

# (3, 6)-接分布の特異曲線とその周辺

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第3 1回 沼津改め 静岡研究会

— 幾何, 複素解析, そして数理物理 —

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2026年3月3日～5日

## 【 (3, 6)-distributions 】

Let  $X$  be a  $C^\infty$  manifold of dimension 6 and  $D \subset TX$  a  $C^\infty$  distribution (= subbundle of  $TX$ ) of rank 3.

We call  $(X, D)$  a **(3, 6) distribution** if Lie brackets of local sections to  $D$  generate  $TX$ , i.e., if

$$D^{(2)} := [D, D] (= D + [D, D]) = TX,$$

or equivalently, if, for any  $x_0 \in X$ , there exists a frame  $\{\xi_1, \xi_2, \xi_3\}$  of  $D$  on a neighbourhood  $U$  of  $x_0$  such that, for any  $x \in U$ ,  $\xi_1, \xi_2, \xi_3$  and  $\xi_4 := [\xi_1, \xi_2]$ ,  $\xi_5 := [\xi_1, \xi_3]$ ,  $\xi_6 := [\xi_2, \xi_3]$  generate the whole tangent space  $T_x X$ .

## 【 Grassmannian of null 3-spaces in $\mathbb{R}^{3,4}$ 】

**Example.** Let  $\mathbb{R}^{3,4}$  denote the vector space  $\mathbb{R}^7$  with an indefinite metric of index  $(3, 4)$ , where  $SO(3, 4)$ , the simple Lie group of type  $B_3$ , acts on naturally.

Consider the **maximal null Grassmannian**

$$X := \{V \mid V : \text{null subspace of } \mathbb{R}^{3,4}, \dim(V) = 3\}.$$

Then we see  $\dim(X) = 12 - 6 = 6$ .

Define  $D \subset TX$  by setting, for  $x = V \in X$ ,  $D_x = D_V$  as the set of velocity vectors  $\frac{d}{dt}V_t|_{t=0}$  for all curves  $\{V_t\}$  in  $X$  starting with  $V_0 = V$  and satisfying  $\frac{d}{dt}V_t \subset V_t^\perp$ ,  $t \in (\mathbb{R}, 0)$ .

Then  $(X, D)$  is a **(3, 6)-distribution**. Any small perturbation of  $D \subset TX$  remains a (3, 6)-distribution.

【 Why (3, 6)-distributions ? 】

Dynkin diagrams of  $G_2$ ,  $B_3$  and  $F_4$



$G_2$  case:

{(2, 3, 5)-distributions}  $\leftarrow - - - \rightarrow$

{(2, 3, 4, 5, 6)-distributions with pseudo-product structures}

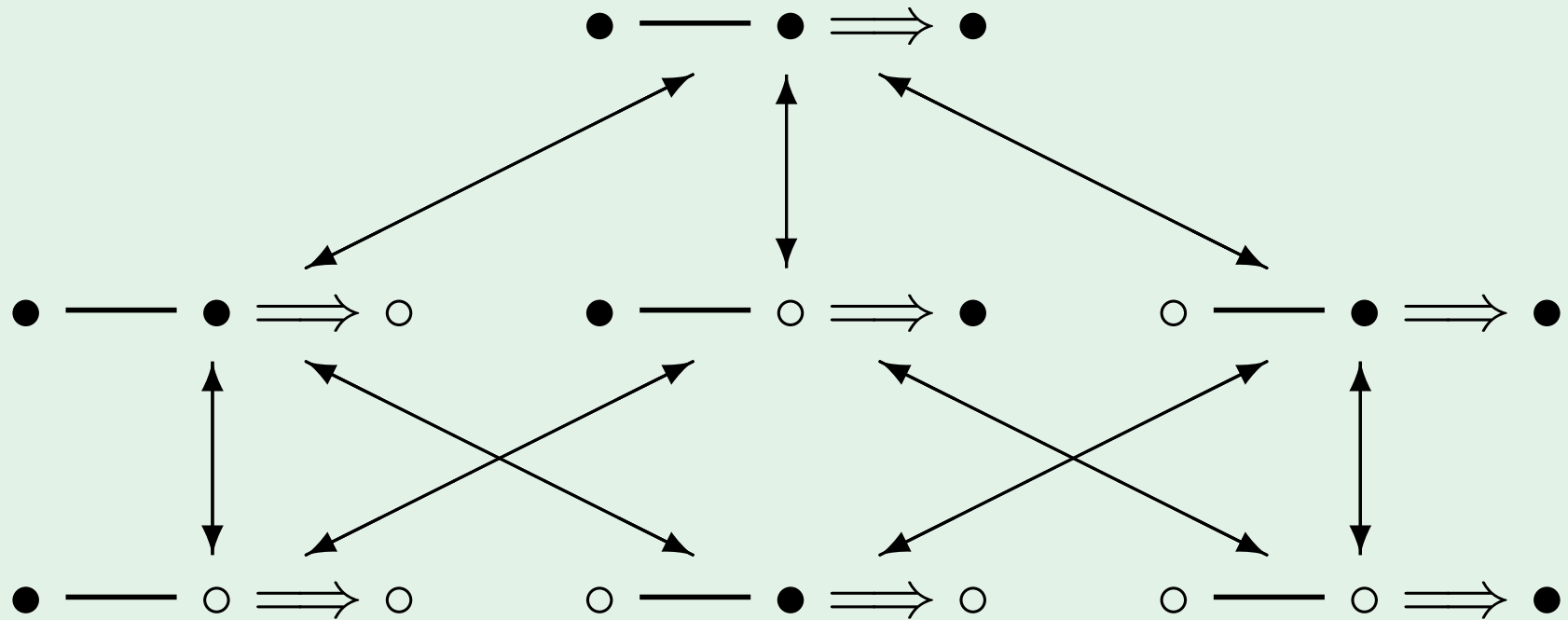
$\leftarrow - - - \rightarrow \left\{ \begin{array}{l} \text{5-dimensional contact manifolds} \\ \text{with Lagrangian cone structures} \end{array} \right\}$



**Question.** How about  $B_3$  ?

{(3, 6)-distributions}  $\leftarrow - - - \rightarrow$  ??

## The $B_3$ Dynkin diagrams



conformal geometry

(2, 3)-conformal str.

contact geometry

7-dim. contact str.  
with Lagrangian cone

control geometry

(3, 6)-distribution

## 【 End-point mappings and singular curves 】

Given a distribution  $D \subset TX$  and a point  $x_0 \in X$ , consider the space  $\mathcal{C}_{x_0}$  consisting of integral curves  $\gamma : [0, 1] \rightarrow X$ ,  $\gamma'(t) \in D_{\gamma(t)}$  for almost every  $t \in [0, 1]$ , with  $\gamma(0) = x_0$ .

The **endpoint mapping**  $\text{End} : \mathcal{C}_{x_0} \rightarrow X$  is defined by  $\text{End}(\gamma) := \gamma(1)$ .

Then a curve  $\gamma \in \mathcal{C}_{x_0}$  is called a **singular curve of  $D$**  if  $\gamma$  is a **singular point** of the **endpoint map**  $\text{End}$ , i.e., the differential  $\text{End}_* : T_\gamma \mathcal{C}_{x_0} \rightarrow T_{\text{End}(\gamma)} X$  is **non-surjective**, i.e., there exists a **cotangent vector**  $p \in T_{\text{End}(\gamma)}^* X$ ,  $p \neq 0$ , satisfying that  $\langle p, \text{End}_*(v) \rangle = 0$ , for any  $v \in T_\gamma \mathcal{C}_{x_0}$ .

## 【 Constrained Hamiltonian equations 】

Let  $(X, D)$  be a  $(3, 6)$ -distribution and  $D = \langle \xi_1, \xi_2, \xi_3 \rangle$ . Define  $H_\xi : T^*X \rightarrow \mathbb{R}$  by  $H_\xi(x, p) := \langle p, \xi(x) \rangle$  and  $H : T^*X \times_X D \rightarrow \mathbb{R}$ , the **Hamiltonian of  $D$** , by

$$H(x, p, u) := u_1 H_{\xi_1}(x, p) + u_2 H_{\xi_2}(x, p) + u_3 H_{\xi_3}(x, p).$$

Consider the Hamiltonian equation with **constraints**:

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}$$

$$H_{\xi_1} = 0, \quad H_{\xi_2} = 0, \quad H_{\xi_3} = 0, \quad \boxed{p \neq 0}.$$

## 【 Singular curves, singular characteristics 】

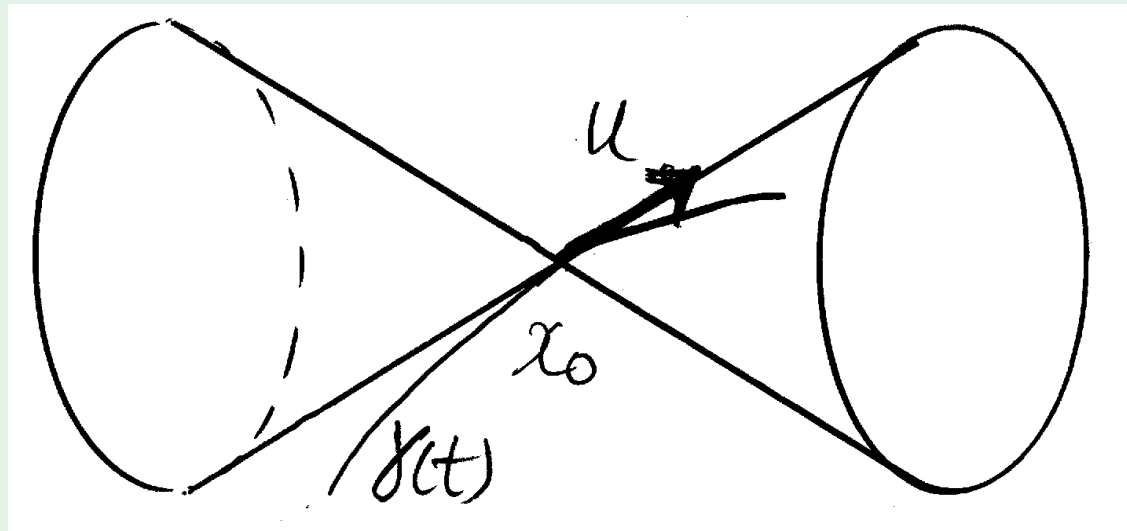
**Definition.** A curve  $\gamma(t) = x(t)$  on  $X$  is called an **integral curve of  $D$**  if  $\gamma'(t) \in T_{\gamma(t)}X$  for almost every  $t$ .

An integral curve  $\gamma$  is called a **singular curve of  $D$**  if there exist  $p(t), u(t)$  such that  $(x(t), p(t), u(t))$  satisfies the above Hamiltonian equation with constraints. Moreover, in this case, the corresponding curve  $(x(t), p(t))$  in  $T^*X$  is called a **singular characteristic of  $D$** .

## 【 Singular velocity cone 】

**Definition.** (Singular Velocity Cone)

We define the singular velocity cone  $SVC(D)$  of  $D$  at  $x_0 \in X$  by the set of tangent vectors  $(x_0, u) \in D$  such that there exists a singular curve  $\gamma : (\mathbb{R}, 0) \rightarrow X$  of  $D$  such that  $\gamma(0) = x_0, \gamma'(0) = u$



**Proposition** (Montgomery(2002)).

Let  $D(\subset TX)$  be a (3, 6)-distribution on  $X$ . Then, for any  $x_0 \in X$  and for any  $u \in D_{x_0}(\subset T_{x_0}X)$ , there exists a singular curve  $\gamma : (\mathbb{R}, 0) \rightarrow X$  of  $D$  satisfying  $\gamma(0) = x_0, \gamma'(0) = u$ . Therefore  $\text{SVC}(D) = D$ .

## 【 Prolongations of (3, 6)-distributions 】

Set  $Z := PD = (D \setminus \text{zero-section})/\mathbb{R}^\times$ , the fibrewise projectivisation of the bundle  $D \rightarrow X$ . Then  $\dim(Z) = 8$ . The induced projection  $\pi_X : Z \rightarrow X$ ,  $\pi(x, [u]) = x$  has the projective plane  $P^2$  as fibres.

We define the distribution  $E \subset TZ$  over  $Z$  by

$$E_{(x, [u])} := \{w \in T_{(x, [u])}Z \mid \pi_*(w) \in [u]\},$$

for  $(x, [u]) \in Z$ , where  $[u]$  is regarded as a line in  $D_x$  through the origin. Note that  $E$  is of rank 3.

## 【 Pseudo-product structures of type $B_3(2, 3)$ 】

**Lemma.** The prolongation  $E \subset TZ$  of  $(X, D)$  has the small growth  $(3, 5, 7, 8)$ .

Moreover there exists **unique complementary subbundle**  $E_1 \subset E$  of rank 1 to  $E_2 := \text{Ker}(\pi_{X*} : TZ \rightarrow TX)$  such that the decomposition  $E = E_1 \oplus E_2$  into integrable subbundles  $E_1, E_2$  of rank 1, 2 respectively gives a “**pseudo-product structure of type  $B_3(2, 3)$** ” on  $E$ , i.e.,

$$\begin{aligned} [E_1, E_2] &= E^{(2)}, & [E_1, E^{(2)}] &= E^{(3)}, & [E_2, E^{(2)}] &= E^{(2)}, \\ [E_1, E^{(3)}] &= E^{(3)}, & [E_2, E^{(3)}] &= TZ. \end{aligned}$$

## 【 The correspondence theorem 】

**Theorem.** The prolongation procedure induces the bijection between

{local isomorphism classes of (3, 6)-distributions}

and

{ local isomorphism classes of (3, 5, 7, 8)-distributions  
with pseudo-product structure of type  $B_3(2, 3)$  }.

**Example** (The null flag manifold of type  $B_3(2, 3)$ ).

$Z := \{V_2 \subset V_3 \subset \mathbb{R}^{3,4} \mid V_3 : \text{null}\}$  and  $E$  : the distribution on  $Z$  defined by  $V_2' \in V_3, V_3' \in V_3^\perp$ . Then  $E$  is a (3, 5, 7, 8)-distribution with pseudo-product structure of type  $B_3(2, 3)$ .

## 【 Pseudo-product structures of type $B_3(1, 2, 3)$ 】

Let  $(Z, E)$  be a  $(3, 5, 7, 8)$ -distribution with pseudo-product structure  $E = E_1 \oplus E_2$  of type  $B_3(2, 3)$ .

**Lemma.** We have  $\text{SVC}(E) = (E_1 \oplus 0) \cup (0 \oplus E_2)$ .

We set  $W := P(0 \oplus E_2) = P(E_2)$ ,  $\dim(W) = 9$ , with projection  $\pi_Z : W \rightarrow Z$ . Define  $F \subset TW$ , for any  $(x, [u], [\ell]) \in P(E_2)$ ,  $(x, [u]) \in Z$ ,  $[\ell] \in P((E_2)_{(x, [u])})$ , by

$$F_{(x, [u], [\ell])} := \{[v] \in T_{(x, [u], [\ell])} \mid \pi_{Z*}([v]) \in E_1 \times [\ell]\}.$$

Then  $F \subset TW$  is a  $(3, 5, 7, 8, 9)$ -distribution with a pseudo-product structure of type  $B_3(1, 2, 3)$ :

**Definition.** A pseudo-product structure of type  $B_3(1, 2, 3)$  on  $F$  is given by a decomposition  $F = F_1 \oplus F_2 \oplus F_3$  into line sub-bundles up to replacing  $F_3$  by another line-subbundle of  $F_1 \oplus F_3$  independent of  $F_1$ , such that, for a generator  $\theta_1, \theta_2, \theta_3$  of  $F_1, F_2, F_3$  respectively,

$$\begin{aligned}
& [\theta_1, \theta_2] =: \theta_4, \quad [\theta_1, \theta_3] \in \langle \theta_1, \theta_3 \rangle, \quad [\theta_2, \theta_3] =: \theta_5, \quad \text{rank}(F^{(2)}) = 5, \\
& [\theta_1, \theta_4] \in \langle \theta_1, \theta_2, \theta_4 \rangle, \quad [\theta_1, \theta_5] \equiv: \theta_6 \text{ mod. } F^{(2)}, \quad [\theta_2, \theta_4] \in \langle \theta_1, \theta_2, \theta_4 \rangle, \\
& [\theta_2, \theta_5] \in \langle \theta_2, \theta_3, \theta_5 \rangle, \quad [\theta_3, \theta_4] \equiv -\theta_6 \text{ mod. } F^{(2)}, \\
& [\theta_3, \theta_5] \equiv: \theta_7 \text{ mod. } F^{(2)}, \quad \text{so that } \text{rank}(F^{(3)}) = 7, \\
& [\theta_1, \theta_6] \in \langle \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6 \rangle, \quad [\theta_1, \theta_7] \equiv: \theta_8 \text{ mod. } F^{(3)}, \\
& [\theta_2, \theta_6] \in \langle \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6 \rangle, \quad [\theta_2, \theta_7] \in \langle \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_7 \rangle, \\
& [\theta_3, \theta_6] \equiv: \theta_8 \text{ mod. } F^{(3)}, \quad [\theta_3, \theta_7] \in F^{(3)}, \quad \text{so that } \text{rank}(F^{(4)}) = 8, \\
& [\theta_1, \theta_8] \in F^{(4)}, \quad [\theta_2, \theta_8] \equiv: \theta_9 \text{ mod. } F^{(4)}, \quad [\theta_3, \theta_8] \in F^{(4)}, \\
& \text{so that } \text{rank}(F^{(5)}) = 9, \quad F^{(5)} = TW. \quad \square
\end{aligned}$$

**Example** (The null full flag manifold of type  $B_3(1, 2, 3)$ ).

Let  $W := \{V_1 \subset V_2 \subset V_3 \subset \mathbb{R}^{3,4} \mid V_3 : \text{null}\}$  and  $F$  the distribution on  $W$  defined by  $V_1' \subset V_2, V_2' \subset V_3, V_3' \subset V_3^\perp$ . Then  $F$  is a  $(3, 5, 7, 8, 9)$ -distribution with pseudo-product structure of type  $B_3(1, 2, 3)$ .

## 【 The correspondence theorem (continued) 】

**Theorem.**

The prolongation procedures induce the bijections between

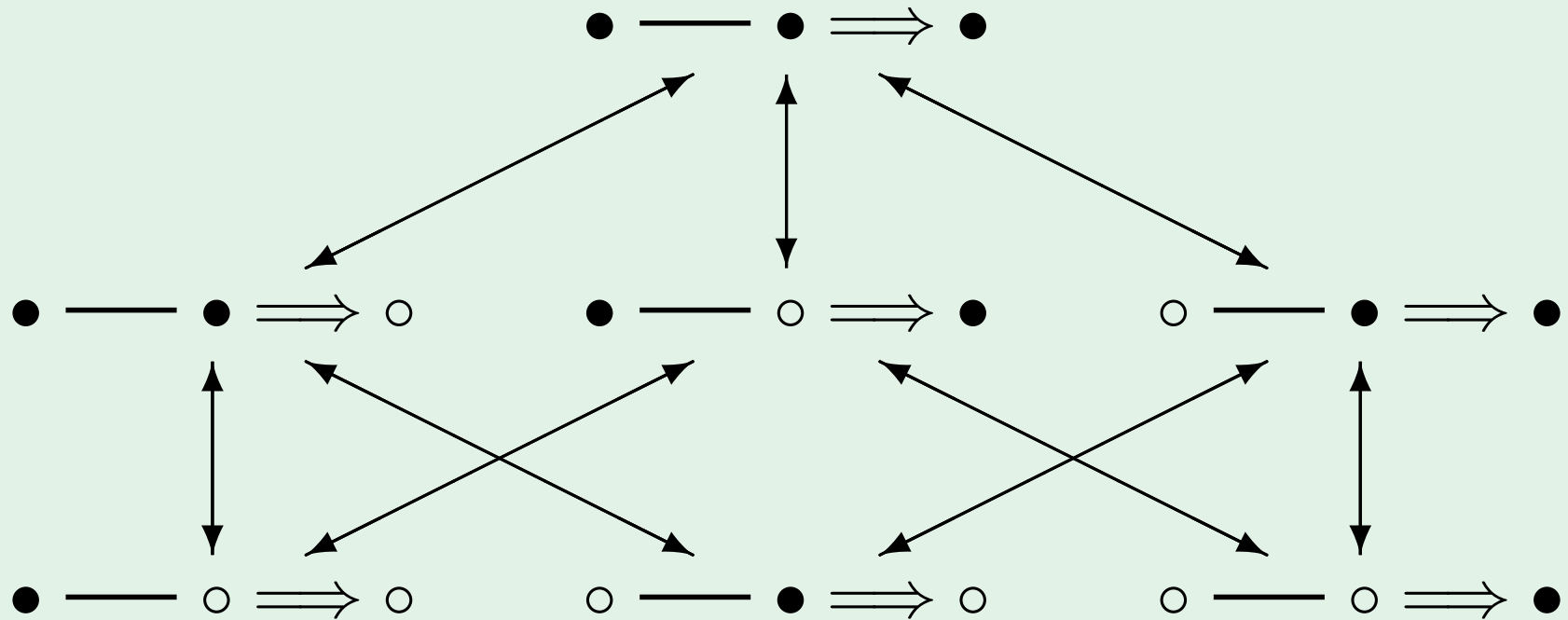
{local isomorphism classes of (3, 6)-distributions},

{ local isomorphism classes of (3, 5, 7, 8)-distributions  
with pseudo-product structure of type  $B_3(2, 3)$  } ,

and

{ local isomorphism classes of (3, 5, 7, 8, 9)-distributions  
with pseudo-product structure of type  $B_3(1, 2, 3)$  } .

## The completed $B_3$ Dynkin diagrams



(3, 6)-distributions

## 【 Another prolongations of (3, 6)-distributions 】

Again, let  $(X, D)$  be a (3, 6)-distribution, but, this time, we consider  $D^* = \text{Hom}(D, \mathbb{R})$ , the dual bundle of  $D$ .

We set  $S := P(D^*) = \{(x, [\alpha]) \mid x \in X, [\alpha] \in P(D_x^*)\}$ , which is identified with the space 2-dimensional subspaces in  $D$  over any point of  $X$ . Note that  $\dim(S) = 8$ . Define the distribution  $L \subset TS$  over  $S$ , for  $(x, [\alpha]) \in S$ , by

$$L_{(x, [\alpha])} := \{v \in T_{(x, [\alpha])}S \mid \langle \alpha, \pi_*(v) \rangle = 0\},$$

where  $\pi : S \rightarrow X$  is the natural projection. Then  $\text{rank}(L) = 4$ . Moreover we see that the small growth of  $L$  is equal to  $(4, 6, 8)$ .

**Example** (The null flag manifold  $B_3(1, 3)$ ).

$S := \{V_1 \subset V_3 \subset \mathbb{R}^{3,4} \mid V_3 : \text{null}\}$  and  $L$  : the distribution on  $S$  defined by  $V_1' \in V_3, V_3' \in V_3^\perp$ . Then  $L$  is a  $(4, 6, 8)$ -distribution with pseudo-product structure of type  $B_3(1, 3)$ .

## [ Pseudo-product structures of type $B_3(1, 3)$ ]

**Definition.** (Pseudo-product structure of type  $B_3(1, 3)$ ).

Let  $L \subset TS$  be a  $(4, 6, 8)$ -distribution.

Then a pseudo-product structure of type  $B_3(1, 3)$  on  $L$  in a generalised sense is a decomposition  $L = L_1 \oplus L_2$  into subbundles of rank 2 respectively,  $L_2$  being integrable, such that there exist generators  $l_1, l_2$  of  $L_1$ , and generators  $l_3, l_4$  of  $L_2$  respectively, satisfying  $[l_1, l_2] \equiv: l_5$ ,  $[l_1, l_3] \equiv: l_6$ ,  $[l_1, l_4] \equiv 0$ ,  $[l_2, l_3] \equiv 0$ ,  $[l_2, l_4] \equiv l_6$ ,  $[l_3, l_4] \equiv 0$ , mod. $L$ , and  $[l_1, l_5] \equiv 0$ ,  $[l_1, l_6] \equiv: l_7$ ,  $[l_2, l_5] \equiv 0$ ,  $[l_2, l_6] \equiv l_8$ ,  $[l_3, l_5] \equiv l_8$ ,  $[l_3, l_6] \equiv 0$ ,  $[l_4, l_5] \equiv -l_7$ ,  $[l_4, l_6] \equiv 0$ , mod. $L^{(2)}$ . □

【 The correspondence theorem (still continued) 】

**Theorem.**

The prolongation procedures induce the bijections between

{local isomorphism classes of (3, 6)-distributions},

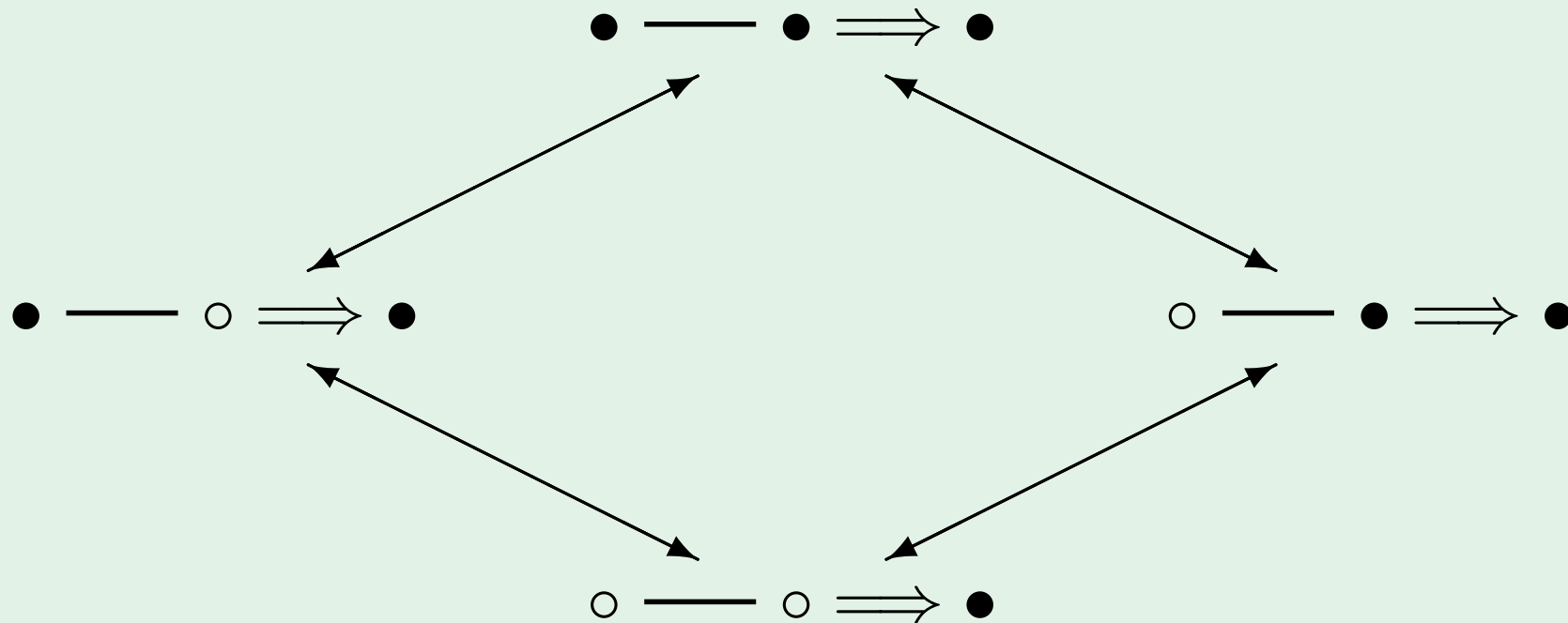
{ local isomorphism classes of (3, 5, 7, 8)-distributions  
with pseudo-product structure of type  $B_3(2, 3)$  } ,

{ local isomorphism classes of (3, 5, 7, 8, 9)-distributions  
with pseudo-product structure of type  $B_3(1, 2, 3)$  } ,

and

{ local isomorphism classes of (4, 6, 8)-distributions  
with pseudo-product structure of type  $B_3(1, 3)$  } .

## Dynkin diagrams of prolongations from (3, 6)-distributions

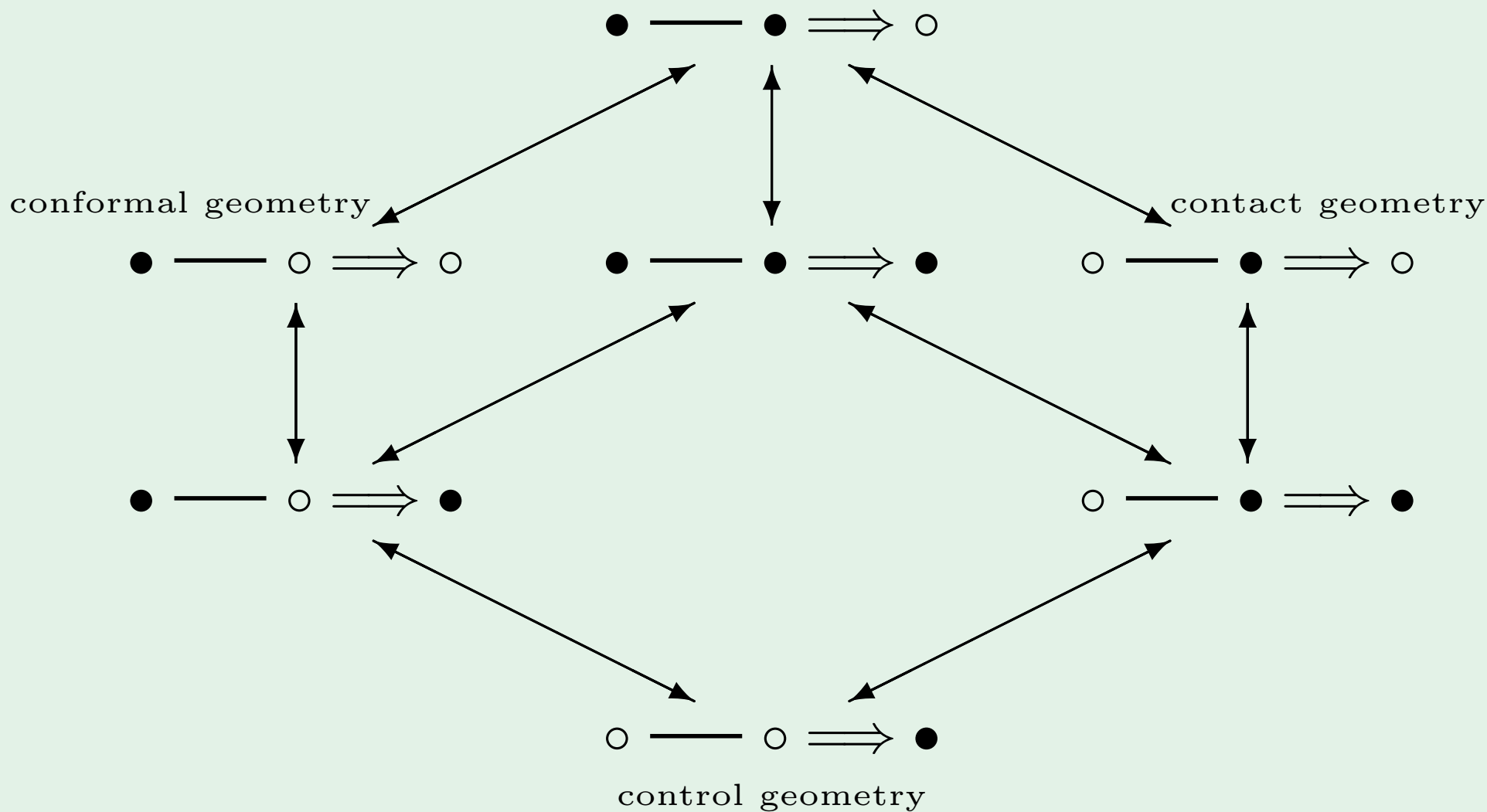


G. Ishikawa, Y. Machida,

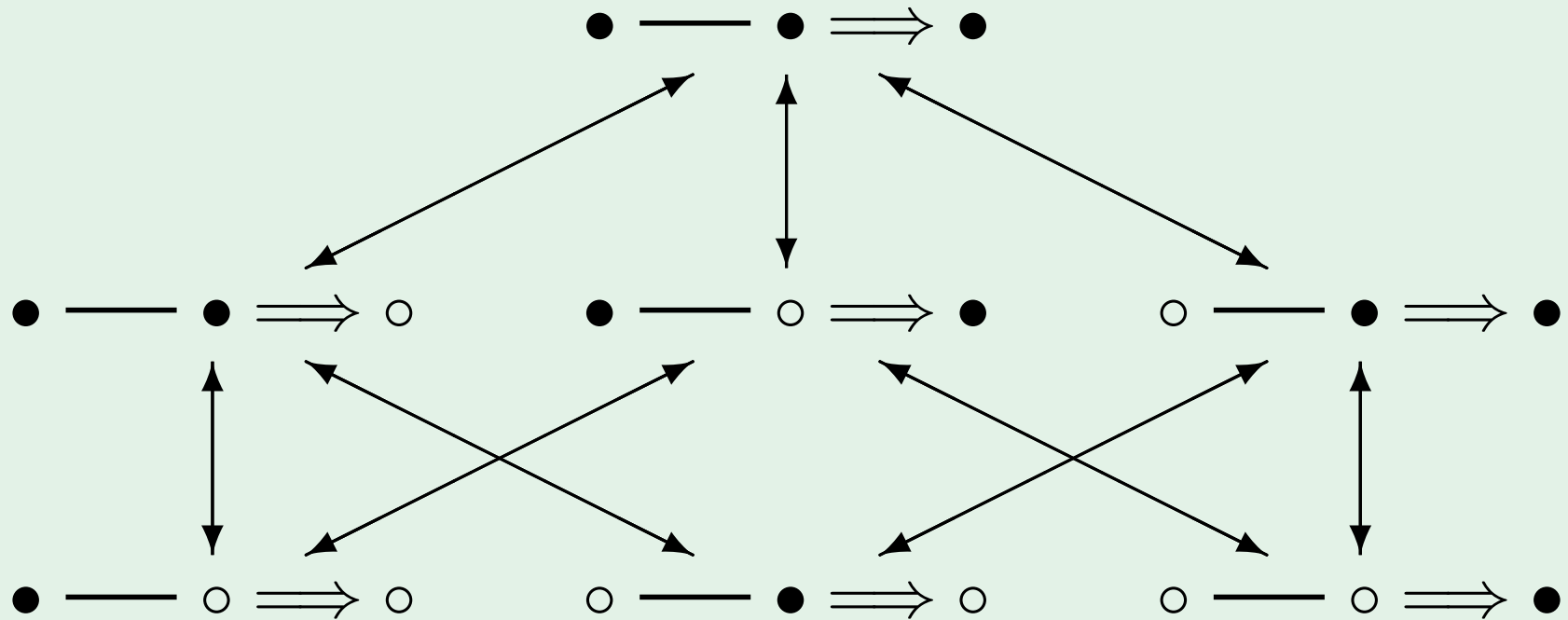
*Prolongations of (3,6)-distributions by singular curves,*

arXiv:2602.10507 [math.DG].

The completed Dynkin- $B_3$ -diagrams  
viewed from (3, 6)-distributions



## The completed $B_3$ Dynkin diagrams



conformal geometry

(2, 3)-conformal str.

contact geometry

7-dim. contact str.  
with Lagrangian cone

control geometry

(3, 6)-distribution

【 Example : Grassmannian of null 2-spaces in  $\mathbb{R}^{3,4}$  】

Let  $\mathbb{R}^{3,4}$  denote the vector space  $\mathbb{R}^7$  with an indefinite metric of index (3, 4), where  $SO(3, 4)$ , the simple Lie group of type  $B_3$ , acts naturally.

Consider the **Grassmannian of null 2-spaces** in  $\mathbb{R}^{3,4}$

$$Y := \{V_2 \mid V_2 : \text{null subspace of } \mathbb{R}^{3,4}, \dim(V_2) = 2\}.$$

Then we see  $\dim(Y) = 10 - 3 = 7$ .

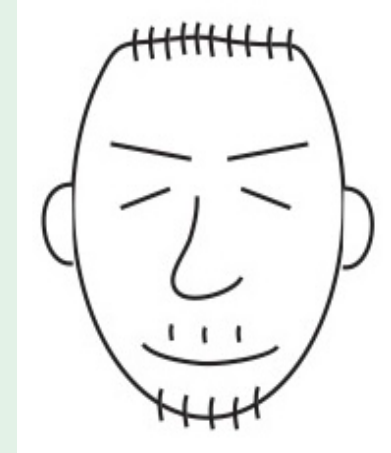
The contact structure  $H \subset TY$  is defined as follows: For  $V_2 \in Y$ ,  $H_{Y_2}$  is the set of  $V_2'(0) \in T_{V_2}Y$  for all curves  $V_2(t) = \langle k_1(t), k_2(t) \rangle$  in  $Y$  satisfying  $V_2(0) = V_2$  and  $V_2(t)^\perp$ . The Lagrangian cone  $C_{V_2} \subset H_{Y_2}$  turns to be an intersection of three quadric hyper-cones.

## [ Lagrangian cone structures of type $B_3(2)$ ]

**Lemma.** The prolongation  $(Z, E = E_1 \oplus E_2)$  of a  $(3, 6)$ -distribution  $(X, D)$  is reduced by  $E_1$  to a Lagrangian cone structure  $(Y, H, C)$  where  $Y = Z/E_1$ ,  $\dim(Y) = 7$ ,  $H = E^{(3)}/E_1 \subset TY$  a contact structure on  $Y$ ,  $C$  a Lagrangian cone field in  $H$  which is generated by a non-degenerate 1-parameter family of plane fields induced by  $E_2$  over  $Y$  along the projection  $Z \rightarrow Y$ . Moreover we have  $\text{rank}(T_s C) = 3$  and  $\text{rank}(O_s^{(2)} C) = 5$  along any generator  $s$  of  $C$ .

**Remark.** The Lagrangian cone field  $C$  is obtained from  $E_2$  under the reduction. In general,  $C_y, (y \in Y)$  is not necessarily an algebraic cone but a transcendental cone.

ご清聴感謝します。



Thank you for your attention !