frac{\partial}{\partial x_{ij}}, (1 \leq i < j \leq 4)$, we get the following bracket relations:

$$
\begin{array}{lll}
[X_1, X_2] = 2X_{12}, & [X_1, X_3] = 2X_{13}, & [X_1, X_4] = 2X_{14}, \\
 & [X_2, X_3] = 2X_{23}, & [X_2, X_4] = 2X_{24}, \\
 & & [X_3, X_4] = 2X_{34}, \\
[Y_1, Y_2] = 2X_{34}, & [Y_1, Y_3] = -2X_{24}, & [Y_1, Y_4] = 2X_{23}, \\
 & [Y_2, Y_3] = 2X_{14}, & [Y_2, Y_4] = -2X_{13}, \\
 & & [Y_3, Y_4] = 2X_{12},
\end{array}
$$

$[Y_1, X_1] = [Y_2, X_2] = [Y_3, X_3] = [Y_4, X_4] = Z$, $[Y_i, X_j] = 0 \, (i \neq j)$,

and

$$[X_i, X_{jk}] = [X_i, Z] = [Y_i, X_{jk}] = [Y_i, Z] = 0.$$

We see D is a $(8, 15)$-distribution.

【 Characteristic matrix of Cartan's (8, 15)-distribution 】

The characteristic matrix A of D is given by

$$\begin{pmatrix} 0 & 2r_{12} & 2r_{13} & 2r_{14} & -s & 0 & 0 & 0 \\ -2r_{12} & 0 & 2r_{23} & 2r_{24} & 0 & -s & 0 & 0 \\ -2r_{13} & -2r_{23} & 0 & 2r_{34} & 0 & 0 & -s & 0 \\ -2r_{14} & -2r_{24} & -2r_{34} & 0 & 0 & 0 & 0 & -s \\ s & 0 & 0 & 0 & 0 & 2r_{34} & -2r_{24} & 2r_{23} \\ 0 & s & 0 & 0 & -2r_{34} & 0 & 2r_{14} & -2r_{13} \\ 0 & 0 & s & 0 & 2r_{24} & -2r_{14} & 0 & 2r_{12} \\ 0 & 0 & 0 & s & -2r_{23} & 2r_{13} & -2r_{12} & 0 \end{pmatrix}$$

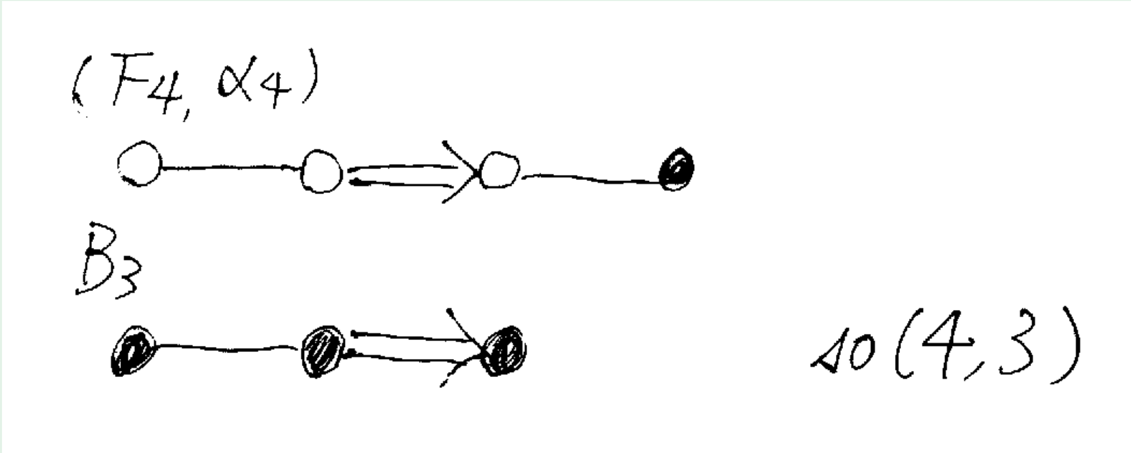
where $r_{ij} := H_{X_{ij}}$ and $s := H_Z$.

In this case we have the Pfaffian

$$\mathrm{Pf}(A) = \{s^2 - 4(r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23})\}^2,$$

and thus we have naturally the conformal $(4, 3)$-metric on $D^{\perp}(\subset T^*M), M = \mathbb{R}^{15}$.

【 F_4 and B_3 】



【 $(4,4)$-metric and SVC of Cartan's $(8,15)$-distribution 】

Write the characteristic matrix as $A = \begin{pmatrix} A_{11} & -sI \\ sI & A_{22} \end{pmatrix}$.
From the constrained Hamiltonian equation for Cartan's $(8,15)$-distribution (M,D), we have on the velocities of D-singular curves,

$$\begin{pmatrix} A_{11} & -sI \\ sI & A_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \mathbf{0}, \cdots\cdots (\star)$$

where

$$\dot{x} = u_1X_1 + u_2X_2 + u_3X_3 + u_4X_4 + v_1Y_1 + v_2Y_2 + v_3Y_3 + v_4Y_4,$$

and $\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = {}^t(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4)$.

If $(\boldsymbol{u}, \boldsymbol{v}) \neq (0,0)$, then we have

$$s^2 - 4(r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23}) = 0.$$

Moreover, the equation $(\star)$ as above is written as $(\star\star)$:

$$\begin{pmatrix} -v_1 & 2u_2 & 2u_3 & 2u_4 & 0 & 0 & 0 \\ -v_2 & -2u_1 & 0 & 0 & 2u_3 & 2u_4 & 0 \\ -v_3 & 0 & -2u_1 & 0 & -2u_2 & 0 & 2u_4 \\ -v_4 & 0 & 0 & -2u_1 & 0 & -2u_2 & -2u_3 \\ u_1 & 0 & 0 & 0 & 2v_4 & -2v_3 & 2v_2 \\ u_2 & 0 & -2v_4 & 2v_3 & 0 & 0 & -2v_1 \\ u_3 & 2v_4 & 0 & -2v_2 & 0 & 2v_1 & 0 \\ u_4 & -2v_3 & 2v_2 & 0 & -2v_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} s \\ r_{12} \\ r_{13} \\ r_{14} \\ r_{23} \\ r_{24} \\ r_{34} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If $(s, r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}) \neq \mathbf{0}$, then we have

$$u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 = 0$$

and naturally we obtain a conformal $(4,4)$-metric on D.

Proposition(Machida-I).
For the above Cartan's $(8, 15)$-distribution $D \subset T\mathbb{R}^{15}$, there exist the conformal $(4, 4)$-metric on D and the conformal $(4, 3)$-metric on $D^{\perp}$ obtained from the data on singular curves of D such that the null-cone $C \subset D$ coincides with the singular velocity cone SVC(D). Moreover the flag manifold of null-subspaces $\{\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset D^{\perp} \subset T^*M\}$ corresponds to a subclass of flags by null-subspaces $\{V_1 \subset V_2 \subset V_4 \subset C \subset D \subset TM\}$ in D and the prolongation (Z, E) of (M, D) by the above null-flags of D turns out to be a $(4, 7, 10, 13, 16, 18, 20, 21, 22, 23, 24)$-distribution. Moreover its symbol algebra realises the nilpotent part of simple Lie algebra F_4, $\mathfrak{m} = \oplus_{i=-11}^{-1}\mathfrak{g}_i$.

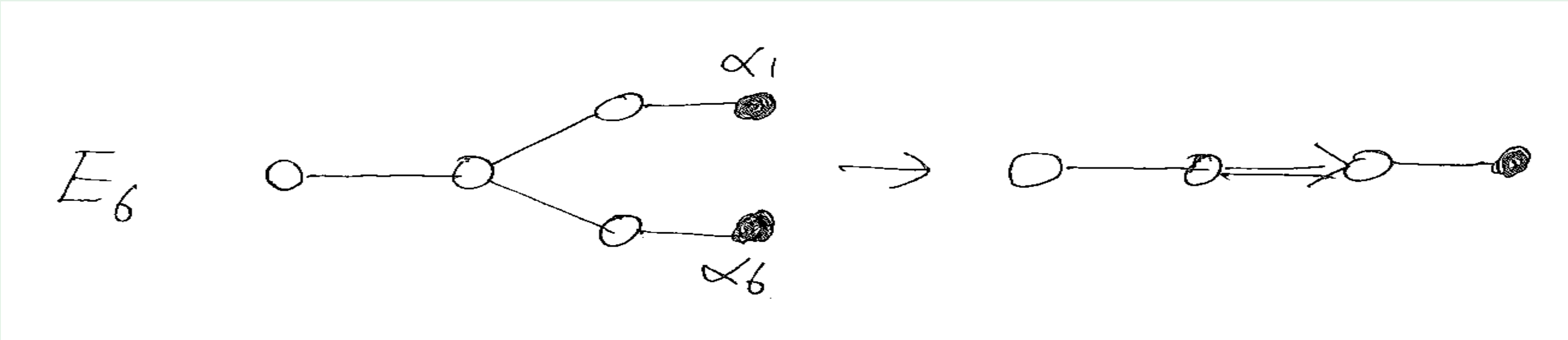
Remark.

Exactly speaking, the simple Lie algebra F_4 has three real forms; one compact type $F_{4(-52)}$ and two non-compact types $F_{4(4)}$ (or $\mathrm{F}_4\mathrm{I}$, $\widetilde{\mathrm{F}}_4$) and $\mathrm{F}_{4(-20)}$ (or $\mathrm{F}_4\mathrm{II}$, F'_4). In fact, Cartan's example gives the normal form of $(8, 15)$-distributions corresponding to the real form $\mathrm{F}_{4(4)}$ (a "hyperbolic" F_4-$(8, 15)$-distributions).

○ Goo Ishikawa, Yoshinori Machida, *Prolongation of* $(8, 15)$*-distribution of type* F_4 *by singular curves*, submitted. arXiv:2501.02789 [math.DG]

【 Nurowski's $(16, 24)$-distribution of type E_6 】

○ P. Nurowski, *Exceptional simple real Lie algebras* $\mathfrak{f}_4$ *and* $\mathfrak{e}_6$ *via contactifications*,
J. Inst. Math. Jussieu **24 (1)** (2024), 157–201.
arXiv:2302.13606 [math.DG]



On $M = \mathbb{R}^{24}$ with coordinates $u_1, \dots, u_8, x_1, \dots, x_8, y_1, \dots, y_8$, we consider the Pfaff system

$$\begin{cases} \lambda_1 = du_1 - x_1 dy_2 + x_2 dy_1 + x_7 dy_8 - x_8 dy_7, \\ \lambda_2 = du_2 - x_2 dy_4 + x_4 dy_2 + x_6 dy_8 - x_8 dy_6, \\ \lambda_3 = du_3 - x_1 dy_4 + x_4 dy_1 + x_5 dy_8 - x_8 dy_5, \\ \lambda_4 = du_4 - x_5 dy_2 + x_6 dy_1 + x_7 dy_4 - x_8 dy_3, \\ \lambda_5 = du_5 - x_2 dy_3 + x_3 dy_2 + x_6 dy_7 - x_7 dy_6, \\ \lambda_6 = du_6 - x_1 dy_3 + x_3 dy_1 + x_5 dy_7 - x_7 dy_5, \\ \lambda_7 = du_7 - x_3 dy_4 + x_4 dy_3 + x_5 dy_6 - x_6 dy_5, \\ \lambda_8 = du_8 - x_1 dy_6 + x_2 dy_5 + x_3 dy_8 - x_4 dy_7, \end{cases}$$

and the distribution $D \subset TM$ of rank 16 defined by

$$D := \{\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 0\}.$$

Then D is generated by

$$
\begin{cases}
X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \frac{\partial}{\partial x_3}, \quad X_4 = \frac{\partial}{\partial x_4}, \\
X_5 = \frac{\partial}{\partial x_5}, \quad X_6 = \frac{\partial}{\partial x_6}, \quad X_7 = \frac{\partial}{\partial x_7}, \quad X_8 = \frac{\partial}{\partial x_8}, \\
Y_1 = \frac{\partial}{\partial y_1} - x_2\frac{\partial}{\partial u_1} - x_4\frac{\partial}{\partial u_3} - x_6\frac{\partial}{\partial u_4} - x_3\frac{\partial}{\partial u_6}, \\
Y_2 = \frac{\partial}{\partial y_2} + x_1\frac{\partial}{\partial u_1} - x_4\frac{\partial}{\partial u_2} + x_5\frac{\partial}{\partial u_4} - x_3\frac{\partial}{\partial u_5}, \\
Y_3 = \frac{\partial}{\partial y_3} + x_8\frac{\partial}{\partial u_4} + x_2\frac{\partial}{\partial u_5} + x_1\frac{\partial}{\partial u_6} - x_4\frac{\partial}{\partial u_7}, \\
Y_4 = \frac{\partial}{\partial y_4} + x_2\frac{\partial}{\partial u_2} + x_1\frac{\partial}{\partial u_3} - x_7\frac{\partial}{\partial u_4} + x_3\frac{\partial}{\partial u_7}, \\
Y_5 = \frac{\partial}{\partial y_5} + x_8\frac{\partial}{\partial u_3} + x_7\frac{\partial}{\partial u_6} + x_6\frac{\partial}{\partial u_7} - x_2\frac{\partial}{\partial u_8}, \\
Y_6 = \frac{\partial}{\partial y_6} + x_8\frac{\partial}{\partial u_2} + x_7\frac{\partial}{\partial u_5} - x_5\frac{\partial}{\partial u_7} + x_1\frac{\partial}{\partial u_8}, \\
Y_7 = \frac{\partial}{\partial y_7} + x_8\frac{\partial}{\partial u_1} - x_6\frac{\partial}{\partial u_5} - x_5\frac{\partial}{\partial u_6} + x_4\frac{\partial}{\partial u_8}, \\
Y_8 = \frac{\partial}{\partial y_8} - x_7\frac{\partial}{\partial u_1} - x_6\frac{\partial}{\partial u_2} - x_5\frac{\partial}{\partial u_3} - x_3\frac{\partial}{\partial u_8}.
\end{cases}
$$

Then we have the bracket relations,

$$
\begin{aligned}
&[X_1, Y_2] = \frac{\partial}{\partial u_1}, [X_1, Y_3] = \frac{\partial}{\partial u_6}, [X_1, Y_4] = \frac{\partial}{\partial u_3}, [X_1, Y_6] = \frac{\partial}{\partial u_8},\\
&[X_2, Y_1] = -\frac{\partial}{\partial u_1}, [X_2, Y_3] = \frac{\partial}{\partial u_5}, [X_2, Y_4] = \frac{\partial}{\partial u_2}, [X_2, Y_5] = -\frac{\partial}{\partial u_8},\\
&[X_3, Y_1] = -\frac{\partial}{\partial u_6}, [X_3, Y_2] = -\frac{\partial}{\partial u_5}, [X_3, Y_4] = \frac{\partial}{\partial u_7}, [X_3, Y_8] = -\frac{\partial}{\partial u_8},\\
&[X_4, Y_1] = -\frac{\partial}{\partial u_3}, [X_4, Y_2] = -\frac{\partial}{\partial u_2}, [X_4, Y_3] = -\frac{\partial}{\partial u_7}, [X_4, Y_7] = \frac{\partial}{\partial u_8},\\
&[X_5, Y_2] = \frac{\partial}{\partial u_4}, [X_5, Y_6] = -\frac{\partial}{\partial u_7}, [X_5, Y_7] = -\frac{\partial}{\partial u_6}, [X_5, Y_8] = -\frac{\partial}{\partial u_3},\\
&[X_6, Y_1] = -\frac{\partial}{\partial u_4}, [X_6, Y_5] = \frac{\partial}{\partial u_7}, [X_6, Y_7] = -\frac{\partial}{\partial u_5}, [X_6, Y_8] = -\frac{\partial}{\partial u_2},\\
&[X_7, Y_4] = -\frac{\partial}{\partial u_4}, [X_7, Y_5] = \frac{\partial}{\partial u_6}, [X_7, Y_6] = \frac{\partial}{\partial u_5}, [X_7, Y_8] = -\frac{\partial}{\partial u_1},\\
&[X_8, Y_3] = \frac{\partial}{\partial u_4}, [X_7, Y_5] = \frac{\partial}{\partial u_3}, [X_7, Y_6] = \frac{\partial}{\partial u_2}, [X_7, Y_7] = \frac{\partial}{\partial u_1},
\end{aligned}
$$

other brackets $[X_i, X_j], [Y_i, Y_j]$ and $[X_i, Y_j]$ being zero.

In particular we see that D is a $(16, 24)$-distribution.

Note that, for the frame

$$X_1, \dots, X_8, \quad Y_1, \dots, Y_8, \quad \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_8},$$

of TM, the dual frame of T^*M is given by

$$dx_1, \dots, dx_8, \ dy_1, \dots, dy_8, \ \lambda_1, \dots, \lambda_8,$$

and that $D^\perp = \langle \lambda_1, \dots, \lambda_8 \rangle$.

Any vector $v \in D$ is written uniquely as

$$v = \sum_{i=1}^8 a_i X_i + \sum_{j=1}^8 b_j Y_j,$$

where $a_1, \dots, a_8, b_1, \dots, b_8$ are regarded as "control parameters", while any co-vector $\alpha \in D^\perp$ is written uniquely as

$$\alpha = \sum_{k=1}^8 \varphi_k \lambda_k,$$

where $\varphi_1, \dots, \varphi_8$ are called "adjoint parameters".

The characteristic matrix is given by

$$A = \begin{pmatrix} O & B \\ -{}^tB & O \end{pmatrix}$$

where

$$B = \begin{pmatrix} 0 & \varphi_1 & \varphi_6 & \varphi_3 & 0 & \varphi_8 & 0 & 0 \\ -\varphi_1 & 0 & \varphi_5 & \varphi_2 & -\varphi_8 & 0 & 0 & 0 \\ -\varphi_6 & -\varphi_5 & 0 & \varphi_7 & 0 & 0 & 0 & -\varphi_8 \\ -\varphi_3 & -\varphi_2 & -\varphi_7 & 0 & 0 & 0 & \varphi_8 & 0 \\ 0 & \varphi_4 & 0 & 0 & 0 & -\varphi_7 & -\varphi_6 & -\varphi_3 \\ -\varphi_4 & 0 & 0 & 0 & \varphi_7 & 0 & -\varphi_5 & -\varphi_2 \\ 0 & 0 & 0 & -\varphi_4 & \varphi_6 & \varphi_5 & 0 & -\varphi_1 \\ 0 & 0 & \varphi_4 & 0 & \varphi_3 & \varphi_2 & \varphi_1 & 0 \end{pmatrix}$$

Lemma. We have that

$$\det(B) = \det(-{}^tB) = (\varphi_1\varphi_7 - \varphi_2\varphi_6 + \varphi_3\varphi_5 + \varphi_4\varphi_8)^4,$$

$$\det(A) = (\varphi_1\varphi_7 - \varphi_2\varphi_6 + \varphi_3\varphi_5 + \varphi_4\varphi_8)^8,$$

and therefore that

$$\mathrm{Pf}(A) = (\varphi_1\varphi_7 - \varphi_2\varphi_6 + \varphi_3\varphi_5 + \varphi_4\varphi_8)^4. \qquad \Box$$

Thus we have intrinsically the conformal $(4,4)$-metric on $D^\perp$ defined by the null cone

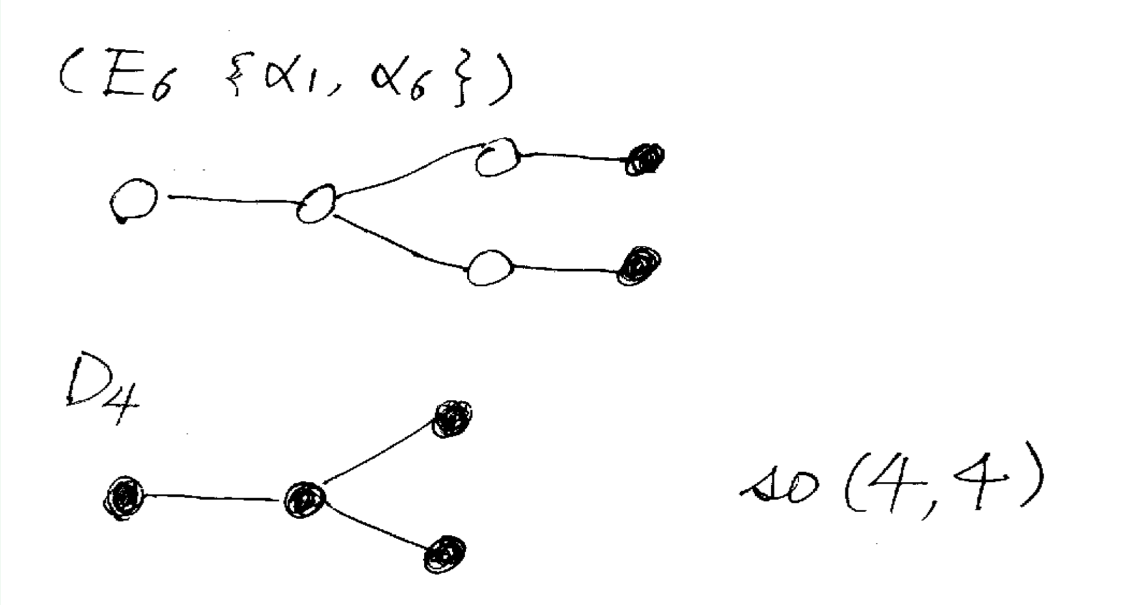
$$C^\perp := \{\textstyle\sum_{k=1}^8 \varphi_k\lambda_k \mid \varphi_1\varphi_7 - \varphi_2\varphi_6 + \varphi_3\varphi_5 + \varphi_4\varphi_8 = 0\},$$

of the quadratic form and associated bi-linear form on $D^\perp$:

$$Q(\varphi) := \varphi_1\varphi_7 - \varphi_2\varphi_6 + \varphi_3\varphi_5 + \varphi_4\varphi_8,$$

$$\langle\varphi|\psi\rangle := \tfrac{1}{2}(\varphi_1\psi_7 + \psi_1\varphi_7 - \varphi_2\psi_6 - \psi_2\varphi_6 + \varphi_3\psi_5 + \psi_3\varphi_5 + \varphi_4\psi_8 + \psi_4\varphi_8).$$

【 E_6 and D_4 】



○ G. Ishikawa, Y. Machida, M. Takahashi, *Geometry of D_4 conformal triality and singularities of tangent surfaces,* in Proc. of Singularities in Geometry and Appl. III, Edinburgh, Scotland, 2013, Journal of Singularities, vol. 12 (2015), 27–52. DOI: 10.5427/jsing.2015.12c

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The singular directions $v = \sum_{i=1}^{8} v_i X_i + \sum_{j=1}^{8} w_j Y_j$ are given by the condition

$$\begin{pmatrix} O & B \\ -{}^tB & O \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is equivalent to that $Bw = 0$ and $(-{}^tB)v = 0$, where $v = {}^t(v_1, \dots, v_8), w = {}^t(w_1, \dots, w_8)$.

The condition $Bw = 0$ is equivalent to that

$$\left\{ \begin{array}{rcl} \varphi_1 w_2 + \varphi_6 w_3 + \varphi_3 w_4 + \varphi_8 w_6 & = & 0 \\ -\varphi_1 w_1 + \varphi_5 w_3 + \varphi_2 w_4 - \varphi_8 w_5 & = & 0 \\ -\varphi_6 w_1 - \varphi_5 w_2 + \varphi_7 w_4 - \varphi_8 w_8 & = & 0 \\ -\varphi_3 w_1 - \varphi_2 w_2 - \varphi_7 w_3 + \varphi_8 w_7 & = & 0 \\ \varphi_4 w_2 - \varphi_7 w_6 - \varphi_6 w_7 - \varphi_3 w_8 & = & 0 \\ -\varphi_4 w_1 + \varphi_7 w_5 - \varphi_5 w_7 - \varphi_2 w_8 & = & 0 \\ -\varphi_4 w_4 + \varphi_6 w_5 + \varphi_5 w_6 - \varphi_1 w_8 & = & 0 \\ \varphi_4 w_3 + \varphi_3 w_5 + \varphi_2 w_6 + \varphi_1 w_7 & = & 0, \end{array} \right.$$

which is equivalent to that $W\varphi = 0$, where

$$
W = \begin{pmatrix}
w_2 & 0 & w_4 & 0 & 0 & w_3 & 0 & w_6 \\
-w_1 & w_4 & 0 & 0 & w_3 & 0 & 0 & -w_5 \\
0 & 0 & 0 & 0 & -w_2 & -w_1 & w_4 & -w_8 \\
0 & -w_2 & -w_1 & 0 & 0 & 0 & -w_3 & w_7 \\
0 & 0 & -w_8 & w_2 & 0 & -w_7 & -w_6 & 0 \\
0 & -w_8 & 0 & -w_1 & -w_7 & 0 & w_5 & 0 \\
-w_8 & 0 & 0 & -w_4 & w_6 & w_5 & 0 & 0 \\
w_7 & w_6 & w_5 & w_3 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

and $\varphi = {}^t(\varphi_1, \dots, \varphi_8)$.

Then we observe that all columns of W are null and orthogonal to each other for the quadratic form

$$q\left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix}\right) := a_1a_6 - a_2a_5 - a_3a_8 + a_4a_7,$$

and for the associated conformal $(4,4)$-metric

$$(\boldsymbol{a}|\boldsymbol{b}) = \frac{1}{2}\left(a_1b_6 + b_1a_6 - a_2b_5 - b_2a_5 - a_3b_8 - b_3a_8 + a_4b_7 + b_4a_7\right).$$

The condition $(-{}^tB)v = 0$ is equivalent to that

$$\left\{\begin{array}{rcl}
\varphi_1 v_2 + \varphi_6 v_3 + \varphi_3 v_4 + \varphi_4 v_6 & = & 0 \\
-\varphi_1 v_1 + \varphi_5 v_3 + \varphi_2 v_4 - \varphi_4 v_5 & = & 0 \\
-\varphi_6 v_1 - \varphi_5 v_2 + \varphi_7 v_4 - \varphi_4 v_8 & = & 0 \\
-\varphi_3 v_1 - \varphi_2 v_2 - \varphi_7 v_3 + \varphi_4 v_7 & = & 0 \\
\varphi_8 v_2 - \varphi_7 v_6 - \varphi_6 v_7 - \varphi_3 v_8 & = & 0 \\
-\varphi_8 v_1 + \varphi_7 v_5 - \varphi_5 v_7 - \varphi_2 v_8 & = & 0 \\
-\varphi_8 v_4 + \varphi_6 v_5 + \varphi_5 v_6 - \varphi_1 v_8 & = & 0 \\
\varphi_8 v_3 + \varphi_3 v_5 + \varphi_2 v_6 + \varphi_1 v_7 & = & 0,
\end{array}\right.$$

which is equivalent to that $V\varphi = 0$, where

$$V = \begin{pmatrix} v_2 & 0 & v_4 & v_6 & 0 & v_3 & 0 & 0 \\ -v_1 & v_4 & 0 & -v_5 & v_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -v_8 & -v_2 & -v_1 & v_4 & 0 \\ 0 & -v_2 & -v_1 & v_7 & 0 & 0 & -v_3 & 0 \\ 0 & 0 & -v_8 & 0 & 0 & -v_7 & -v_6 & v_2 \\ 0 & -v_8 & 0 & 0 & -v_7 & 0 & v_5 & -v_1 \\ -v_8 & 0 & 0 & 0 & v_6 & v_5 & 0 & -v_4 \\ v_7 & v_6 & v_5 & 0 & 0 & 0 & 0 & v_3 \end{pmatrix}$$

and $\varphi = {}^t(\varphi_1, \dots, \varphi_8)$. Then we see that all columns of V are null and orthogonal to each other for the same quadratic form q and the associated $(4, 4)$-metric as above.

Proposition. Let $D \subset TM = T\mathbb{R}^{24}$ be the Nurowski's $(16, 24)$-distribution of type E_6. Then we have:
(1) Naturally there arise the conformal $(4, 4)$-metric Q on $D^{\perp}$, the decomposition $D = D_1 \oplus D_2$ into distributions D_1, D_2 of rank 8, and conformal $(4, 4)$-metrics q_1 on D_1 and q_2 on D_2, respectively.
(2) Associated to flag bundle $\mathcal{F}'_{1,2,3}$ over M which consists of Q-null-flags $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset C'_{\mathbf{x}} \subset D^{\perp}_{\mathbf{x}}$, $\mathbf{x} \in M = \mathbb{R}^{24}$, there arises the flag-manifold $\mathcal{F}$ consisting of null-flags $V_2 \subset V_4 \subset V_6 \subset C_{\mathbf{x}} \subset (D_1)_{\mathbf{x}} \oplus (D_2)_{\mathbf{x}} \subset D_{\mathbf{x}}$, for $q_1 \oplus q_2$, $\mathbf{x} \in M$.
(3) The nilpotent graded Lie algebra of the prolongation $(\mathcal{F}, E)$ of (M, D) is isomorphic to the negative part of the real form of the exceptional simple graded Lie algebra $E_6 I$.

Thank you !