Symplectic singularities of differentiable mappings

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Dedicated to Professor Shing-Tung Yau on his 70th birthday

Abstract

Our purpose is to survey the recent results concerning the study of germs of singular varieties in a symplectic space. We formulate the theory of symplectic bifurcations with the symplectic group action on the reduced space and provide the complete symplectic classification of simple and unimodal singularities of planar curves. Their differential and symplectic invariants, e.g. symplectic defect and δ-invariant were distinguished and the corresponding cyclic moduli spaces were calculated. The classification problem of singular differentiable mappings to the symplectic space was considered and basic invariants for classification, symplectic codimension, symplectic isotropic codimension, symplectic multiplicity were constructed. The methods of geometric and algebraic restrictions of differential forms to singular varities were presented and applied in symplectic classification of space curves and singular surfaces.

1 Introduction

The basic concepts of symplectic geometry were introduced mostly by Lagrange [49] in connection with his study of motion of planets in the framework of analytical mechanics. During the long eclipse of mechanics, the language of modern geometry was developed and symplectic geometry with its symplectic group of transformations appeared to be a very universal and useful part of it (cf. [76], [30]). In the quasiclassical methods of quantum mechanics the notion of Lagrangian or isotropic submanifold (cf. [55]) was introduced and soon appeared to play an especially important role in symplectic geometry and its applications. It appeared that the idea of Lagrangian submanifold to be the morphism of the symplectic 'category' is very unifying for symplectic geometry itself as well cantral object of studies in various contexts. The closely related focal sets of systems of rays, the wave-front evolution and caustics were investigated long ago by Huygens, Leibnitz, Bernoulli, Jacobi, and Morse (cf. [46]). However, R. Thom [70] emphasized the fundamental importance of the theory of stable singularities of smooth mappings in investigation of basic symplectic invariants. He initiated the application of singularity theory to these systems and suggested the extended use of stable Lagrangian submanifolds to model the internal peculiarities of physical systems in general. Following Thom's proposal, Arnol'd (cf. [2], [6]) gave the classification of simple, stable singularities of Lagrange

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projections and gave a new approach to study fundamental symplectic invariants of varieties induced purely by singularity, symplectic "ghosts" (cf. [4]). Then it became obvious that the symplectic singularities of varieties, smooth mappings into symplectic space and their stable or generic properties, are indispensable for understanding a large part of geometrical optics, classical mechanics, variational calculus, geometric quantization, optimization and control theory, Hamilton-Jacobi equations, holonomic systems, phase transitions and field theory. All these directions indicate their own problems and methods of solution, as well suggested the new investigations in singularity theory itself.

The recently found local symplectic and contact algebras (cf. [3], [82], [15]) connects J. Mather theory of singularities of differentiable mappings with the basic symplectic invariants of curves and surfaces. In the present paper we give a report on the representative directions of the theory of symplectic singularities and emphasize the basic results, which seem very important from the point of view of applications to algebraic and geometric investigations of singularities. Section 2 starts with the symplectic bifurcation theory of varieties with their special case of bifurcations which are liftable as isotropic liftings to the symplectic ambient space. By the canonical liftable equivalence the generic one-parameter bifurcation problems of Morse-type planar curves and finite sets of points were classified. In Section 3, we introduce the notion of symplectic equivalence, isotopy equivalence and symplectic versality and stability of planar curves. The basic invariants, symplectic codimension, symplectic defect and δ-invariant were defined and used to get the lists of symplectic classification of simple and uni-modal planar curves. Each symplectic moduli space of the classified families, which is a Hausdorff space in the natural topology, is extended to a cyclic quotient singularity. The section concludes with the complete classification list of simple and unimodal planar curves under diffeomorphism equivalency which finally solves the Zariski question. Symplectic invariants of mappings and details of classification using the Puiseux characteristics are quickly reviewed in Section 4. The multigerm formulation of basic invariants for symplectic classification in higher dimensions is provided. Section 5 is mostly an attempt to construct the basic symplectic invariants for isotropic mappings. We also clarify the role of the isotropic double points and 'open umbrellas' in classification of symplectic-isotropic singularities. Especially that the image of an isotropically stable perturbation is homotopically equivalent to the bouquet of circles. The symplectic invariants of surfaces are also expressed by configurations of open umbrellas. In the final section we present the use of an another method to investigate the differentiable and symplectic invariants of singular varieties and mappings. The notions of geometric and algebraic restriction of a differential form to the subset are defined and the residual algebraic restrictions are introduced. In the particular case of symplectic forms the residues represents the hidden symplectic invariants, which is a purely singularity product. Again in this context we approach to symplectic classification of map-germs and Whitney's umbrellas ending with the formulation of residues via a mapping and then used to get symplectic residues of planar curves and residues of Lagrangian varieties and hypersurfaces.

2 Symplectic bifurcations

2.1 Symplectic bifurcation problem

Let (M^{2n}, ω), $\quad n \geq 2$, be a symplectic manifold and $H : M^{2n} \to \mathbf{R}^{n-k}$, $\quad 0 \leq k \leq n-1$, a fibration of M. Suppose that H is expressed as $H = (H_1, \dots, H_{n-k}) : M^{2n} \to \mathbf{R}^{n-k}$ with $\{H_i\}$ in involution, i.e. the Poisson product $\{H_i, H_j\} \equiv 0$ for all $1 \leq i, j \leq n-k$, where $\{H_i, H_j\}$ is defined by an equality $\{H_i, H_j\}\omega^n = dH_i \wedge dH_j \wedge \omega^{n-1}$. Then each fiber $H^{-1}(\bar{q})$ of H is a coisotropic submanifold of (M^{2n}, ω). By the Jacobi-Liouville theorem, locally there exist relative Darboux coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ on M^{2n} such that $\omega = \sum_{i=1}^n dp_i \wedge dq_i$ and $H(p, q) = (\bar{q}) = (q_{k+1}, \dots, q_n)$. To each coisotropic submanifold $H^{-1}(\bar{q})$, or equivalently to each $\bar{q} \in \mathbf{R}^{n-k}$ we prescribe the symplectic space, the reduced symplectic space $M_{\bar{q}}$ given by the canonical reduction (Marseden-Weinstein reduction) $\pi_{\bar{q}}$;

$$\bar{q} \mapsto \pi_{\bar{q}} : H^{-1}(\bar{q}) \to H^{-1}(\bar{q}) / \sim_{\bar{q}} \equiv M_{\bar{q}}$$

along each fiber $H^{-1}(\bar{q})$ endowed with the symplectic structure $\mu_{\bar{q}}$ uniquely defined by the reduction formula

$$\pi_{\bar{q}}^* \mu_{\bar{q}} = \omega|_{H^{-1}(\bar{q})}.$$

In Darboux coordinates we assume

$$\pi_{\bar{q}}(p_1, \dots, p_k, q_1, \dots, q_k, p_{k+1}, \dots, p_n) = (p_1, \dots, p_k, q_1, \dots, q_k)$$

and identify $(M_{\bar{q}}, \mu_{\bar{q}})$ with $T^*\mathbf{R}^k$ endowed with its canonical Liouville form.

Collecting all the reduction projections we get the total projection

$$\pi : M^{2n} \to T^*\mathbf{R}^k \times \mathbf{R}^{n-k} =: N^{n+k},$$

$$\pi(p, q) = (p_1, \dots, p_k, q_1, \dots, q_k, q_{k+1}, \dots, q_n)$$

such that the following diagram commutes

$$\begin{array}{ccc} (M^{2n}, \omega) & & \\ \downarrow{\scriptstyle \pi} & \searrow^{H} & \\ T^*\mathbf{R}^k \times \mathbf{R}^{n-k} & \xrightarrow[\Pi]{} & \mathbf{R}^{n-k}, \end{array}$$

where Π is the projection to the second component.

Definition 2.1 ([36]) Any map germ $F : (\mathbf{R}^m, 0) \to T^*\mathbf{R}^k \times \mathbf{R}^{n-k}$, $\quad n - k \leq m < n + k$ is called a *bifurcation in symplectic space* $T^*\mathbf{R}^k$ (or shortly a *symplectic bifurcation*).

Bifurcating family of varieties in $T^*\mathbf{R}^k$ is defined, from the above diagram as

$$\mathbf{R}^{n-k} \ni \bar{q} \mapsto F(\mathbf{R}^m) \cap \Pi^{-1}(\bar{q}) \subset T^*\mathbf{R}^k \times \{\bar{q}\}.$$

In [36], it is treated mainly the case $m - n + k = 1$, i.e. symplectic bifurcations of curves.

Definition 2.2 Let $F : (\mathbf{R}^m, 0) \to T^*\mathbf{R}^k \times \mathbf{R}^{n-k}$ be a symplectic bifurcation. The germ F is called *transverse* if F is transverse to $T^*\mathbf{R}^k \times \{0\}$ at 0. Moreover we call it an *isotropic* (resp. *Lagrangian*) bifurcation if F lifts to a smooth map-germ $\widetilde{F} : (\mathbf{R}^m, 0) \to (M^{2n}, \omega)$ which is isotropic i.e. $\widetilde{F}^*\omega = 0$, (and $m = n$) and the following diagram commutes

$$\begin{array}{ccccc} & & (M^{2n},\omega) & & \\ & \overset{\widetilde{F}}{\nearrow} & \downarrow \pi & \overset{H}{\searrow} & \\ (\mathbf{R}^m,0) & \xrightarrow[F]{} & T^*\mathbf{R}^k\times\mathbf{R}^{n-k} & \xrightarrow[\Pi]{} & \mathbf{R}^{n-k}, \end{array}$$

The lifting $\widetilde{F}$ is called an *isotropic lifting* (resp. *Lagrangian lifting*) of F.

By setting $L = \widetilde{F}(\mathbf{R}^m)$, we regard the projection $\pi|_L : L \to N$ as a bifurcation of isotropic varieties $\pi_{\bar{q}}(H^{-1}(\bar{q}) \cap L)$ with expected dimension $m-n+k$. Then $\pi(L)$ is a family of $(m-n+k)$-dimensional isotropic varieties in $T^*\mathbf{R}^k$ parametrized by $\mathbf{R}^{n-k}$,

$$\mathbf{R}^{n-k} \ni \bar{q} \mapsto \pi(H^{-1}(\bar{q}) \cap L) \subset T^*\mathbf{R}^k \times \{\bar{q}\}.$$

If $m = n$ then this is a family of Lagrangian varieties in $T^*\mathbf{R}^k$.

V. M. Zakalyukin [78] classified the simple stable Lagrangian submanifold-germs ($m = n$) by symplectomorphisms which preserve a given coisotropic fibration. Then, admitting Lagrangian or isotropic varieties, we study the liftability and the classification problem of varieties in the reduced space. In other words, we consider the "bottom-up" construction. The idea appeared before in the PhD thesis of M. Mikosz and part of it is published in [59].

Proposition 2.3 ([36]) *Let $m = n - k + 1$ and $F : (\mathbf{R}^{n-k+1}, 0) \to (N, 0) = (T^*\mathbf{R}^k \times \mathbf{R}^{n-k}, 0)$ be a transverse symplectic bifurcation. Then F is isotropic, i.e. there exists a isotropic lifting $\widetilde{F} : (\mathbf{R}^{n-k+1}, 0) \to M = \mathbf{R}^{2n}$ of F, which is unique up to liftable equivalence (in the sense of Definition 2.5 below).*

We see that there exist (even in the simplest case $n = 2, k = 1$) many examples of non-transverse bifurcation problems of curves $F : (\mathbf{R}^2, 0) \to N = \mathbf{R}^2 \times \mathbf{R}$ which are not liftable to isotropic mappings into M (See Proposition 2.9).

On the other hand we have:

Proposition 2.4 ([36]) *If $F : (\mathbf{R}^n, 0) \to N = T^*\mathbf{R} \times \mathbf{R}^{n-1}$ is an immersion germ, then F is a transverse symplectic bifurcation if and only if F possesses a Lagrangian lifting.*

2.2 Liftable equivalence

Definition 2.5 Let $F_1, F_2 : (\mathbf{R}^m, 0) \to (T^*\mathbf{R}^n \times \mathbf{R}^{n-k} =: N, 0)$ be two symplectic bifurcation germs. We say that the two map-germs F_1, F_2 are *liftably equivalent* if there are diffeomorphism-germs ψ, ϕ and a symplectomorphism-germ Φ such that the following diagram commutes

$$\begin{array}{ccccc} (\mathbf{R}^m,0) & \xrightarrow{F_1} & (N,0) & \xleftarrow[\pi]{} & ((M,0),\omega) \\ \downarrow \psi & & \downarrow \phi & & \downarrow \Phi \\ (\mathbf{R}^m,0) & \xrightarrow{F_2} & (N,0) & \xleftarrow[\pi]{} & ((M,0),\omega), \end{array}$$

Diffeomorphism-germ ϕ in such a diagram is called *symplectically liftable*.

Thus we consider the classification of bifurcation-germs singularities according to the group $\mathcal{G}_{symp}$ of symplectically liftable diffeomorphisms of $(N,0)$ which is the subgroup of the group $\mathcal{G}$ of diffeomorphisms-germs. The group $\mathcal{G}_{symp}$ was explicitly described in [36].

Proposition 2.6 ([36]) *For a diffeomorphism-germ $\phi:(N,0)=(T^*\mathbf{R}^k\times\mathbf{R}^{n-k},0)\to(T^*\mathbf{R}^k\times\mathbf{R}^{n-k},0)$, the following conditions are equivalent:*

(1) *ϕ is a symplectically liftable diffeomorphism.*

(2) *ϕ is a Poisson diffeomorphism (for the Poisson structure on N induced from M by π).*

(3) *ϕ is a family of symplectic diffeomorphisms on $T^*\mathbf{R}^k$ with parameter $\bar q=(q_{k+1},\dots,q_n)$. Namely, if we set*

$$\phi(q_1,p_1,\dots,q_k,p_k,\bar q)=(Q_1,P_1,\dots,Q_k,P_k,\bar Q),$$

then $\overline{Q}=(Q_{k+1},\dots,Q_n)$ depends only on $\bar q$, and

$$(q_1,p_1,\dots,q_k,p_k)\mapsto(Q_1,P_1,\dots,Q_k,P_k)$$

is a symplectomorphism on $(T^*\mathbf{R}^k,0)$ for each fixed $\bar q=(q_{k+1},\dots,q_n)$.

(4) *ϕ has a symplectic lifting $\Phi:(M,0)\to(M,0)$ preserving fibers of H, namely, there exists a diffeomorphism-germ $\sigma:(\mathbf{R}^{n-k},0)\to(\mathbf{R}^{n-k},0)$ such that the following diagram commutes:*

$$\begin{array}{ccccc}(N,0) & \xleftarrow{\pi} & ((M,0),\omega) & \xrightarrow{H} & (\mathbf{R}^{n-k},0)\\ \phi\downarrow & & \Phi\downarrow & & \downarrow\sigma\\ (N,0) & \xleftarrow{\pi} & ((M,0),\omega) & \xrightarrow{H} & (\mathbf{R}^{n-k},0).\end{array}$$

Moreover we have the following description: Any vector field X over N generating a liftable equivalence is given by

$$X(q_1,p_1,\dots,q_k,p_k;\bar q)=X_{h_{\bar q}}(q_1,p_1,\dots,q_k,p_k)+\sum_{i=1}^{n-k}a_i(\bar q)\frac{\partial}{\partial q_{k+i}}$$

for some functions $a_i(\bar q),(i=1,\dots,n-k)$ and the Hamiltonian

$$h_{\bar q}(q_1,p_1,\dots,q_k,p_k)=h(q_1,p_1,\dots,q_k,p_k;\bar q)$$

for the Hamiltonian vector-field over $T^*\mathbf{R}^k$ for each separate $\bar q\in\mathbf{R}^{n-k}$. We easily see that the lifted Hamiltonian vector-field $\widetilde{X}$ over (M^{2n},ω) generating the lifted symplectic equivalence is defined by the Hamiltonian

$$\tilde h(p,q)=h(q_1,p_1,\dots,q_k,p_k;\bar q)+\sum_{i=1}^{n-k}p_{k+i}a_i(\bar q),\quad \widetilde{X}=X_{\tilde h}.$$

Definition 2.7 Let $L_1,L_2:(\mathbf{R}^m,0)\to(M^{2n},0)$ be two map-germs into symplectic space (M^{2n},ω). Then L_1,L_2 are called *H-symplectically equivalent* if there exist a symplectomorphism $\Phi:((M^{2n},0),\omega)\to((M^{2n},0),\omega)$ and diffeomorphisms $\psi:(\mathbf{R}^m,0)\to(\mathbf{R}^m,0)$, $\sigma:(\mathbf{R}^{n-k},0)\to(\mathbf{R}^{n-k},0)$ such that the following diagram commutes

$$\begin{array}{ccccc}(\mathbf{R}^m,0) & \xrightarrow{L_1} & (M,0) & \xrightarrow{H} & (\mathbf{R}^{n-k},0)\\ \phi\downarrow & & \Phi\downarrow & & \downarrow\sigma\\ (\mathbf{R}^m,0) & \xrightarrow{L_2} & (M,0) & \xrightarrow{H} & (\mathbf{R}^{n-k},0),\end{array}$$

Now we have immediately from Proposition 2.3 the following

Corollary 2.8 *If $F_1, F_2 : (\mathbf{R}^{n-k+1}, 0) \to (T^*\mathbf{R}^n \times \mathbf{R}^{n-k}, 0)$ are liftably equivalent, then their isotropic liftings $\widetilde{F}_1, \widetilde{F}_2 : (\mathbf{R}^{n-k+1}, 0) \to (M^{2n}, 0)$ are H-symplectically equivalent.*

A map-germ $F : (\mathbf{R}^2, 0) \to (\mathbf{R}^2\times\mathbf{R}, 0)$ is called *of Morse type* if $F(x_1, x_2) = (q_1(x), p_1(x), q_2(x))$ such that $q_2 : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ is a submersion or a Morse function at 0. Then we have the generic classification of liftable germs among the class of map-germs of Morse type.

Proposition 2.9 ([36]) *Let $F : (\mathbf{R}^2, 0) \to (\mathbf{R}^2 \times \mathbf{R}, 0)$ be a generic liftable map-germ of Morse type, namely, a generic one-parameter bifurcation problem of planar curves of Morse type. Then F is liftably equivalent to one of the following types(cf. Fig. 1):*

(1) $(x_1, x_2) \mapsto (x_1, 0, x_2)$*: The transverse immersion.*

(2) $(x_1, x_2) \mapsto (x_1^2, x_1^3+x_1x_2, x_2)$*: The transverse Whitney umbrella, or, the cusp bifurcation.*

(3) $(x_1, x_2) \mapsto (x_1, x_1x_2 + O(3), \frac{1}{2}(x_1^2 - x_2^2))$*: The hyperbolic Whitney umbrella, or, the X-pinch bifurcation.*

(4) $(x_1, x_2) \mapsto (x_1, x_1x_2 + O(3), \frac{1}{2}(x_1^2 + x_2^2))$*: The elliptic Whitney umbrella, or, the figure eight bifurcation.*

Here $O(3)$ means the terms in x_1, x_2 at least of third order.

Moreover the germ $F(x_1, x_2) = (x_1, x_1x_2 + \varphi(x_1, x_2), \frac{1}{2}(x_1^2 \pm x_2^2)), \mathrm{ord}\varphi \geq 3$ is liftable if and only if φ is of the form

$$\varphi(x_1, x_2) = \int_0^{x_2} \left(\mp x_2\frac{\partial\psi}{\partial x_1} + x_1\frac{\partial\psi}{\partial x_2}\right) dx_2 + \kappa(x_1),$$

for some smooth function $\psi(x_1, x_2)$ of order ≥ 2 and $\kappa(x_1)$ of order ≥ 3.

See Figure 1.

2.3 Bifurcations of finite sets of points

Now let us consider the case $m = n - k$.

Any map-germ $F : (\mathbf{R}^{n-k}, 0) \to T^*\mathbf{R}^k \times \mathbf{R}^{n-k}$ will be called a *finite point set symplectic bifurcation* in $(T^*\mathbf{R}^k, \omega_k)$ provided $f = \Pi \circ F$ is finite-to-one map-germ. In this case the symplectic bifurcations are classified by singularities of map-germs $f : (\mathbf{R}^{n-k}, 0) \to (\mathbf{R}^{n-k}, 0)$. In local coordinates

$$f(x_1, \ldots, x_{n-k}) = (q_{k+1}(x), \ldots, q_n(x))$$

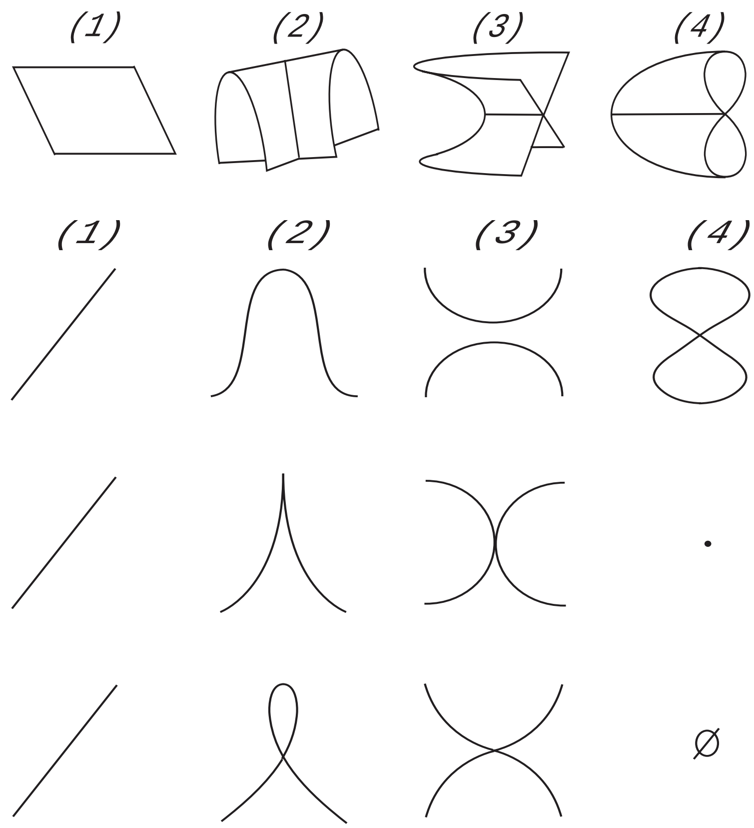

Figure 1: Typical isotropic bifurcations

and we have a bifurcating family of collection of points

$$\bar{q} \mapsto \{F(\{f^{-1}(\bar{q})\})\}$$

all staying in the image of F projected into $T^*\mathbf{R}^k$. It will be an isotropic bifurcation if F lifts to an isotropic map-germ $\tilde{F} : (\mathbf{R}^{n-k}, 0) \to (\mathbf{R}^{2n}, \omega)$, $\quad \tilde{F}^*\omega = 0$.

Moreover if $n - k = 1$, then all one-parameter symplectic bifurcations of finite sets of points in $T^*\mathbf{R}^k$ are obviously isotropic and classified by critical values of a smooth function $f(x) = q_{k+1}(x)$.

Singularities of F are classified by the symplectic singularities of divergent mapping diagrams

$$T^*\mathbf{R}^k \xleftarrow{\ \rho\ } \mathbf{R}^{n-k} \xrightarrow{\ f\ } \mathbf{R}^{n-k},$$

where $F = (\rho, f)$. They are initially classified by singularities of mappings f. It is natural to ask if generic singularities of map-germs F can be lifted to isotropic map-germs $\widetilde{F}$.

For the case $n - k = 2$ we have

Proposition 2.10 ([39]) *All smooth map-germs $F : (\mathbf{R}^2, 0) \to (T^*\mathbf{R}^k \times \mathbf{R}^2, 0)$ such that $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ is a singularity of corank one are regular isotropic bifurcations, i.e. F is liftable to an immersive isotropic map-germ $\tilde{F} : (\mathbf{R}^2, 0) \to (\mathbf{R}^{2(k+2)}, \omega)$, $\tilde{F}^*\omega = 0$.*

Moreover we have

Proposition 2.11 ([39]) *Normal forms of corank one generic isotropic bifurcations $F = (\rho, f) : (\mathbf{R}^2, 0) \to (T^*\mathbf{R} \times \mathbf{R}^2, 0)$ of fold and cusp type are liftable equivalent to*
1. (fold) $\quad f(x) = (x_1, x_2^2) : \quad \rho(x) = (x_1\phi(x_1, x_2), x_1 + x_2)$.
2. (cusp) $\quad f(x) = (x_1, x_2x_1 + x_2^3) : \quad \rho(x) = (x_1\phi_1(x_1, x_2), x_2 + \phi_2(x_1, x_1x_2 + x_2^3)$.

2.4 Symplectic bifurcation of planar curves

In what follows, we concentrate on the symplectic bifurcation problem of curves on the symplectic plane ($m = n, k = 1$). Thus we are going to consider map-germs $F : (\mathbf{R}^n, 0) \to (T^*\mathbf{R} \times \mathbf{R}^{n-1}, 0)$.

Proposition 2.12 ([36]) *Let $F : (\mathbf{R}^n, 0) \to (N, 0) = (T^*\mathbf{R} \times \mathbf{R}^{n-1}, 0)$ be a a transverse map-germ. Then F is liftably equivalent to*

$$\phi \circ F \circ \psi = (f_1(x_1, x_2, \dots, x_n), f_2(x_1, x_2, \dots, x_n), x_2, \dots, x_n),$$

for some function-germs f_1, f_2. Moreover we have the followings: (a1) *If F is an immersion at* 0*, then F is liftably equivalent to*

$$(0, x_1, x_2, \dots, x_n).$$

(a2) *Suppose F is not an immersion at* 0*. Then F is liftably equivalent to the germ $\phi \circ F \circ \psi$ such that the 2-jet $j^2(\phi \circ F \circ \psi)$ is equal to (x_1^2, x_1x_2, x'), $(x_1^2, 0, x')$, $(x_1x_2, 0, x')$ or $(0, 0, x')$, where $x' = (x_2, \dots, x_n)$.*

Then we have the following prenormal form of F.

Proposition 2.13 ([36]) *Let* $F : (\mathbf{R}^n, 0) \to (N, 0) = (\mathbf{R}^2 \times \mathbf{R}^{n-1}, 0)$, $F(x) = (q_1(x), p_1(x), q'(x))$, *be a smooth map-germ. Assume that* F *is transverse to* $\mathbf{R}^2 \times \{0\}$ *and that* F *is finite, namely, the ideal generated by components of* F *is of finite codimension. Then* F *is liftably equivalent to one of the following forms, for some* $m \geq 2$,

$$F_m(x) = (x_1^m + \sum_{i=1}^{m-2} a_i(x')x_1^i, x_1 c(x), x'),$$

where $x' = (x_2, \dots, x_n)$, *and* $a_i(x'), c(x)$ *are smooth function-germs.*

For the case $m = 2$, we have the normal forms, by using the versality theorem in the symplectic case [12][13].

Proposition 2.14 ([36]) *Let* $F : (\mathbf{R}^n, 0) \to (\mathbf{R}^2 \times \mathbf{R}^{n-1}, 0)$ *be a finite and transverse map-germ. Assume the* 2*-jet of* F *is equal to* $(x_1^2, x_1x_2, 0)$ *or* $(x_1^2, 0, 0)$. *Then* F *is liftably equivalent to*

$$(q_1, p_1, q') = (x_1^2, x_1^{2\ell+1} + \lambda_1(x')x_1^{2\ell-1} + \lambda_2(x')x_1^{2\ell-3} + \cdots + \lambda_\ell(x')x_1, x'),$$

for some positive integer ℓ, *and for some functions* $\lambda_1(x'), \dots, \lambda_\ell(x')$ *of* $x' = (x_2, \dots, x_n)$ *with* $\lambda_j(0) = 0, 1 \leq j \leq \ell$.

3 Symplectic classification of planar curves

3.1 Symplectic equivalence and isotopy equivalence of planar curves

A transversal map-germ $F : (\mathbf{R}^n, 0) \to (\mathbf{R}^2 \times \mathbf{R}^{n-1}, 0)$ is liftably equivalent to an unfolding $(t, \lambda) \mapsto (f_\lambda(t), \lambda)$, where $\lambda \in (\mathbf{R}^{n-1}, 0)$ and f_λ is a family of parametrized curves in the symplectic plane $\mathbf{R}^2$, t being the inner variable and $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ the outer variables (Proposition 2.12). Therefore we proceed to consider the classification problems of bifurcations (unfoldings) of curves in the symplectic plane.

Definition 3.1 Two families of planar curves $f_\lambda, f'_\lambda, (\lambda \in (\mathbf{R}^\ell, 0))$ are called *symplectically equivalent* if there exist a family of diffeomorphisms $\Sigma = (\sigma_\lambda) : (\mathbf{R} \times \mathbf{R}^\ell, 0) \to (\mathbf{R}, 0)$, a family of symplectomorphisms $T = (\tau_\lambda) : (\mathbf{R}^2 \times \mathbf{R}^\ell, 0) \to (\mathbf{R}^2, 0)$, and a diffeomorphism $\varphi : (\mathbf{R}^\ell, 0) \to (\mathbf{R}^\ell, 0)$ such that $\tau_\lambda \circ f'_\lambda \circ \sigma_\lambda = f_{\varphi(\lambda)}$, for some representatives of germs. Then, setting $F : (\mathbf{R} \times \mathbf{R}^\ell, 0) \to (\mathbf{R}^2 \times \mathbf{R}^\ell, 0)$, $F(t, \lambda) = (f_\lambda(t), \lambda)$ and $F' : (\mathbf{R} \times \mathbf{R}^\ell, 0) \to (\mathbf{R}^2 \times \mathbf{R}^\ell, 0)$, $F(t, \lambda) = (f'_\lambda(t), \lambda)$, we see that if f_λ and f'_λ are symplectically equivalent then F and F' are liftably equivalent.

In the ordinary singularity theory, the versal unfolding of a singularity dominates any other unfoldings. To seek the versal unfolding of curves on the symplectic plane for symplectic equivalence, we must first study the symplectic classification problem of planar curves.

For example, consider the simple cusp (A_2) $f = (t^2, t^3) : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$. Then the unfolding $F : (\mathbf{R} \times \mathbf{R}, 0) \to (\mathbf{R}^2 \times \mathbf{R}, 0)$ defined by $F(t, \lambda) = (t^2, t^3 + \lambda t, \lambda)$ is versal with respect to the right-left equivalence. Then we ask: Is it a symplectically versal unfolding?

Now first we consider the basic problem: Let $C, C' \subset (\mathbf{R}^2, 0)$ be two curve-germs. Assume that there exist a diffeomorphism-germ $\sigma : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ with $\sigma(C) = C'$. Then does there exist a symplectic (area-preserving) diffeomorphism $\sigma' : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ with $\sigma'(C) = C'$?

Definition 3.2 We call two map-germs $f, f' : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ *isotopic* (resp. *equivalent*) if there exist a smooth family $\tau_s : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ of diffeomorphism-germs starting from the identity τ_0 (resp. a diffeomorphism-germ $\tau : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$) and a diffeomorphism-germ $\sigma : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that $f' \circ \sigma = \tau_1 \circ f$ (resp. $f' \circ \sigma = \tau \circ f$). Moreover f and f' are called *symplectically isotopic* (resp. *symplectically equivalent*) if we can take, in the above definitions, τ_s (resp. τ) to be symplectic.

A map-germ $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$, is called *achiral* (resp. *chiral*) if f and $\bar{f}$ are isotopic (resp. non-isotopic). Here we denote by $\bar{f}$ the map-germ $(\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ defined by $\bar{f}(t) = (f_1(t), -f_2(t))$.

Here we give several illustrating examples for the notions introduced above.

Example 3.3 (About the definition of isotopy. [36]) Consider curves $f(t) = (t^2, t^3)$ and $f'(t) = (t^2, -t^3)$ of type A_2 (resp. $f(t) = (t^3, t^4)$ and $f'(t) = (t^3, -t^4)$ of type E_6) . Then we see f and f' are symplectically isotopic, by just taking τ_s identity (resp. the rotation by $s\pi$) and $\sigma(t) = -t$. Therefore germs (t^2, t^3) and (t^3, t^4) are achiral. However, there does not exist a smooth family of pairs of diffeomorphism-germs (σ_s, τ_s) starting from $(\mathrm{id}_{\mathbf{R}}, \mathrm{id}_{\mathbf{R}^2})$ with $f' \circ \sigma_1 = \tau_1 \circ f$.

Example 3.4 (The difference between equivalence and isotopy. [36]) Consider curves $f(t) = (t^3, t^5)$ and $f'(t) = (t^3, -t^5)$ of type E_8. Then f and f' are equivalent but not isotopic. Therefore the germ (t^3, t^5) is chiral.

Lemma 3.5 ([36]) *Let m, k be positive integers and k even. Then the two curve-germs $f = (t^m, t^{m+k} + o(t^{m+k}))$ and $f' = (t^m, -t^{m+k} + o(t^{m+k}))$ are not isotopic.*

Since any symplectomorphism-germ can be connected to the identity through symplectomorphism-germs, we see that f and f' are symplectically isotopic if and only if they are symplectically equivalent. Therefore the following is clear.

Lemma 3.6 ([36]) *If $f, f' : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ are symplectically equivalent, then they are isotopic.*

Now naturally we are led to the following question: Are $f, f' : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ symplectically equivalent if they are isotopic?

The question is solved in the following subsections.

3.2 Symplectic versality and stability.

Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a C^∞ map-germ, in other words, a planer curve. Recall the codimension i.e. $\mathcal{A}_e$-codimension ([74]), of f is defined by

$$\mathrm{codim}(f) := \dim_{\mathbf{R}} \frac{V_f}{tf(V_1) + wf(V_2)},$$

where $V_f := \{v : (\mathbf{R}, 0) \to T\mathbf{R}^2 \mid \pi \circ v = f\}$, $\pi : T\mathbf{R}^2 \to \mathbf{R}^2$ being the natural projection, is the space of vector field-germs along f, V_1 (resp. V_2) is the space of vector field-germs over $(\mathbf{R}, 0)$ (resp. $(\mathbf{R}^2, 0)$), and $tf : V_1 \to V_f$ (resp. $wf : V_2 \to V_f$) is the homomorphism defined by $tf(\xi) := f_*(\xi)$ (resp. $wf(\eta) := \eta \circ f$). A planar curve f is called $\mathcal{A}$-finite if $\mathrm{codim}(f) < \infty$.

Then f has an $\mathcal{A}$-versal unfolding with the parameter dimension codim(f). If f is analytic, the condition of $\mathcal{A}$-finiteness is equivalent to, for instance, that the complexification of f has an injective representative.

Moreover we define, in the case $\mathbf{R}^2$ is regarded as $T^*\mathbf{R}$,

$$\text{sp-codim}(f) := \dim_{\mathbf{R}} \frac{V_f}{tf(V_1) + wf(VH_2)},$$

where $VH_2 \subseteq V_2$ means the space of Hamiltonian vector field-germs over the symplectic plane $(\mathbf{R}^2, 0)$. Then clearly we have

$$\text{sp-codim}(f) \geq \text{codim}(f).$$

Definition 3.7 An unfolding $F : (\mathbf{R}\times\mathbf{R}^\ell, 0) \to (\mathbf{R}^2\times\mathbf{R}^\ell, 0)$ of f is called *symplectically versal* if, any unfolding $G : (\mathbf{R} \times \mathbf{R}^s, 0) \to (\mathbf{R}^2 \times \mathbf{R}^s, 0)$, of f is symplectically equivalent to φ^*F for some C^∞ map-germ $\phi : (\mathbf{R}^s, 0) \to (\mathbf{R}^\ell, 0)$.

The following result is a special case of the versality theorem in [13]:

Proposition 3.8 ([36]) *An unfolding $F : (\mathbf{R} \times \mathbf{R}^\ell, 0) \to (\mathbf{R}^2 \times \mathbf{R}^\ell, 0)$ of $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ is symplectically versal if and only if F is infinitesimally symplectically versal, that is,*

$$V_f = \left\langle \left.\frac{\partial \bar{F}}{\partial \lambda_1}\right|_{\mathbf{R}\times 0}, \dots, \left.\frac{\partial \bar{F}}{\partial \lambda_\ell}\right|_{\mathbf{R}\times 0} \right\rangle_{\mathbf{R}} + tf(V_1) + wf(VH_2).$$

Moreover two versal unfoldings F and F' of f with the same parameter dimension are liftably equivalent.

A map-germ $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ has a symplectically versal unfolding if and only if sp-codim$(f) < \infty$.

Remark 3.9 ([36]) By Damon's theory [13], we have the characterization of "symplectic finite determinacy". A map-germ $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ is called *symplectically finitely determined* if there exists a positive integer k such that any $f' : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ with $j^k f'(0) = j^k f(0)$ is symplectically equivalent to f. Then f is symplectically finitely determined if and only if sp-codim$(f) < \infty$.

We have a close relation between symplectic versality and symplectic stability ([33]) via the notion of Lagrangian liftings.

Theorem 3.10 ([36]) (Symplectic versality and stability.) *Let $F : (\mathbf{R}\times\mathbf{R}^\ell, 0) \to (\mathbf{R}^2\times\mathbf{R}^\ell, 0)$ be a symplectically versal unfolding. Then the Lagrangian lifting $\widetilde{F} : (\mathbf{R}\times\mathbf{R}^\ell, 0) \to (\mathbf{R}^2\times\mathbf{R}^{2\ell}, 0)$ is symplectically stable, that is, any isotropic deformation of $\widetilde{F}$ is trivialized by symplectic equivalences. Therefore $\widetilde{F}$ is symplectically equivalent to an open Whitney umbrella and satisfies that*

$$VI_{\widetilde{F}} = t\widetilde{F}(V_{1+\ell}) + w\widetilde{F}(VH_{2+2\ell}).$$

in terms of [33]. *In particular $\widetilde{F}$ has an injective representative.*

For a mapping $\varphi : (\mathbf{R}^s, 0) \to (\mathbf{R}^\ell, 0)$, and an unfolding $F : (\mathbf{R} \times \mathbf{R}^\ell, 0) \to (\mathbf{R}^2 \times \mathbf{R}^\ell, 0)$, $F(t, \lambda) = (f_\lambda(t), \lambda)$, we define the pull-back unfolding $\varphi^* F : (\mathbf{R} \times \mathbf{R}^s, 0) \to (\mathbf{R}^2 \times \mathbf{R}^s, 0)$ by $(\varphi^* F)(t, \mu) = (f_{\varphi(\mu)}(t), \mu)$.

Proposition 3.11 ([36]) *Let $F : (\mathbf{R} \times \mathbf{R}^\ell, 0) \to (\mathbf{R}^2 \times \mathbf{R}^\ell, 0)$ be an unfolding of $f = F|_{\mathbf{R}\times 0}$ and $\widetilde{F}$ a Lagrangian lifting of F. Let $\varphi : (\mathbf{R}^s, 0) \to (\mathbf{R}^\ell, 0)$ be a map-germ. Then the lifting $\widetilde{\varphi^* F} : (\mathbf{R} \times \mathbf{R}^s, 0) \to (\mathbf{R}^2 \times \mathbf{R}^s, 0)$ of $\varphi^* F$ defined by*

$$p_j := \sum_{1 \le k \le \ell} \frac{\partial \varphi_k}{\partial \mu_j} (p_{k+1} \circ \widetilde{F}),$$

$(2 \le j \le s+1)$*, is a Lagrangian lifting of $\varphi^* F$. In fact we have*

$$\widetilde{\varphi^* F}^* \theta_{\mathbf{R}^2 \times \mathbf{R}^{2s}} = (\mathrm{id}_{\mathbf{R}} \times \varphi)^* \widetilde{F}^* \theta_{\mathbf{R}^2 \times \mathbf{R}^{2\ell}},$$

for the Liouville form $\theta_{\mathbf{R}^2 \times \mathbf{R}^{2s}}$ (resp. $\theta_{\mathbf{R}^2 \times \mathbf{R}^{2\ell}}$) on $\mathbf{R}^2 \times \mathbf{R}^{2s} = T^*(\mathbf{R} \times \mathbf{R}^s)$ (resp. $\mathbf{R}^2 \times \mathbf{R}^{2\ell} = T^*(\mathbf{R} \times \mathbf{R}^\ell)$).

The above Proposition 3.11 means the Lagrangian lifting of the pull-back unfolding can be obtained by reduction from the Lagrangian lifting of the original unfolding.

In particular we have:

Corollary 3.12 ([36]) *Let $G : (\mathbf{R}^n, 0) \to (N, 0) = (\mathbf{R}^2 \times \mathbf{R}^{n-1}, 0)$ be a transverse map-germ to $\mathbf{R}^2 \times \{0\}$. Assume the restriction $(G^{-1}(\mathbf{R}^2 \times \{0\}), 0) \to \mathbf{R}^2 \times \{0\}$ is $\mathcal{A}$-finite, then G is obtained from an open Whitney umbrella by a reduction process.*

3.3 Symplectic defect.

For a map-germ $f = (f_1, f_2) : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$, we set

$$\begin{aligned} \mathcal{G}_f &:= \{h \in \mathcal{E}_1 \mid dh \in \langle df_1, df_2 \rangle_{f^* \mathcal{E}_2}\}, \\ &= \{h \in \mathcal{E}_1 \mid dh \in f^*(\Lambda^1_2)\}, \end{aligned}$$

where $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) is the $\mathbf{R}$-algebra of C^∞ map-germs on $(\mathbf{R}, 0)$ (resp. $(\mathbf{R}^2, 0)$), Λ^1_2 is the space of differential 1-forms on $(\mathbf{R}^2, 0)$ and the homomorphism $f^* : \mathcal{E}_2 \to \mathcal{E}_1$ (resp. $f^* : \Omega^1_2 \to \Omega^1_1$) is defined by the pull-back by f. Moreover we set

$$\mathcal{R}_f := \{h \in \mathcal{E}_1 \mid dh \in \langle df_1, df_2 \rangle_{\mathcal{E}_1}\},$$

(cf. [33]). Thus we have defined intrinsically the sequence of vector spaces:

$$\mathcal{E}_1 \supseteq \mathcal{R}_f \supseteq \mathcal{G}_f \supseteq f^* \mathcal{E}_2.$$

The above sequence is understood as follows.

For each element $h \in \mathcal{G}_f$, the exterior differential dh is written as $(b \circ f) df_1 - (a \circ f) df_2 = f^*(b dq_1 - a dp_1)$ for some functions $a, b \in \mathcal{E}_2$. Through the symplectic structure $dp_1 \wedge dq_1$ on the (q_1, p_1)-plane $\mathbf{R}^2$, the 1-form $b dq_1 - a dp_1$ on $(\mathbf{R}^2, 0)$ corresponds to the vector field

$\eta = a\dfrac{\partial}{\partial q_1} + b\dfrac{\partial}{\partial p_1}$ over $(\mathbf{R}^2, 0)$. The vector field $wf(\eta)$ along f is regarded as an infinitesimal isotropic deformation of f. In this case we say that h is a generating function of $wf(\eta)$.

In general, a function $h(t)$ is called *a generating function* of a vector field

$$v = v_1(t)\left(\frac{\partial}{\partial q} \circ f\right) + v_2(t)\left(\frac{\partial}{\partial p} \circ f\right) : (\mathbf{R}, 0) \to T\mathbf{R}^2$$

along $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$, if $dh = v_2 df_1 - v_1 df_2 (= v^*\widetilde{\theta})$, the pull-back by the isotropic map $v : (\mathbf{R}, 0) \to T\mathbf{R}^2 \cong T^*\mathbf{R}^2$ of the Liouville 1-form $\widetilde{\theta}$ on $T\mathbf{R}^2$.

Thus, $\mathcal{G}_f$ is the space of generating functions of infinitesimal deformations of f induced from diffeomorphisms on the plane $\mathbf{R}^2$. Then we see that $\mathcal{G}_f$ is an $\mathbf{R}$-vector subspace of $\mathcal{E}_1$ and that $\mathcal{G}_f$ contains $f^*\mathcal{E}_2$. Similarly, $f^*\mathcal{E}_2$ is regarded as the space of generating functions of infinitesimal deformations of f induced from symplectomorphisms on the plane $\mathbf{R}^2$. Moreover, $\mathcal{R}_f$ is the space of generating functions of all infinitesimal deformations of f.

Then the following is clear:

Lemma 3.13 ([36]) *Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a map-germ. For a diffeomorphism $\tau : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$, we have $\mathcal{R}_{\tau\circ f} = \mathcal{R}_f$ and $\mathcal{G}_{\tau\circ f} = \mathcal{G}_f$. Moreover, for a diffeomorphism $\sigma : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$, $\sigma^* : \mathcal{E}_1 \to \mathcal{E}_1$ maps $\mathcal{R}_f$ to $\mathcal{R}_{f\circ\sigma}$ and $\mathcal{G}_f$ to $\mathcal{G}_{f\circ\sigma}$ respectively.*

Then the following is the key lemma of [36].

Lemma 3.14 ([36]) *There exists a vector space isomorphism*

$$\frac{tf(V_1) + wf(V_2)}{tf(V_1) + wf(VH_2)} \cong \frac{\mathcal{G}_f}{f^*\mathcal{E}_2}.$$

Note that the dimension of $\mathcal{G}_f / f^*\mathcal{E}_2$ depends only on the right-left equivalence class of f. Thus we have

Theorem 3.15 ([36]) *Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be an $\mathcal{A}$-finite map-germ. Then the symplectic defect*

$$\mathrm{sd}(f) := \text{sp-codim}(f) - \mathrm{codim}(f)$$

is equal to $\dim(\mathcal{G}_f / f^*\mathcal{E}_2)$, and it depends only on the right-left equivalence class of f, that is, the symplectic defect is an $\mathcal{A}$-invariant. Hence sp-codim(f) *is an $\mathcal{A}$-invariant.*

Then we have for instance

Corollary 3.16 ([36]) *If a planar curve is right-left equivalent to $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0), f(t) = (t^m, t^{m+k})$ for some positive integers m, k, then its symplectic defect is equal to zero.*

Remark 3.17 If f is $\mathcal{A}$-finite, then, by Mather-Gaffney's theorem, we see f is $\mathcal{L}$-finite ([74], p.494). Then we have that the vector space $\mathcal{E}_1 / f^*\mathcal{E}_2$ is of finite dimension. So, if f is $\mathcal{A}$-finite, namely, if codim(f) is finite, then sp-codim(f), so is the symplectic defect, is necessarily finite.

Remark 3.18 The symplectic codimension is not an $\mathcal{A}$-invariant (a diffeomorphism invariant) for map-germs $\mathbf{R} \to \mathbf{R}^4$. For example, consider map-germs

$$A_{2,0} : (q_1 = t^2, p_1 = t^3, q_2 = 0, p_2 = 0),$$

and

$$A_{2,1} : (q_1 = t^2, p_1 = t^5, q_2 = t^3, p_2 = 0),$$

from Arnold's classification [3]. Then $A_{2,0}$ and $A_{2,1}$ are clearly $\mathcal{A}$-equivalent. However we have sp-codim$(A_{2,0}) = 3$, and sp-codim$(A_{2,1}) = 4$. In fact, when $f = A_{2,0} : (\mathbf{R}.0) \to (\mathbf{R}^4, 0)$, we can take ${}^t(0,t,0,0), {}^t(0,0,t,0), {}^t(0,0,0,t)$ as a basis of the vector space $V_f/(tf(V_1) + wf(VH_4))$. For $f = A_{2,1}$ we need ${}^t(0,t^2,0,0)$ in addition.

From the definition of the symplectic defect, we have:

Proposition 3.19 ([36]) *Let* $F : (\mathbf{R} \times \mathbf{R}^\ell, 0) \to (\mathbf{R}^2 \times \mathbf{R}^\ell, 0)$ *be an* $\mathcal{A}$*-versal unfolding of* $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ *with* $\mathrm{sd}(f) = 0$. *Then* F *is a symplectically versal unfolding of* f.

Lastly we show that the symplectic codimension of an $\mathcal{A}$-finite map-germ is, actually, equal to the classical δ-invariant.

Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be an $\mathcal{A}$-finite map-germ. Then we set $\delta(f) := \dim_{\mathbf{R}} \mathcal{E}_1/f^*\mathcal{E}_2$. And we have:

Theorem 3.20 ([36]) *For an* $\mathcal{A}$*-finite map-germ* $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$,

$$\text{sp-codim}(f) = \delta(f).$$

3.4 Simple planar curves

Bruce and Gaffney [10] classified simple planer curves: The $\mathcal{A}$-equivalence class of simple (0-modal) planar curves are given in the following list:

$A_{2\ell} : t \mapsto (t^2, t^{2\ell+1})$;
$E_{6\ell} : t \mapsto (t^3, t^{3\ell+1} \pm t^{3(\ell+p)+2}), 0 \le p \le \ell - 2; t \mapsto (t^3, t^{3\ell+1})$;
$E_{6\ell+2} : t \mapsto (t^3, t^{3\ell+2} \pm t^{3(\ell+p)+4}), 0 \le p \le \ell - 2; t \mapsto (t^3, t^{3\ell+2})$;
$W_{12} : t \mapsto (t^4, t^5 \pm t^7); t \mapsto (t^4, t^5)$;
$W_{18} : t \mapsto (t^4, t^7 \pm t^9); t \mapsto (t^4, t^7 \pm t^{13}); t \mapsto (t^4, t^7)$;
$W^{\#}_{1,2q-1} : t \mapsto (t^4, t^6 + t^{2q+5}), q \ge 1$.

Note that, in the above list, the germs $(t^3, t^4 \pm t^5)$ and (t^3, t^4) of type E_6 (resp. $(t^3, t^5 \pm t^7)$ and (t^3, t^5) of type E_8) are actually $\mathcal{A}$-equivalent. See also [5] pp. 57–59.

Then we calculate symplectic defect of simple planar curves.

Theorem 3.21 ([36])

(1) *If* f *is equivalent to* $A_{2\ell}, E_6, E_8$ *or* $E_{6\ell} : (t^3, t^{3\ell+1}); E_{6\ell+2} : (t^3, t^{3\ell+2}); W_{12} : (t^4, t^5); W_{18} : (t^4, t^7)$ *then* $\mathrm{sd}(f) = 0$.

(2) *If f is equivalent to* $E_{6\ell}:(t^3,t^{3\ell+1}\pm t^{3(\ell+p)+2}), 0\le p\le \ell-2, \ell\ge 2$, *then* $t^{3(\ell+p+1)+2},\dots,t^{6\ell-1}$ *form a basis of* $G_f/f^*\mathcal{E}_2$ *and* $\mathrm{sd}(f)=\ell-p-1$. *The family* $(t^3,(\pm1)^{\ell+1}t^{3\ell+1}+\sum_{j=1}^{\ell-1}\lambda_j t^{3(\ell+j)-1})$ *contains all symplectic classes of type* $E_{6\ell}$. *If f is equivalent to* $E_{6\ell+2}:(t^3,t^{3\ell+2}\pm t^{3(\ell+p)+4})$, *then* $t^{3(\ell+p+1)+4},\dots,t^{6\ell+1}$ *form a basis of* $G_f/f^*\mathcal{E}_2$ *and* $\mathrm{sd}(f)=\ell-p-1$. *The family* $(t^3,(\pm1)^{\ell}t^{3\ell+2}+\sum_{j=1}^{\ell-1}\lambda_j t^{3(\ell+j)+1})$ *contains all symplectic classes of type* $E_{6\ell+2}$.

(3) *If f is equivalent to* $W_{12}:(t^4,t^5\pm t^7)$, *then* t^{11} *forms a basis of* $G_f/f^*\mathcal{E}_2$ *and* $\mathrm{sd}(f)=1$. *The family* $(t^4,t^5+\lambda t^7)$ *contains all symplectic classes of type* W_{12}.

(4) *If f is equivalent to* $W_{18}:(t^4,t^7\pm t^9)$, *then* t^{13},t^{17} *form a basis of* $G_f/f^*\mathcal{E}_2$ *and* $\mathrm{sd}(f)=2$. *If f is equivalent to* $W_{18}:(t^4,t^7\pm t^{13})$, *then* t^{17} *forms a basis of* $G_f/f^*\mathcal{E}_2$ *and* $\mathrm{sd}(f)=1$. *The family* $(t^4,t^7+\lambda t^9+\mu t^{13})$ *contains all symplectic classes of type* W_{18}.

(5) *If f is equivalent to* $W^{\#}_{1,2q-1}:(t^4,t^6\pm t^{2q+5})$, *then* t^{2q+9},t^{2q+13} *form a basis of* $G_f/f^*\mathcal{E}_2$ *and* $\mathrm{sd}(f)=2$. *The family* $(t^4,\pm t^6+\lambda t^{2q+5}+\mu t^{2q+9}), \lambda\neq 0$, *contains all symplectic classes of type* $W^{\#}_{1,2q-1}$.

To examine the symplectic equivalence classes of simple planar curves, we note the following results:

Lemma 3.22 *Let* $f:(\mathbf{R},0)\to(\mathbf{R}^2,0)$ *be* $\mathcal{A}$*-finite. If* $\mathrm{ord}(f)=m$, *then f is symplectically equivalent to* $(t^m,t^{m\ell+j}+o(t^{m\ell+j}))$ *for some* $\ell\ge 1$ *and j with* $1\le j\le m-1$. *In particular, if* $\mathrm{ord}(f)=2$, *then f is symplectically equivalent to* $(t^2,t^{2\ell+1}+o(t^{2\ell+1}))$ *for some* $\ell\ge 1$. *If* $\mathrm{ord}(f)=3$, *then f is symplectically equivalent to* $(t^3,t^{3\ell+1}+o(t^{3\ell+1}))$ *or* $(t^3,t^{3\ell+2}+o(t^{3\ell+2}))$ *for some* $\ell\ge 1$.

If $\mathrm{ord}(f)=4$ *and f is* $\mathcal{A}$*-simple, then f is symplectically equivalent to* $(t^4,t^5+o(t^5))$, $(t^4,t^6+o(t^6))$ *or* $(t^4,t^7+o(t^7))$.

Theorem 3.23 (1) *Let* $\ell\ge 2$. *Then any planar curve germ of type* $E_{6\ell}$ *is symplectically equivalent to*

$$f_\lambda=(t^3,(\pm1)^{\ell+1}t^{3\ell+1}+\sum_{j=1}^{\ell-1}\lambda_j t^{3(\ell+j)-1}),$$

for some $\lambda=(\lambda_1,\dots,\lambda_{\ell-1})\in\mathbf{R}^{\ell-1}$. *Moreover* f_λ *and* f'_λ *are symplectically equivalent if and only if* $\lambda'=(\pm1)^{\ell-1}\lambda$.

(2) *Let* $\ell\ge 2$. *Then any planar curve germ of type* $E_{6\ell+2}$ *is symplectically equivalent to*

$$f_\lambda=(t^3,(\pm1)^{\ell}t^{3\ell+2}+\sum_{j=1}^{\ell-1}\lambda_j t^{3(\ell+j)+1}),$$

for some $\lambda=(\lambda_1,\dots,\lambda_{\ell-1})\in\mathbf{R}^{\ell-1}$. *Moreover* f_λ *and* f'_λ *are symplectically equivalent if and only if* $\lambda'=(\pm1)^{\ell}\lambda$.

(3) *Any planar curve germ of type* W_{12} *is symplectically equivalent to*

$$f_\lambda=(t^4,t^5+\lambda t^7)$$

for some $\lambda\in\mathbf{R}$. *Moreover* f_λ *and* f'_λ *are symplectically equivalent if and only if* $\lambda'=\lambda$.

(4) *Any planar curve germ of type* W_{18} *is symplectically equivalent to*

$$f_{\lambda,\mu}=(t^4,t^7+\lambda t^9+\mu t^{13})$$

for some $(\lambda, \mu) \in \mathbf{R}^2$*. Moreover* $f_{\lambda,\mu}$ *and* $f_{\lambda',\mu'}$ *are symplectically equivalent if and only if* $(\lambda', \mu') = (\lambda, \mu)$*.*

(5) *Let* $q \geq 1$*. Then any planar curve germ of type* $W^{\#}_{1,2q-1}$ *is symplectically equivalent to*

$$f_{\lambda,\mu} = (t^4, \pm t^6 + \lambda t^{2q+5} + \mu t^{2q+9}),$$

for some $(\lambda, \mu) \in (\mathbf{R} - \{0\}) \times \mathbf{R}$*. Moreover* $f_{\lambda,\mu}$ *and* $f_{\lambda',\mu'}$ *are symplectically equivalent if and only if* $(\lambda', \mu') = \pm(\lambda, \mu)$*.*

We summerize the results in table 1.

	DIFF. NORMAL FORM	DEFECT	SYM. NORMAL FORM
$A_{2\ell}$	$(t^2, t^{2\ell+1})$	0	$(t^2, t^{2\ell+1})$
E_6	(t^3, t^4)	0	(t^3, t^4)
$E_{6\ell}(\ell \geq 2)$	$(t^3, t^{3\ell+1} \pm t^{3(\ell+p)+2}), 0 \leq p \leq \ell-2$ $(t^3, t^{3\ell+1})$	$\ell - p - 1$ 0	$(t^3, (\pm 1)^{\ell+1} t^{3\ell+1} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1})$
E_8	(t^3, t^5)	0	$(t^3, \pm t^5)$
$E_{6\ell+2}(\ell \geq 2)$	$(t^3, t^{3\ell+2} \pm t^{3(\ell+p)+4}), 0 \leq p \leq \ell-2$ $(t^3, t^{3\ell+2})$	$\ell - p - 1$ 0	$(t^3, (\pm 1)^{\ell} t^{3\ell+2} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1})$
W_{12}	$(t^4, t^5 \pm t^7)$ (t^4, t^5)	1 0	$(t^4, t^5 + \lambda t^7)$
W_{18}	$(t^4, t^7 \pm t^9)$ $(t^4, t^7 \pm t^{13})$ (t^4, t^7)	2 1 0	$(t^4, t^7 + \lambda t^9 + \mu t^{13})$
$W^{\#}_{1,2q-1}$	$(t^4, t^6 + t^{2q+5}), q \geq 1$	2	$(t^4, \pm t^6 + \lambda t^{2q+5} + \mu t^{2q+9})$

Table 1: The symplectic classification of simple planar curves.

For the symplectically versal unfoldings we have:

Proposition 3.24 *The symplectically versal unfolding with the minimal number of parameters for each $\mathcal{A}$-simple planar curve is given by:*

$A_{2\ell}$ *(*sp-codim $= \ell$*) :*

$$(t^2, t^{2\ell+1} + \sum_{j=1}^{\ell} \lambda_j t^{2\ell-2j+1}),$$

$(\lambda_1, \ldots, \lambda_\ell) \in (\mathbf{R}^\ell, 0)$.

$E_{6\ell}$ *(*sp-codim $= 3\ell$*) :*

$$\left(\begin{array}{l} t^3 + \lambda t, \\ (\pm 1)^{\ell+1} t^{3\ell+1} + \sum_{j=1}^{\ell} \mu_j t^{3\ell-3j+1} + \sum_{j=1}^{2\ell-1} \nu_j t^{6\ell-3j-1} \end{array} \right),$$

$(\nu_1, \ldots, \nu_{\ell-1}) \in \mathbf{R}^{\ell-1}, (\lambda, \mu_1, \ldots, \mu_\ell, \nu_\ell, \ldots, \nu_{2\ell-1}) \in (\mathbf{R}^{2\ell+1}, 0)$.

$E_{6\ell+2}$ *(*sp-codim $= 3\ell+1$*) :*

$$\left(\begin{array}{l} t^3 + \lambda t, \\ (\pm 1)^{\ell} t^{3\ell+2} + \sum_{j=1}^{\ell} \mu_j t^{3\ell-3j+2} + \sum_{j=1}^{2\ell} \nu_j t^{6\ell-3j+1} \end{array} \right),$$

$(\nu_1, \ldots, \nu_{\ell-1}) \in \mathbf{R}^{\ell-1}, (\lambda, \mu_1, \ldots, \mu_\ell, \nu_\ell, \ldots, \nu_{2\ell}) \in (\mathbf{R}^{2\ell+2}, 0)$.

W_{12} *(*sp-codim $= 6$*) :*

$$\begin{pmatrix} t^4 + \lambda_1 t^2 + \lambda_2 t, \\ t^5 + \mu_1 t^7 + \mu_2 t^3 + \mu_3 t^2 + \mu_4 t \end{pmatrix},$$

$\mu_1 \in \mathbf{R}, (\lambda_1, \lambda_2, \mu_2, \mu_3, \mu_4) \in (\mathbf{R}^5, 0)$.

W_{18} *(*sp-codim $= 9$*) :*

$$\begin{pmatrix} t^4 + \lambda_1 t^2 + \lambda_2 t, \\ t^7 + \mu_1 t^{13} + \mu_2 t^9 + \mu_3 t^6 + \mu_4 t^5 + \mu_5 t^3 + \mu_6 t^2 + \mu_7 t \end{pmatrix},$$

$(\mu_1, \mu_2) \in \mathbf{R}^2, (\lambda_1, \lambda_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7) \in (\mathbf{R}^7, 0)$.

$W^{\#}_{1,2q-1}$ *(*sp-codim $= q + 7$*) :*

$$\begin{pmatrix} t^4 + \lambda t^2 + \rho t, \\ \pm t^6 + t^{2q+5} + \mu t^{2q+9} + \sum_{j=0}^{q+2} \nu_j t^{2q+5-2j} + \theta t^2 + \rho t^{2q+2} \end{pmatrix},$$

$(\nu_0, \mu) \in \mathbf{R}^2, \nu_0 \neq -1, (\lambda, \nu_1, \ldots, \nu_{q+2}, \theta, \rho) \in (\mathbf{R}^{q+5}, 0)$.

Remarkably the symplectic versal unfolding can be taken uniformly for each class of simple planar curves; this is not the case for the $\mathcal{A}$-versal unfoldings. This is natural because the A-E-W-$W^{\#}$-classification is based on the constancy of the Milnor number μ, and the μ-constant strata coincide with the sp-codim constant strata (cf. Theorem 3.20).

In particular we have:

Proposition 3.25 *Let $F : (\mathbf{R}^n, 0) \to (\mathbf{R}^2 \times \mathbf{R}^{n-1})$ be a symplectically versal unfolding of $A_{2\ell} : (t^2, t^{2\ell+1}), \ell \leq n-1$. Then F is liftable equivalent to*

$$(x_1, x_2, \ldots, x_n) = (t, \lambda_1, \ldots, \lambda_{n-1}) \mapsto$$
$$(q_1, p_1, q_2, \ldots, q_n) = (t^2, t^{2\ell+1} + \lambda_1 t^{2\ell-1} + \lambda_2 t^{2\ell-3} + \cdots + \lambda_\ell t, \lambda_1, \ldots, \lambda_{n-1}).$$

Example 3.26 (Opening of the Whitney umbrella) Any symplectically versal unfolding of $A_1 : (t^2, t^3)$ is liftable equivalent to

$$(x_1, x_2, \ldots, x_n) = (t, \lambda_1, \ldots, \lambda_{n-1}) \mapsto$$
$$(q_1, p_1, q_2, \ldots, q_n) = (t^2, t^3 + \lambda_1 t, \lambda_1, \ldots, \lambda_{n-1}).$$

The Lagrangian lifting is symplectically equivalent to

$$(x_1, x_2, \ldots, x_n) = (t, \lambda_1, \ldots, \lambda_{n-1}) \mapsto$$
$$(q_1, p_1, q_2, \ldots, q_n, p_2, \ldots, p_n) = (t^2, t^3 + \lambda_1 t, \lambda_1, \frac{2}{3} t^3, 0, \ldots, 0).$$

3.5 Isotopy and symplectic classifications.

We show that if the symplectic defect vanishes, then the classifications by isotopy and by symplectomorphism coincide.

Lemma 3.27 *There are isomorphisms of the vector spaces:*

$$\frac{tf(m_1V_1) + wf(m_2V_2)}{tf(m_1V_1) + wf(m_2V_2 \cap VH_2)} \cong \frac{G'_f}{f^* m_2^2} \cong \frac{G_f}{f^*\mathcal{E}_2},$$

where m_1 (resp. m_2) is the maximal ideal of $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) consisting of functions H with $H(0) = 0$, and $G'_f = \{h \in m_1 \mid dh \in \langle df_1, df_2\rangle_{f^* m_2}\}$.

Corollary 3.28 *The symplectic defect of a planar curve-germ measures the codimension of the symplectic equivalence orbit in the $\mathcal{A}$-equivalence orbit of the germ (in the jet space of sufficiently high order).*

Remark 3.29 Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a map-germ. Then, for diffeomorphisms $\tau : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ and $\sigma : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$, we have $G'_{\tau\circ f} = G'_f$ and $\sigma^*(G'_f) = G'_{f\circ\sigma}$.

Theorem 3.30 *Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be an $\mathcal{A}$-finite map-germ with* $\mathrm{sd}(f) = 0$. *If a map-germ $f' : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ is isotopic to f, then f' is symplectically equivalent to f.*

Example 3.31 The planar curves of type E_8 are classified up to isotopy into $E_8^+ : t \mapsto (t^3, t^5)$ and $E_8^- : t \mapsto (t^3, -t^5)$, because they are chiral. Then, since the symplectic defect vanishes in this case, this gives also the symplectic classification.

3.6 Lagrangian liftings of the swallowtails.

Let M^{2k} be the space of polynomials of degree $2k + 1$ of the form (cf. [1])

$$M^{2k} = \{\frac{x^{2k+1}}{(2k+1)!} + q_1\frac{x^{2k-1}}{(2k-1)!} + \ldots + q_k\frac{x^k}{(k)!} - p_k\frac{x^{k-1}}{(k-1)!} + \ldots + (-1)^k p_1\}$$

endowed with the symplectic Darboux form $\sum_{i=1}^k dp_i \wedge dq_i$ (reduction of the sl_2-invariant symplectic form on the space of binary forms of $2k + 3$-degree).

The canonical projection into N is given by the derivative

$$D^{k-1} = \frac{d^{k-1}}{dx^{k-1}},$$

which projects M^{2k} into the space of polynomials

$$N = \{\frac{x^{k+2}}{(k+2)!} + q_1\frac{x^k}{(k)!} + \ldots + q_k x - p_k\}.$$

The standard (generalized) swallowtail in N is defined as the space $\Sigma_k \subset N$ of polynomials having at least one root of multiplicity ≥ 2.

The derivative $\dfrac{d}{dx}$ of the polynomial decreases the multiplicities of its roots, however the difference of the degree of polynomial and the multiplicity of the root, called the co-multiplicity, is not affected by the derivative. So the polynomials of Σ_k have roots of comultiplicity $\leq k$.

The canonical Lagrangian variety, which is a Lagrangian lifting of Σ_k is defined by V.I. Arnold as the space $\tilde{\Sigma}_k$ of polynomials in M^{2k} having a root of multiplicity at least $k+1$. This lifting is most regular (stabilisation in the sense of Arnold) because the multiplicity is at least $k+1$ and the degree of the polynomial is $2k+1$ and finally the polynomials of $\tilde{\Sigma}_k$ have only unique root of this multiplicity. So the intersection points of Σ_k are avoided.

A parametrisation of Σ_k is given in the form

$$F : (\mathbf{R}^k, 0) \to (N, 0)$$

$$\begin{aligned} F(s) &= (s_1, \ldots, s_{k-1}, -\frac{s_k^{k+1}}{(k+1)!} - \sum_{i=1}^{k-1} s_i \frac{s_k^{k-i}}{(k-i)!}, \\ &\quad -\frac{s_k^{k+2}}{(k+2)k!} - \sum_{i=1}^{k-1} s_{k-i} \frac{s_k^{k-i+1}}{(k-i+1)(k-i-1)!}). \end{aligned}$$

Its Lagrangian lifting $\widetilde{\Sigma}_k$, $\widetilde{F} : (\mathbf{R}^k, 0) \to (M^{2k}, \omega)$ is generated by the following generating family (cf. [44], p. 106),

$$P_k(q, \lambda) = \frac{1}{2} \int_0^{\ell} (\frac{k+2}{(k+1)!} x^{k+1} + \sum_{i=1}^{k} q_i \frac{x^{k-i}}{(k-i)!})^2 dx.$$

Thus the associated symplectic bifurcating family of curves (swallowtail bifurcation family) in $(\mathbf{R}^2, dp_k \wedge dq_k)$ is defined by:

$$q_k = -\frac{k+2}{(k+1)!} x^{k+1} - \sum_{i=1}^{k-1} \frac{1}{(k-i)!} q_i x^{k-i},$$

$$p_k = -\frac{1}{k!} x^{k+2} - \sum_{i=1}^{k-1} \frac{1}{(k-i+1)(k-i-1)!} q_i x^{k-i}.$$

where x is the curve parameter and $(q_1, \ldots, q_{k-1})$ are the bifurcation parameters of the family. We see that this is an unfolding of the curve

$$(q_k, p_k) = (-\frac{k+2}{(k+1)!} x^{k+1}, -\frac{1}{k!} x^{k+2}).$$

In a more general setting, this result may be formulated in the following way.

Proposition 3.32 *Let $G : (\mathbf{R} \times N^{k+1}, 0) \to \mathbf{R}$ be a function family germ with A_{k+1}-type singularity. Let Σ_k be the discriminant set of G,*

$$\Sigma_k = \{u \in N^{k+1} \mid G(x, u) = 0, G'_x(x, u) = 0, \text{for some } x \in (\mathbf{R}, 0)\}.$$

Then there exist a symplectic space $(\mathbf{R}^{2k},\omega)$ *and an isotropic fibration*

$$\pi : ((\mathbf{R}^{2k},\omega),0) \to (N^{k+1},0); (p,q) \mapsto (q_1,\dots q_k,p_k)$$

and a Lagrangian lifting $\widetilde{\Sigma}$ *of* Σ*. Moreover* $\widetilde{\Sigma}$ *is uniquely defined by the conditions:*

$$\widetilde{\Sigma} = \{\bar{u} \in \mathbf{R}^{2k} \mid D^{-(k-l)}G(x,u) + \sum_{i=1}^{k-l}(-1)^{i-1}p_{k-i}\frac{x^{k-i-l}}{(k-i-l)!} = 0, 1 \le \ell \le k\},$$

where $D^0G(x,u) = G(x,u)$ *and* $\bar{u} = (u,p_1,\dots,p_{k-1})$.

3.7 Frontal-symplectic versality and open swallowtails.

In the case $k=2$, we interpret Givental's construction from the versality viewpoint of "frontal-symplectic" category, based on the fact that the swallowtail surface provides the versal unfolding of planar curve of type $E_6 : t \mapsto (t^3,t^4)$, among wave-front curves.

Here we give a direct method to construct a versal unfolding in the frontal-symplectic category.

Let $f : (\mathbf{R},0) \to (\mathbf{R}^2,0)$ be a non-flat map-germ. After using a symplectomorphism of $\mathbf{R}^2$, we assume $\mathrm{ord} f_1 < \mathrm{ord} f_2$. Let $(f,\varphi) : (\mathbf{R},0) \to (\mathbf{R}^3,0)$ be the Legendrian liftings of f for the contact form $\alpha = dy - pdx$. In fact $\varphi = (\frac{df_2}{dt})/(\frac{df_1}{dt})$, and then $df_2 - \varphi df_1 = 0$. Note that $\mathrm{ord}\varphi = \mathrm{ord} f_2 - \mathrm{ord} f_1$. Let $w = (f,\varphi;\xi,\eta,\psi) : (\mathbf{R},0) \to T\mathbf{R}^3$ be an infinitesimal deformation of (f,φ) among Legendrian (integral) mappings. Then $d\eta - \psi df_1 - \varphi d\xi = 0$, that is, $d(\eta - \varphi\xi) = -\xi d\varphi + \psi df_1$. Set $k = \eta - \varphi\xi$. Then $k - k(0)$ has order $\ge \min\{\mathrm{ord} f_1, \mathrm{ord}\varphi\}$. For the induced infinitesimal deformation $v = (f;\xi,\eta) : (\mathbf{R},0) \to T\mathbf{R}^2$ of f, take a function h with $dh = \eta df_1 - \xi df_2$, a *generating function* of v. Then $dh = \eta df_1 - \xi\varphi df_1 = (\eta - \xi\varphi)df_1 = kdf_1$. So $h - h(0)$ is a sum of a monomial of order $\mathrm{ord} f_1$ and a function of order $\ge \min\{2\mathrm{ord} f_1, \mathrm{ord} f_2\}$.

Set $m = \mathrm{ord} f_1, k = \min\{2\mathrm{ord} f_1, \mathrm{ord} f_2\}$ and set

$$S = \mathbf{R} + \mathbf{R}t^m + m_1^k.$$

Then S is a vector subspace of $\mathcal{E}_1$ containing $f^*\mathcal{E}_2$.

Lemma 3.33 *Let* $F : (\mathbf{R}^1 \times \mathbf{R}^\ell,0) \to (\mathbf{R}^2 \times \mathbf{R}^\ell,0), F(t,\lambda) = (\bar{F}(t,\lambda),\lambda)$ *be a frontal unfolding of a non-flat map-germ* $f : (\mathbf{R},0) \to (\mathbf{R}^2,0)$*. Assume* $\mathrm{ord}(f_1) < \mathrm{ord}(f_2)$*. If* $\left.\frac{\partial \bar{F}}{\partial \lambda_1}\right|_{\mathbf{R}\times 0}, \dots, \left.\frac{\partial \bar{F}}{\partial \lambda_\ell}\right|_{\mathbf{R}\times 0}$ *generate* $S/f^*\mathcal{E}_2$ *via generating functions over* $\mathbf{R}$*, and also generate vector fileds* $t^i\frac{\partial}{\partial q_1}\circ f + \varphi t^i\frac{\partial}{\partial p_1}\circ f, (2\mathrm{ord}(f_1) - \mathrm{ord}(f_2) \le i \le \mathrm{ord}(f_1) - 2)$ *over* $\mathbf{R}$*, then* F *is a frontal-symplectically versal unfolding of* f*. Frontal-symplectically versal unfoldings are unique up to liftable equivalence.*

Example 3.34 (The open swallowtail.) Let $f : (\mathbf{R},0) \to (\mathbf{R}^2,0), f(t) = (t^3,t^4)$ be a map-germ of type E_6. Then the one-parameter unfolding $F : (\mathbf{R}^2,0) \to (\mathbf{R}^3,0), F(t,\lambda) = (q_1,p_1,q_2) = (t^3 + 3\lambda t, t^4 + 2\lambda t^2, \lambda)$ of f is a frontal-symplectic versal unfolding of f. The image of F is the swallowtail surface and has the double point locus. The Lagrangian lifting

$$\widetilde{F}(t,\lambda) = (q_1,p_1,q_2,p_2) = (t^3 + 3\lambda t, t^4 + 2\lambda t^2, \lambda, \frac{6}{5}t^5 + 2\lambda t^3),$$

of F coincides with the open swallowtail surface, which has no self-intersections.

Example 3.35 (The open folded umbrella.) Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0), f(t) = (t^2, t^5)$ be a map-germ of type A_4. Then the one-parameter unfolding $F : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0), F(t, \lambda) = (q_1, p_1, q_2) = (t^2, t^5 + \lambda t^3, \lambda)$ of f is a frontal-symplectic versal unfolding of f. The image of F is the folded umbrella and has the double point locus. The Lagrangian lifting

$$\widetilde{F}(t, \lambda) = (q_1, p_1, q_2, p_2) = (t^2, t^5 + \lambda t^3, \lambda, \frac{2}{5}t^5),$$

of F has no self-intersections and may be called "the open folded umbrella".

3.8 Symplectic classification of uni-modal planar curves

We have proceed to the symplectic classification in [36] to uni-modal planar curves in [40] in the complex analytic case. Here we give the result in the real case.

Proposition 3.36 (cf. [36][40]) *Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a simple or a uni-modal map-germ under diffeomorphism equivalence to the symplectic plane with the symplectic form $\Omega = dx_1 \wedge dx_2$. Then f is symplectomorphic to one of the following normal forms of map-germs $(x_1(t), x_2(t)) : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$:*

$$\begin{array}{ll}
A_{2\ell} : & (t^2, t^{2\ell+1}), \\
E_{6\ell} : & (t^3, (\pm)^{\ell+1} t^{3\ell+1} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1}), \\
E_{6\ell+2} : & (t^3, (\pm)^{\ell} t^{3\ell+2} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1}), \\
W_{12} : & (t^4, t^5 + \lambda_1 t^7), \\
W_{18} : & (t^4, t^7 + \lambda_1 t^9 + \lambda_2 t^{13}), \\
W^{\#}_{1,2\ell-1} : & (t^4, \pm t^6 + \lambda_1 t^{2\ell+5} + \lambda_2 t^{2\ell+9}), \lambda_1 \neq 0, \quad (\ell = 1, 2, \dots)
\end{array}$$

$$\begin{array}{ll}
N_{20} : & (t^5, t^6 + \lambda_1 t^8 + \lambda_2 t^9 + \lambda_3 t^{14}), \\
N_{24} : & (t^5, \pm t^7 + \lambda_1 t^8 + \lambda_2 t^{11} + \lambda_3 t^{13} + \lambda_4 t^{18}), \\
N_{28} : & (t^5, t^8 + \lambda_1 t^9 + \lambda_2 t^{12} + \lambda_3 t^{14} + \lambda_4 t^{17} + \lambda_5 t^{22}), \\
W_{24} : & (t^4, t^9 + \lambda_1 t^{10} + \lambda_2 t^{11} + \lambda_3 t^{15} + \lambda_4 t^{19}), \\
W_{30} : & (t^4, t^{11} + \lambda_1 t^{13} + \lambda_2 t^{14} + \lambda_3 t^{17} + \lambda_4 t^{21} + \lambda_5 t^{25}), \\
W^{\#}_{2,2\ell-1} : & (t^4, \pm t^{10} + \lambda_1 t^{2\ell+9} + \lambda_2 t^{2\ell+11} + \lambda_3 t^{2\ell+13} \\
& \qquad + \lambda_4 t^{2\ell+17} + \lambda_5 t^{2\ell+21}), \lambda_1 \neq 0, \quad (\ell = 1, 2, \dots)
\end{array}$$

Note that the difference of the classifications in the complex case and in the real case appears as "plus-minus" in some appropriate points.

3.9 Symplectic moduli spaces.

The symplectic moduli spaces are determined by the following result:

Theorem 3.37 ([40]) *Let $f_\lambda(t) = (t^m, t^n + \lambda_1 t^{r_1} + \lambda_2 t^{r_2} + \cdots + \lambda_s t^{r_s})$ be one of the symplectic normal forms in Theorem 3.36. Then two curve-germs f_λ and $f_{\lambda'}$ belonging to the same family are symplectomorphic if and only if there exists an $(m+n)$-th root $\zeta \in \mathbf{C}$ of unity satisfying*

$$\lambda'_1 = \zeta^{r_1 - n}\lambda_1, \ \lambda'_2 = \zeta^{r_2 - n}\lambda_2, \ \dots, \ \lambda'_s = \zeta^{r_s - n}\lambda_s.$$

In particular each symplectic moduli space of a family is a Hausdorff space in the natural topology and it is extended to a cyclic quotient singularity.

The symplectic moduli spaces are given in Tables 2 and 3.

3.10 Symplectic rigidity

In the process of symplectic classification, we observe a kind of rigidity. Let $f_\lambda, (\lambda \in \mathbf{C}^s)$ be one of the symplectic normal forms of simple or uni-modal parametric planar curve singularities. Since the symplectic normal form gives a mini-transversal to symplectic orbits in a sufficiently higher order jet space, we see that each symplectomorphism equivalence class is isolated in the parameter space $\mathbf{C}^s$. Moreover we have stronger rigidity, *symplectic rigidity*, which implies Theorem 3.37. To see that we need a series of conditions on non-linear symplectomorphisms, which is obtained via straightforward calculations.

Then we have the symplectic rigidity:

Proposition 3.38 *Let f_λ and $f_{\lambda'}$ be germs belonging to one of the symplectic normal forms of simple or uni-modal parametric planar curve singularities. If f_λ and $f_{\lambda'}$ are symplectomorphic, then they are linearly symplectomorphic: If there exists a symplectomorphism equivalence (σ, τ) satisfying $\tau \circ f_{\lambda'} = f_\lambda \circ \sigma$, then there exists a symplectomorphism equivalence (Σ, T) such that $T \circ f_{\lambda'} = f_\lambda \circ \Sigma,\ \Sigma : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ is a complex linear transformation, and $T : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ is a complex linear symplectic transformation.*

Remark 3.39 If two curve-germs $f,\ g : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ are symplectomorphic, then they are *symplectically isotopic*, that is, there exist C^∞ families of bi-holomorphic diffeomorphisms σ_s and bi-holomorphic symplectomorphisms τ_s $(s \in [0, 1])$ on $(\mathbf{C}, 0)$ and $(\mathbf{C}^2, 0)$ respectively such that $\sigma_0(t) = t,\ \tau_0(x, y) = (x, y)$ and $\tau_1(g(t)) = f(\sigma_1(t))$. This fact is a feature of the complex case and it is proved by using the fact that $\mathrm{SL}(2, \mathbf{C})$ is arc-wise connected and the group of symplectomorphisms with identity linear part is arc-wise connected (cf. [34]). Thus our symplectic moduli space in Tables 1 and 2 are also moduli spaces for the symplectic isotopy equivalence (cf. [72]).

3.11 Differential normal forms.

The classification of simple singularities by Bruce-Gaffney [10] is extended to the following:

Theorem 3.40 *Under diffeomorphism equivalences the simple and uni-modal singularities of*

	DIFF. NORMAL FORM	SYMP. NORMAL FORM	SYMP. MODULI SPACE
$A_{2\ell}$	$(t^2, t^{2\ell+1})$	$(t^2, t^{2\ell+1})$	
$E_{6\ell}$ $(\ell \geq 1)$	$(t^3, t^{3\ell+1} + t^{3(\ell+p)+2})$ $(0 \leq p \leq \ell - 2)$ $(t^3, t^{3\ell+1})$	$(t^3, t^{3\ell+1} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1})$	$\mathbf{C}^{\ell-1}/G,\ G = \mathbf{Z}/(3\ell+4)\mathbf{Z}$ $(\lambda_1, \ldots, \lambda_{\ell-1}) \mapsto$ $(\zeta\lambda_1, \ldots, \zeta^{3j-2}\lambda_j, \ldots, \zeta^{3\ell-5}\lambda_{\ell-1})$ $(\zeta^{3\ell+4} = 1, \text{primitive})$
$E_{6\ell+2}$ $(\ell \geq 1)$	$(t^3, t^{3\ell+2} + t^{3(\ell+p)+4}),$ $(0 \leq p \leq \ell - 2)$ $(t^3, t^{3\ell+2})$	$(t^3, t^{3\ell+2} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1})$	$\mathbf{C}^{\ell-1}/G,\ G = \mathbf{Z}/(3\ell+5)\mathbf{Z}$ $(\lambda_1, \ldots, \lambda_{\ell-1}) \mapsto$ $(\zeta^2\lambda_1, \ldots, \zeta^{3j-1}\lambda_j, \ldots, \zeta^{3\ell-4}\lambda_{\ell-1})$ $(\zeta^{3\ell+5} = 1, \text{primitive})$
W_{12}	$(t^4, t^5 + t^7)$ (t^4, t^5)	$(t^4, t^5 + \lambda t^7)$	$\mathbf{C}/G,\ G = \mathbf{Z}/9\mathbf{Z}$ $\lambda \mapsto \zeta\lambda,\ (\zeta^9 = 1)$
W_{18}	$(t^4, t^7 + t^9)$ $(t^4, t^7 + t^{13})$ (t^4, t^7)	$(t^4, t^7 + \lambda t^9 + \mu t^{13})$	$\mathbf{C}^2/G,\ G = \mathbf{Z}/11\mathbf{Z}$ $(\lambda, \mu) \mapsto (\zeta\lambda, \zeta^3\mu),\ (\zeta^{11} = 1)$
$W^{\#}_{1,2\ell-1}$ $(\ell \geq 1)$	$(t^4, t^6 + t^{2\ell+5})$	$(t^4, t^6 + \lambda t^{2\ell+5} + \mu t^{2\ell+9})$ $(\lambda \neq 0)$	$(\mathbf{C}^* \times \mathbf{C})/G,\ G = \mathbf{Z}/10\mathbf{Z},$ $(\lambda, \mu) \mapsto (\zeta^{2\ell-1}\lambda, \zeta^{2\ell+3}\mu),$ $(\zeta^{10} = 1, \text{primitive})$

Table 2: The complex symplectic moduli spaces of simple parametric planar curve singularities.

	SYMP. NORMAL FORM	SYMP. MODULI SPACE
N_{20}	$(t^5, t^6 + \lambda_1 t^8 + \lambda_2 t^9 + \lambda_3 t^{14})$	$\mathbf{C}^3/G,\ G = \mathbf{Z}/11\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3) \mapsto (\zeta^2\lambda_1, \zeta^3\lambda_2, \zeta^8\lambda_3),$ $(\zeta^{11} = 1)$
N_{24}	$(t^5, t^7 + \lambda_1 t^8 + \lambda_2 t^{11} + \lambda_3 t^{13} + \lambda_4 t^{18})$	$\mathbf{C}^4/G,\ G = \mathbf{Z}/12\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto$ $(\zeta\lambda_1, \zeta^4\lambda_2, \zeta^6\lambda_3, \zeta^{11}\lambda_4),$ $(\zeta^{12} = 1, \text{primitive})$
N_{28}	$(t^5, t^8 + \lambda_1 t^9 + \lambda_2 t^{12} + \lambda_3 t^{14} + \lambda_4 t^{17} + \lambda_5 t^{22})$	$\mathbf{C}^5/G,\ G = \mathbf{Z}/13\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \mapsto$ $(\zeta\lambda_1, \zeta^4\lambda_2, \zeta^6\lambda_3, \zeta^9\lambda_4, \zeta^{14}\lambda_5),$ $(\zeta^{13} = 1, \text{primitive})$
W_{24}	$(t^4, t^9 + \lambda_1 t^{10} + \lambda_2 t^{11} + \lambda_3 t^{15} + \lambda_4 t^{19})$	$\mathbf{C}^4/G,\ G = \mathbf{Z}/13\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto$ $(\zeta\lambda_1, \zeta^2\lambda_2, \zeta^6\lambda_3, \zeta^{10}\lambda_4),$ $(\zeta^{13} = 1)$
W_{30}	$(t^4, t^{11} + \lambda_1 t^{13} + \lambda_2 t^{14} + \lambda_3 t^{17}$ $+ \lambda_4 t^{21} + \lambda_5 t^{25})$	$\mathbf{C}^5/G,\ G = \mathbf{Z}/15\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \mapsto$ $(\zeta^2\lambda_1, \zeta^5\lambda_2, \zeta^6\lambda_3, \zeta^{10}\lambda_4, \zeta^{14}\lambda_5),$ $(\zeta^{15} = 1, \text{primitive})$
$W^{\#}_{2,2\ell-1}$	$(t^4, t^{10} + \lambda_1 t^{2\ell+9} + \lambda_2 t^{2\ell+11} + \lambda_3 t^{2\ell+13}$ $+ \lambda_4 t^{2\ell+17} + \lambda_5 t^{2\ell+21}), \quad (\lambda_1 \neq 0).$	$(\mathbf{C}^* \times \mathbf{C}^4)/G,\ G = \mathbf{Z}/14\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \mapsto$ $(\zeta^{2\ell-1}\lambda_1, \zeta^{2\ell+1}\lambda_2, \zeta^{2\ell+3}\lambda_3, \zeta^{2\ell+7}\lambda_4, \zeta^{2\ell+11}\lambda_5),$ $(\zeta^{14} = 1, \text{primitive})$

Table 3: The complex symplectic moduli spaces of uni-modal parametric planar curve singularities.

parametric planar curves $f:(\mathbf{C},0)\to(\mathbf{C}^2,0)$ *are classified completely into the following list:*

$$
\begin{array}{ll}
A_{2\ell}: & (t^2,t^{2\ell+1}),\quad (\ell=1,2,3,\dots),\\
E_{6\ell}: & (t^3,t^{3\ell+1}+t^{3(\ell+p)+2}),(0\le p\le \ell-2),\ (t^3,t^{3\ell+1}),\\
E_{6\ell+2}: & (t^3,t^{3\ell+2}+t^{3(\ell+p)+4}),(0\le p\le \ell-2),\ (t^3,t^{3\ell+2}),\\
W_{12}: & (t^4,t^5+t^7),\ (t^4,t^5),\\
W_{18}: & (t^4,t^7+t^9),\ (t^4,t^7+t^{13}),\ (t^4,t^7),\\
W^{\#}_{1,2\ell-1}: & (t^4,t^6+t^{2\ell+5}),\\
N_{20}: & (t^5,t^6+t^8+\lambda t^9)\ (-\lambda\sim\lambda),\ (t^5,t^6+t^9),\ (t^5,t^6+t^{14}),\ (t^5,t^6),\\
N_{24}: & (t^5,t^7+t^8+\lambda t^{11}),\ (t^5,t^7+t^{11}+\lambda t^{13})\ (-\lambda\sim\lambda),\\
 & (t^5,t^7+t^{13}),\ (t^5,t^7+t^{18}),\ (t^5,t^7),\\
N_{28}: & (t^5,t^8+t^9+\lambda t^{12}),\ (t^5,t^8+t^{12}+\lambda t^{14})\ (-\lambda\sim\lambda),\\
 & (t^5,t^8+t^{14}+\lambda t^{17})\ (-\lambda\sim\lambda),\ (t^5,t^8+t^{17}),\ (t^5,t^8+t^{22}),\ (t^5,t^8),\\
W_{24}: & (t^4,t^9+t^{10}+\lambda t^{11})\ (\lambda\ne\frac{19}{18}),\ (t^4,t^9+t^{10}+\frac{19}{18}t^{11}+\lambda t^{15}),\\
 & (t^4,t^9+t^{11}),\ (t^4,t^9+t^{15}),\ (t^4,t^9+t^{19}),\ (t^4,t^9),\\
W_{30}: & (t^4,t^{11}+t^{13}+\lambda t^{14})\ (-\lambda\sim\lambda),\ (t^4,t^{11}+t^{14}+\lambda t^{17})\ (\lambda\ne\frac{25}{22}),\\
 & (t^4,t^{11}+t^{14}+\frac{25}{22}t^{17}+\lambda t^{21})\ (\omega\lambda\sim\lambda,\omega^3=1),\\
 & (t^4,t^{11}+t^{17}),\ (t^4,t^{11}+t^{21}),\ (t^4,t^{11}+t^{25}),\ (t^4,t^{11}),\\
W^{\#}_{2,2\ell-1}: & (t^4,t^{10}+t^{2\ell+9}+\lambda t^{2\ell+11})\ (\omega\lambda\sim\lambda,\omega^{2\ell-1}=1)\quad (\ell=1,2,3,\dots).
\end{array}
$$

In the list, for instance $-\lambda\sim\lambda$ means that $(t^5,t^6+t^8+\lambda' t^9)$ is diffeomorphic to $(t^5,t^6+t^8+\lambda t^9)$ if and only if $\lambda'=\pm\lambda$.

Remark 3.41 Ebey [22] gave the diffeomorphism classifications of the cases $(2,2\ell+1),(3,3\ell+1),(3,3\ell+2),(4,5),(4,6,2\ell+5)$ $(4,7)$ and $(5,9)$. Also he erroneously classified also the cases $(4,9),(4,10,2\ell+9)$ and $(4,11)$; his classification have several omissions and errors, which are corrected in our classification. Also note that several cases with modality ≥ 2 are classified by diffeomorphisms by several authors: $(5,9)$ ([22]); $(6,7)$ ([79]); $(5,11)$ ([51]); $(2p,2q,2q+\ell)$ ([53]); $(6,9,10)$ ([28]).

Remark 3.42 The classification of planar curve singularities is closely related to the classification of Legendre curve singularities and the classification of Goursat distributions ([81][82][66][64][65]).

Lemma 3.43 *Let* N *denote* $\max\{r\in\mathbf{N}\setminus S(f)\}$. *Then we have:*

(1) *For any* $\psi(t)$ *with* $\mathrm{ord}(\psi)>N$, *there exists a holomorphic function* $h_\lambda(x,y)$ *on* $(\mathbf{C}^2,0)$ *depending on* λ *holomorphically and satisfying* $\psi(t)=h_\lambda(f_\lambda(t))$ *and* $\mathrm{ord}(h_\lambda)\ge 2$.

(2) *Any vector field* $v=(0,\rho(t))$ *along* f_λ *is symplectically solvable, i.e.* $v=tf_\lambda(\xi)+wf(\eta)$ *for some vector fields* ξ,η *with* $\xi(0)=0,\eta(0,0)=0$, *if* $\mathrm{ord}(\rho)>N-m$.

Also we have

Lemma 3.44 ([38]) *Let* $f:(\mathbf{C},0)\to(\mathbf{C}^2,0)$ *be a curve-germ with the Puiseux characteristic* $(m,\beta_1,\dots)$. *If* $m=4$ *and* $\beta_1\ge 13$, *or* $m=5$ *and* $\beta_1\ge 9$, *or* $m\ge 6$ *then the modality of* f *is at least* 2.

Remark 3.45 In general, for each equi-singularity class, the symplectic moduli space is mapped canonically onto the differential moduli space, i.e. the ordinary moduli space. The dimension of the fiber over a diffeomorphism class $[f]$ is called the *symplectic defect* and denoted by $\mathrm{sd}(f)$ in [36]. It is known that $\mathrm{sd}(f) = \mu(f) - \tau(f)$, where $\mu(f) = 2\delta(f)$ is the Milnor number of f and $\tau(f)$ is the Tyurina number of f ([72][50][18]). Let $s(f)$ (resp. $c(f)$) be the symplectic modality, that is, the number of parameters in the symplectic normal form of f (resp. the codimension of the locus in the parameter space corresponding to germs diffeomorphic to f). Then $s(f) - c(f) = \mathrm{sd}(f)$. Thus we have the formula for the Tjurina number (by means of Varchenko-Lando's formula) as

$$\tau(f) = 2\delta(f) + c(f) - s(f).$$

For example, for $f = (t^4, t^{11} + t^{21})$ in the case of W_{30}, we have $\delta(f) = 15,\ c(f) = 3,\ s(f) = 5$ and in fact $\tau(f) = 28$.

Note that the differential moduli space is not a Hausdorff space, while the symplectic moduli space is, at least for 0-modal and 1-modal cases, as we clearly observe in Theorems 3.40 and 3.37. Therefore the symplectic moduli space can be called a *Hausdorffication* of the differential moduli space.

Remark 3.46 The adjacency of simple and uni-modal singularities of parametric planar curves is generated (as an ordering) by $A_{2\ell} \leftarrow A_{2\ell+2}, E_{6\ell} \leftarrow E_{6\ell+2} \leftarrow E_{6\ell+6}$ $(\ell = 1, 2, \dots), A_{6s-2} \leftarrow E_{12s-6}, A_{6s} \leftarrow E_{12s}, A_{6s-2} \leftarrow E_{12s-4}, A_{6s+2} \leftarrow E_{12s+2}$ $(s = 1, 2, \dots), E_8 \leftarrow W_{12} \leftarrow W_{18}, W_{12} \leftarrow W^{\#}_{1,1}, E_{12} \leftarrow W^{\#}_{1,1} \leftarrow W_{18}, W^{\#}_{1,2\ell-1} \leftarrow W^{\#}_{1,2\ell+1}$ $(\ell = 1, 2, \dots), W^{\#}_{1,1} \leftarrow N_{20} \leftarrow N_{24} \leftarrow N_{28}, W_{18} \leftarrow N_{24}, W_{24} \leftarrow N_{28}, W_{18} \leftarrow W_{24} \leftarrow W_{30}, E_{18} \leftarrow W_{24} \leftarrow W^{\#}_{2,1}, E_{20} \leftarrow W_{30}, W^{\#}_{2,2\ell-1} \leftarrow W^{\#}_{2,2\ell+1}$ $(\ell = 1, 2, \dots)$.

Remark 3.47 (Classification of curves with characteristic (6, 7)).

Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ be a plane branch of characteristic $(6, 7), m = 6, \beta_1 = 7$. The quotient $\mathcal{O}_1/f^*\mathcal{O}_2$ has the monomial basis

$$t, t^2, t^3, t^4, t^5, t^8, t^9, t^{10}, t^{11}, t^{15}, t^{16}, t^{17}, t^{22}, t^{23}, t^{29}.$$

The symplectic normal form is given by

$$f_\lambda(t) = (t^6, t^7 + \lambda_1 t^9 + \lambda_2 t^{10} + \lambda_3 t^{11} + \lambda_4 t^{16} + \lambda_5 t^{17} + \lambda_6 t^{23}),$$

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \in \mathbf{C}^6$. Moreover we can check that f_λ and $f_{\lambda'}$ are symplectomorphic if and only if there exists a $\zeta \in \mathbf{C}$ with $\zeta^{13} = 1$ satisfying

$$\lambda'_1 = \zeta^2\lambda_1, \lambda'_2 = \zeta^3\lambda_2, \lambda'_3 = \zeta^4\lambda_3, \lambda'_4 = \zeta^9\lambda_4, \lambda'_5 = \zeta^{10}\lambda_5, \lambda'_6 = \zeta^{16}\lambda_6.$$

Thus the symplectic moduli space $\mathcal{M}_{\mathrm{symp}}(6, 7)$ is homeomorphic to $\mathbf{C}^6/G$ for $G = \mathbf{Z}/13\mathbf{Z}$ with the representation $G \to \mathrm{GL}(6, \mathbf{C})$ given by

$$\zeta \mapsto ((\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \mapsto (\zeta^2\lambda_1, \zeta^3\lambda_2, \zeta^4\lambda_3, \zeta^9\lambda_4, \zeta^{10}\lambda_5, \zeta^{16}\lambda_6)).$$

As a by-product we get the exact diffeomorphism classification of planar curves of characteristic $(6, 7)$ due to Zariski using our symplectic method:

Theorem 3.48 (Zariski [79]) *Any planar curve-germ $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ of Puiseux characteristic $(6, 7)$ is diffeomorphic to one of the following normal forms:*

$$
\begin{array}{ll}
Z^0_{\lambda,\mu}: & (t^6, t^7 + t^9 + \lambda t^{10} + \mu t^{11}),\ \mu \neq \frac{9}{8}\lambda^2 + \frac{23}{14},\ (\pm\lambda, \mu) \sim (\lambda, \mu), \\
Z^1_{\lambda,\nu}: & (t^6, t^7 + t^9 + \lambda t^{10} + (\frac{9}{8}\lambda^2 + \frac{23}{14})t^{11} + \nu t^{17}),\ (\pm\lambda, \nu) \sim (\lambda, \nu), \\
Z'^1_\lambda: & (t^6, t^7 + t^{10} + \lambda t^{11}),\ \omega\lambda \sim \lambda, \omega^3 = 1, \\
Z^2_\lambda: & (t^6, t^7 + t^{11} + \lambda t^{16}),\ \omega\lambda \sim \lambda, \omega^4 = 1, \\
Z^3_\lambda: & (t^6, t^7 + t^{16} + \lambda t^{17}),\ \omega\lambda \sim \lambda, \omega^9 = 1, \\
Z^4: & (t^6, t^7 + t^{17}), \\
Z^5: & (t^6, t^7 + t^{23}), \\
Z^6: & (t^6, t^7).
\end{array}
$$

4 Symplectic invariants of mappings

4.1 Symplectic equivalence.

Let $\omega = \sum_{i=1}^n dp_i \wedge dx_i$ be the standard symplectic form on $\mathbf{K}^{2n} = T^*\mathbf{K}^n$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Mappings are assumed to be real analytic or C^∞ for $\mathbf{K} = \mathbf{R}$ and complex analytic for $\mathbf{K} = \mathbf{C}$. Multi-germs $f : (\mathbf{K}^m, S) \to (\mathbf{K}^{2n}, 0)$ and $f' : (\mathbf{K}^m, S') \to (\mathbf{K}^{2n}, 0)$ to the symplectic space are called *symplectomorphic* (resp. *diffeomorphic*, *homeomorphic*) if the diagram

$$
\begin{array}{ccc}
(\mathbf{K}^m, S) & \xrightarrow{f} & (\mathbf{K}^{2n}, 0) \\
\sigma \downarrow & & \downarrow \tau \\
(\mathbf{K}^m, S') & \xrightarrow{f'} & (\mathbf{K}^{2n}, 0)
\end{array}
$$

is commutative for some diffeomorphism-germ σ and some symplectomorphism-germ τ, $\tau^*\omega = \omega$ (resp. for some diffeomorphism-germs σ, τ, for some homeomorphism-germs σ, τ). Here S, S' are finite sets.

For a map-germ $f : (\mathbf{K}^m, S) \to (\mathbf{K}^{2n}, 0)$, the diffeomorphism class of the pull-back form $f^*\omega$ on $(\mathbf{K}^m, S)$ of the symplectic form ω is an obvious symplectic invariant of f: If f and f' are symplectomorphic, then $f^*\omega$ and $f'^*\omega$ are diffeomorphic, that is, for a diffeomorphism $\sigma : (\mathbf{K}^m, S) \to (\mathbf{K}^m, S')$, we have $\sigma^*(f'^*\omega) = f^*\omega$. We call $f^*\omega$ the *geometric restriction* of ω by f. In this connection, we mention a theorem which contains the classical Darboux theorem as the special case $m = 0$:

Theorem 4.1 (Darboux-Givental [6]) *Two immersion-mono-germs $f, f' : (\mathbf{K}^m, 0) \to (\mathbf{K}^{2n}, 0)$ are symplectomorphic if and only if the geometric restrictions $f^*\omega$ and $f'^*\omega'$ are diffeomorphic.*

Thus in the *non-singular case* (the case of immersion-mono-germs), the classification problem is reduced to that of the geometric restrictions of the symplectic form to the sources. Note that the pull-backs of symplectic forms are not arbitrary. To explain this, recall the standard notions: A submanifold M in the symplectic space $(\mathbf{K}^{2n}, \omega)$ is called coisotropic (resp. isotropic, symplectic) if the skew-orthogonal in $\mathbf{K}^{2n}$ to each tangent space T_pM, $p \in M$, to M contains

T_pM (resp. the geometric restriction $\omega|_M$ is zero, $\omega|_M$ is symplectic). By the classical Darboux theorem, for a coisotropic submanifold, the local diffeomorphism class of the geometric restriction $\omega|_M$ is determined by just the dimension of M. Moreover, we know that a non-singular hypersurface is coisotropic. Then we have

Corollary 4.2 *All non-singular hypersurface-germs in* $\mathbf{K}^{2n}$ *are symplectomorphic. All coisotropic (resp. isotropic, symplectic) submanifold-germs of fixed dimension in* $\mathbf{K}^{2n}$ *are symplectomorphic.*

Note that all immersion-germs on a fixed dimensional source are diffeomorphic in our sense.

In the *singular case*, however, even if f and f' are diffeomorphic and $f^*\omega$ and $f'^*\omega$ are diffeomorphic, f and f' are not necessarily symplectomorphic. Therefore the symplectic classification is very different from the differential classification.

A mapping f is called *isotropic* if $f^*\omega = 0$, that is, if $\sum_{i=1}^n d(p_i \circ f) \wedge d(x_i \circ f) = 0$. If $m = 1$, then any germ $f : (\mathbf{K}, S) \to (\mathbf{K}^{2n}, 0)$ is isotropic. Moreover if $f : \mathbf{K}^n \to \mathbf{K}^{2n}, m = n$, then we often call isotropic f *Lagrangian*.

For the class of Lagrangian map-germs $f : (\mathbf{K}^n, 0) \to (\mathbf{K}^{2n}, 0)$, the basic theory is established by Givental [26].

Let $N \subset (\mathbf{K}^{2n}, 0)$ be a germ of analytic variety. We assume the regular locus of N is dense in N. Consider de Rham complex (Λ^*_{2n}, d), the algebra of germs of differential forms on $(\mathbf{K}^{2n}, 0)$ and the exterior differential $d : \Lambda^*_{2n} \to \Lambda^*_{2n}$. Then de Rham complex $(\Lambda^*(N), d)$ for N is defined as the quotient cochain complex of (Λ^*_{2n}, d) by the differential graded ideal $I^*(N)$ consisting of differential forms vanishing on the regular locus of N and the cohomology algebra $H^*(N) = H^*(\Lambda^*_{2n}, d)$ from the cochain complex $(\Lambda^*(N), d)$.

We say (N, ω) is *Lagrangian* if $\dim N = n$ and the restriction of a symplectic form ω to the regular locus of N vanishes. If (N, ω) is Lagrangian and $\omega = d\alpha$, then we have the well-defined cohomology class $[\alpha]$ in $H^1(N)$, which is called the *characteristic class* of (N, ω).

We call N *reduced* if it is not a product of an analytic set and a non-singular manifold of positive dimension. Then we have:

Theorem 4.3 ([26]) *Let* (N, ω) *be a reduced Lagrangian variety for a symplectic form* $\omega = d\alpha$ *on* $(\mathbf{K}^{2n}, 0)$*. Then any Lagrangian variety* (N, ω') *is symplectomorphic to* (N, ω)*, provided the symplectic form* $\omega' = d\alpha'$ *is sufficiently near* ω *and* $[\alpha'] = [\alpha] \in H^1(N)$*.*

In general (N, ω) and (N', ω') are called *symplectomorphic* if there exists a diffeomorphism-germ $T : (\mathbf{K}^{2n}, 0) \to (\mathbf{K}^{2n}, 0)$ satisfying $T(N) = N'$ and $T^*\omega' = \omega$.

Moreover Givental ([26]) shows that, if N is quasi-homogeneous (for a positive weight), then de Rham complex $(\Lambda^*(N), d)$ is acyclic (see also [17]). Therefore we have

Theorem 4.4 ([26]) *Suppose* $N \subset (\mathbf{K}^{2n}, 0)$ *is reduced and quasi-homogeneous. Then any Lagrangian varieties* (N, ω) *and* (N, ω') *are symplectomorphic, provided* $\mathbf{K} = \mathbf{C}$*.*

Suppose, in the parametric form, two map-germs $f, f' : (\mathbf{K}^m, 0) \to (\mathbf{K}^{2n}, 0)$ are diffeomorphic by (σ, τ). If f, f' are symplectomorphic for a fixed symplectic form ω, then $(f(\mathbf{K}^n), \omega)$ and $(f(\mathbf{K}^n), \tau^*\omega)$ are symplectomorphic. Moreover, under the condition that f is a normalization of the image, if $(f(\mathbf{K}^n), \omega)$ and $(f(\mathbf{K}^n), \tau^*\omega)$ are symplectomorphic, then f and f' are symplectomorphic (cf. [37]).

Corollary 4.5 *Suppose two isotropic map-germs $f, f' : (\mathbf{C}^n, 0) \to (\mathbf{C}^{2n}, 0)$ are diffeomorphic. Assume that f is a normalization of the image, which is reduced, and f, f' are quasi-homogeneous for the same weight. Then f and f' are symplectomorphic.*

Example 4.6 ([33]) Let $f : (\mathbf{K}^2, 0) \to (\mathbf{K}^4, 0)$ be isotropic. Suppose f is diffeomorphic to

$$f_{\mathrm{ou}}(t, u) := \left(t^2, u, ut, \frac{2}{3}t^3\right) = (x_1, x_2, p_1, p_2).$$

Then f is symplectomorphic to f_{ou} (Whitney's open umbrella). Moreover for any n there exists a class of open umbrellas characterised by the symplectic structural stability, and for them the Darboux-type theorem holds.

Note that the Darboux-type theorem follows from Givental's theory (Corollary 4.5) directly in the case $\mathbf{K} = \mathbf{C}$. The method is applied also to the case $\mathbf{K} = \mathbf{R}$.

Another generalization of the Darboux-Givental theorem ([6]) to the singular case is given by the following result:

Theorem 4.7 ([18]) *For any $N, N' \subset \mathbf{K}^{2n}$ quasi-homogeneous, (N, ω) and (N', ω') are symplectomorphic if and only if the algebraic restrictions $[\omega]_N$ and $[\omega']_{N'}$ are diffeomorphic.*

The algebraic restriction $[\omega]_N$ is defined as the residue class of ω modulo the differential ideal $J^*(N) \subset \Lambda^*_{2n}$ generated by functions vanishing on N. Note that $J^*(N) \subset I^*(N)$. We set $\Lambda^*_{\mathrm{alg}}(N) = \Lambda^*_{2n}/J^*(N)$ and $H^*_{\mathrm{alg}}(N) = H^*(\Lambda^*_{\mathrm{alg}}(N), d)$. Therefore $[\omega]_N \in H^2_{\mathrm{alg}}(N)$. Note that there exists the canonical surjection $\pi : \Lambda^*_{\mathrm{alg}}(N) \to \Lambda^*(N)$ of cochain complexes.

For the classification of curves in a symplectic space $\mathbf{K}^{2n}$ ($m = 1, n \geq 2$), Arnold initiated the investigation on the difference between diffeomorphism and symplectic classifications ([3]). Then Kolgushkin [48] has completed the symplectic classification of simple multi-germs $(\mathbf{C}, S) \to (\mathbf{C}^{2n}, 0)$. Moreover Domitrz [15] has given several results on symplectic classification of multi-germs of curves by the method of algebraic restrictions.

Restricting ourselves to the case $m = n = 1$, namely to planar mono-curves $(\mathbf{K}, 0) \to (\mathbf{K}^2, 0)$, we have given both symplectic and differential exact classifications of differentially simple and uni-modal planar curve singularities, and clarified the difference between the differential and symplectic classifications ([36][38][40]). In our formulation, we do not fix diffeomorphism types but fix homeomorphism types of planar curve singularities. Actually we fix Puiseux characteristics and then we have symplectic classification results in a unified manner (§4.4).

4.2 Basic invariants for classification.

For the exact classification problem of singularities, the notion of *codimension* is the most basic one to measure the complexity or degeneracy of singularities. For instance, the classification of a class of singularities of mappings proceeds from small codimension to large. In general, for a map-germ $f : (\mathbf{K}^n, S) \to (\mathbf{K}^p, 0)$, the $\mathcal{A}_e$-codimension of f is defined by

$$\mathcal{A}_e\text{-cod}(f) = \dim_{\mathbf{K}} V_f / [f_*(V_S) + (V_p) \circ f],$$

the dimension of the quotient of the infinitesimal deformations of f by those induced from right-left equivalences [56][74]. We often write $\mathrm{cod}(f) = \mathcal{A}_e\text{-cod}(f)$ briefly. The codimension

$\mathcal{A}_e$-cod(f) is finite if and only if f is finitely $\mathcal{A}$-determined. Moreover the codimension is estimated by other geometric invariants such as 0-stable invariants in terms of "disentanglement" ([45][61][62]). For instance, the $\mathcal{A}_e$-codimension of an $\mathcal{A}$-finite germ $f : (\mathbf{C}, S) \to (\mathbf{C}^2, 0)$ is estimated as

$$\mathcal{A}_e\text{-cod}(f) \leq \delta(f) - r + 1, \quad \cdots\cdots (*)$$

where $r = \#S$ and $\delta(f) = \dim_{\mathbf{K}} \mathcal{O}_S/f^*\mathcal{O}_2$, the number of double points of f. Here $\mathcal{O}_S$ (resp. $\mathcal{O}_n$) denotes the $\mathbf{K}$-algebra of C^∞ or holomorphic function-germs on $(\mathbf{K}, S)$ (resp. $(\mathbf{K}^n, 0)$). Moreover the equality holds if and only if f is quasi-homogeneous ([63]). See also [14][29].

In [36], we introduced the notion of symplectic codimension sp-cod(f) for a germ $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ of planar branch under the symplectomorphism equivalence and showed that sp-cod(f) coincides with $\delta(f)$ (See Theorem 3.20). The result is easily generalized to multi-germs and in fact we have

$$\text{sp-cod}(f) = \delta(f) - r + 1,$$

for a multi-germ $f : (\mathbf{C}, S) \to (\mathbf{C}^2, 0)$. Therefore the inequality (*) is rewritten as

$$\mathcal{A}_e\text{-cod}(f) \leq \text{sp-cod}(f),$$

and the difference sp-cod$(f) - \mathcal{A}_e$-cod(f) represents the difference of symplectomorphism and diffeomorphism classifications of planar curves, as well as the grade of non-homogeneity $\mu - \tau$, the difference of Milnor and Tjuria numbers.

Let $f : (\mathbf{K}^n, S) \to (\mathbf{K}^{2n}, 0)$ be a multi-germ of isotropic mapping (or Lagrangian immersion with singularities). Then we set

$$\text{sp-cod}(f) = \dim_{\mathbf{K}} VI_f/[f_*(V_S) + (VH_{2n}) \circ f],$$

and call it the *symplectic codimension* (or the *symplectic-isotropic codimension*) of $f : (\mathbf{K}^n, S) \to (\mathbf{K}^{2n}, 0)$. Here VI_f is the space of infinitesimal *isotropic* deformations of f:

$$VI_f = \{v : (\mathbf{K}^n, S) \to T\mathbf{K}^{2n} \mid v^*\dot{\omega} = 0, \pi \circ v = f\},$$

for the natural symplectic lifting $\dot{\omega}$ of ω on $T\mathbf{K}^{2n}$, $\dot{\omega} = \sum_{i=1}^n d\varphi_i \wedge dx_i + dp_i \wedge d\xi_i$ for the coordinates $(x, p; \xi, \varphi)$ of $T\mathbf{K}^{2n}$, and $\pi : T\mathbf{K}^{2n} \to \mathbf{K}^{2n}$ is the bundle projection. Moreover we denote by VH_{2n} the space of holomorphic Hamiltonian vector fields over $(\mathbf{K}^{2n}, 0)$, and by V_S the space of holomorphic vector fields over $(\mathbf{K}^n, S)$. The symplectic codimension sp-cod(f) is regarded as the minimal number of parameters for "the symplectically versal isotropic unfolding" of f, if f is of corank one.

4.3 New symplectic invariants in higher dimensions.

For $n \geq 2$, there is no such simple relation between the $\mathcal{A}_e$-codimension and the symplectic codimension, because the symplectic-isotropic codimension indicates the codimension in a subspace of map-germs of an orbit of a subgroup of $\mathcal{A}$. To measure the difference between symplectomorphism equivalence and diffeomorphism equivalence for isotropic map-germs we introduce another symplectic invariant diff-cod(f) = diff-cod$_I(f)$, the differential-isotropic codimension instead of the symplectic-isotropic codimension sp-cod(f) = sp-cod$_I(f)$ of f. Then we set

$$\text{sd}(f) = \text{sp-cod}(f) - \text{diff-cod}(f).$$

We give an algebraic description of sd(f) and show that both sp-cod(f) and diff-cod(f) are $\mathcal{A}$-invariants, hence so is sd(f). Moreover, we show an example of quasi-homogeneous isotropic map-germs $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^4, 0)$ with sd$(f) > 0$.

In this subsection, we consider new geometric symplectic invariants of isotropic mappings for $\mathbf{K} = \mathbf{C}$. If a multi-germ of isotropic mapping $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ is of corank ≤ 1, and sp-cod$(f) < \infty$, then f can be perturbed to a symplectically stable isotropic mapping $\tilde{f}$ whose singularities consist of open umbrellas and transverse self-intersection points (double points). See §5.1. The number of transverse self-intersection points of the perturbation $\tilde{f}$ does not depend on the perturbation. It is called *the number of isotropic double points* of f and denoted by $\delta_I = \delta_I(f)$. Note that, for $n = 1$, $\delta_I(f) = \delta(f)$.

We give a relation between the two symplectic invariants sp-cod$(f) =$ sp-cod$_I(f)$ and $\delta_I(f)$ for isotropic map-germs $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$. Moreover, we introduce another invariant $u_I(f)$, the number of open umbrellas, for isotropic map-germs $f : (\mathbf{C}^2, S) \to (\mathbf{C}^4, 0)$ and provide a relation of $\delta_I(f)$ and $u_I(f)$ with the Segre number of the image variety of f using Gaffney's result [24].

Let $f : (\mathbf{K}, S) \to (\mathbf{K}^2, 0)$ be a multi-germ of planar curve. We assume that the base point set S consists of r points.

Theorem 4.8 ([41]) *Let $f : (\mathbf{K}, S) \to (\mathbf{K}^2, 0)$ be an $\mathcal{A}$-finite planar curve with r components. Then* sp-cod(f) *and* $\delta(f)$ *are both finite and we have*

$$\text{sp-cod}(f) = \delta(f) - r + 1,$$

where $r = \#S$ and $\delta(f) = \dim_{\mathbf{C}} \mathcal{O}_S / f^*\mathcal{O}_2$, the number of double points of a stable perturbation of f.

Remark 4.9 If we set

$$\mathcal{G}_f = \{h \in \mathcal{O}_S \mid dh \in \langle d(x \circ f), d(p \circ f)\rangle_{f^*\mathcal{O}_2}\},$$

then we have

$$\mathcal{A}_e\text{-cod}(f) = \dim_{\mathbf{K}} \frac{\mathcal{O}_S}{\mathcal{G}_f}.$$

Moreover

$$\text{sd}(f) = \dim_{\mathbf{K}} \frac{\mathcal{G}_f}{f^*\mathcal{O}_2} - r + 1.$$

Note that $\mathcal{O}_S, \mathcal{R}_f$ and $\mathcal{G}_f$ are defined via the exterior derivative and any locally constant functions belong to them, which is not the case for $f^*\mathcal{O}_2$.

In general, for each homeomorphism class of planar curves, the symplectic moduli space is mapped canonically onto the differential moduli space. The dimension of the fiber over a diffeomorphism class $[f]$ equals sd(f). It is known that sd$(f) = \mu(f) - \tau(f)$, where $\mu(f) = 2\delta(f)$ is the Milnor number of f and $\tau(f)$ is the Tyurina number of f ([72][50][17]). Let $s(f)$ be the symplectic modality, that is, the number of parameters in the symplectic normal form of f. Moreover let $c(f)$ be the codimension of the locus in the parameter space corresponding to germs diffeomorphic to f. Then $s(f) - c(f) =$ sd(f). Thus we have the formula, even for multi-germs, for the Tyurina number (by means of Varchenko-Lando's formula): $\tau(f) = 2\delta(f) + c(f) - s(f)$. See [38][40] for details.

In particular, for the planar branch, differential classifications of planar curves are obtained by [22], [79], [10], [51], [53], [28], and so on.

4.4 Puiseux characteristics

Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$, $f(t) = (x(t), y(t))$, be a germ of a holomorphic parametric planar curve. Let m be the minimum of the order of $x(t)$ and that of $y(t)$ at $t = 0$. Then, using a re-parametrization and the symplectomorphism $(x, y) \mapsto (y, -x)$ if necessary, we see that f is symplectomorphic to $(t^m, \sum_{k=m}^{\infty} a_k t^k)$. Suppose $m \geq 2$, that is, f is not an immersion.

Set $\beta_1 = \min\{k \mid a_k \neq 0, m \nmid k\}$ and let e_1 be the greatest common divisor of m and β_1. Inductively set $\beta_j = \min\{k \mid a_k \neq 0, e_{j-1} \nmid k\}$, and let e_j be the greatest common divisor of β_j and e_{j-1}, $j \geq 2$. Then $e_{q-1} > 1$, $e_q = 1$ for a sufficiently large q, and we call $(m = \beta_0, \beta_1, \beta_2, \dots, \beta_q)$ the *Puiseux characteristic* of f. The Puiseux characteristic is a basic diffeomorphism invariant, and it determines exactly the homeomorphism class of f ([79]). For example, setting $e_0 = m$, we have the number of double points $\delta(f)$ described by $\delta(f) = \frac{1}{2}\sum_{j=1}^{q}(\beta_j - 1)(e_{j-1} - e_j)$ ([60][75]).

Then f is symplectomorphic to a germ of the form

$$(t^m, t^{\beta_1} + \sum_{k=\beta_1+1}^{\infty} b_k t^k).$$

As is stated already, in [40], we characterize simple and uni-modal singularities by means of their Puiseux characteristics using an infinitesimal method.

Let f be of Puiseux characteristic $(m, \beta_1, \dots, \beta_q)$. A monomial basis of $\mathcal{O}_1/f^*\mathcal{O}_2$ can be calculated by considering the *order semigroup*

$$S(f) = \{\mathrm{ord}(k) \mid k \in f^*\mathcal{O}_2\} \subseteq \mathbf{N}.$$

In fact $\{t^r \mid r \in \mathbf{N} \setminus S(f),\ r > 0\}$ forms a monomial basis of $\mathcal{O}_1/f^*\mathcal{O}_2$.

Then we have the following general result ([40]) on symplectic classification via the order semigroup:

Theorem 4.10 *Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$, $f(t) = (t^m,\ t^{\beta_1} + \sum_{k=\beta_1+1}^{\infty} b_k t^k)$, be a germ of Puiseux characteristic $(m, \beta_1, \dots, \beta_q)$. Let $r_1 + m, \dots, r_s + m (r_1 < \cdots < r_s)$ be all elements of $\mathbf{N} \setminus S(f)$ with $r_j > \beta_1 (1 \leq j \leq s)$. Then f is symplectomorphic to*

$$f_\lambda(t) = (t^m,\ t^{\beta_1} + \lambda_1 t^{r_1} + \lambda_2 t^{r_2} + \cdots + \lambda_s t^{r_s})$$

for some $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathbf{C}^s$.

A family $f_\lambda(t) (\lambda \in \mathbf{C}^s)$, is called a *symplectic normal form* for the Puiseux characteristic $(m, \beta_1, \dots, \beta_q)$ if any planar curve-germ of Puiseux characteristic $(m, \beta_1, \dots, \beta_q)$ is symplectomorphic to $f_\lambda(t)$ for some $\lambda \in \mathbf{C}^s$. And those $\lambda \in \mathbf{C}^s$ for which f_λ is symplectomorphic to a given plane branch form a discrete subset of $\mathbf{C}^s$.

If there exists a symplectic normal form, then we have a surjective mapping of $\mathbf{C}^s$ into the space of symplectic moduli with discrete fibers.

Then we have the following results on symplectic normal forms

Proposition 4.11 *Under the same notation as in Theorem 4.10, we have the following:*

(1) *If the Puiseux characteristic is (m, β_1), then the family*

$$f_\lambda(t) = (t^m,\ t^{\beta_1} + \lambda_1 t^{r_1} + \cdots + \lambda_s t^{r_s}),$$

$\lambda = (\lambda_1, \dots, \lambda_s) \in \mathbf{C}^s$, *is a symplectomorphic normal form.*

(2) *If the Puiseux characteristic is* $(4, 6, 2\ell+5)$, *then* $s = \ell+1$ *and* $r_1 = 7, r_2 = 9, \dots, r_{\ell-1} = 2\ell+3, r_\ell = 2\ell+5, r_{\ell+1} = 2\ell+7$. *Within the family*

$$f_c(t) = (t^4, t^6 + c_1 t^7 + c_2 t^9 + \cdots + c_{\ell-1} t^{2\ell+3} + c_\ell t^{2\ell+5} + c_{\ell+1} t^{2\ell+7}),$$

the subfamily

$$f_\lambda(t) = (t^4, t^6 + \lambda_1 t^{2\ell+5} + \lambda_2 t^{2\ell+7}),$$

$\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2, \lambda_1 \neq 0$, *is a symplectic normal form.*

(3) *If the Puiseux characteristic is* $(4, 10, 2\ell+9)$, *then* $s = \ell+4$ *and* $r_1 = 11, r_2 = 13, r_3 = 15, \dots, r_{\ell-1} = 2\ell+7, r_\ell = 2\ell+9, r_{\ell+1} = 2\ell+11, r_{\ell+2} = 2\ell+13, r_{\ell+3} = 2\ell+17, r_{\ell+4} = 2\ell+21$. *Within the family*

$$\begin{aligned} f_c(t) \;=\; & (t^4, t^{10} + c_1 t^{11} + c_2 t^{13} + c_3 t^{15} + \cdots + c_{\ell-1} t^{2\ell+7} \\ & + c_\ell t^{2\ell+9} + c_{\ell+1} t^{2\ell+11} + c_{\ell+2} t^{2\ell+13} + c_{\ell+3} t^{2\ell+17} + c_{\ell+4} t^{2\ell+21}), \end{aligned}$$

the subfamily

$$f_\lambda(t) = (t^4, t^{10} + \lambda_1 t^{2\ell+9} + \lambda_2 t^{2\ell+11} + \lambda_3 t^{2\ell+13} + \lambda_4 t^{2\ell+17} + \lambda_5 t^{2\ell+21}),$$

$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in \mathbf{C}^5, \lambda_1 \neq 0$, *is a symplectic normal form.*

The above Proposition 4.11 implies the following exact list of normal forms under symplectomorphic equivalence:

Theorem 4.12 *A simple or uni-modal singularity* $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ *is symplectomorphic to a germ which belongs to one of the following families (called "symplectic normal forms"):*

$$\begin{array}{ll}
A_{2\ell}: & (t^2, t^{2\ell+1}), \\
E_{6\ell}: & (t^3, t^{3\ell+1} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1}), \\
E_{6\ell+2}: & (t^3, t^{3\ell+2} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1}), \\
W_{12}: & (t^4, t^5 + \lambda t^7), \\
W_{18}: & (t^4, t^7 + \lambda t^9 + \mu t^{13}), \\
W^{\#}_{1,2\ell-1}: & (t^4, t^6 + \lambda t^{2\ell+5} + \mu t^{2\ell+9}), \lambda \neq 0 (\ell = 1, 2, \dots), \\
N_{20}: & (t^5, t^6 + \lambda_1 t^8 + \lambda_2 t^9 + \lambda_3 t^{14}), \\
N_{24}: & (t^5, t^7 + \lambda_1 t^8 + \lambda_2 t^{11} + \lambda_3 t^{13} + \lambda_4 t^{18}), \\
N_{28}: & (t^5, t^8 + \lambda_1 t^9 + \lambda_2 t^{12} + \lambda_3 t^{14} + \lambda_4 t^{17} + \lambda_5 t^{22}), \\
W_{24}: & (t^4, t^9 + \lambda_1 t^{10} + \lambda_2 t^{11} + \lambda_3 t^{15} + \lambda_4 t^{19}), \\
W_{30}: & (t^4, t^{11} + \lambda_1 t^{13} + \lambda_2 t^{14} + \lambda_3 t^{17} + \lambda_4 t^{21} + \lambda_5 t^{25}), \\
W^{\#}_{2,2\ell-1}: & (t^4, t^{10} + \lambda_1 t^{2\ell+9} + \lambda_2 t^{2\ell+11} + \lambda_3 t^{2\ell+13} + \lambda_4 t^{2\ell+17} + \lambda_5 t^{2\ell+21}), \\
 & \lambda_1 \neq 0 \ (\ell = 1, 2, \dots).
\end{array}$$

Considering the symplectomorphism equivalence, we have given the classification of uni-modal planar curve-germs and we observe that there exists the difference (or "quotient") between differential and symplectic classifications:

Theorem 4.13 ([38]) *For planar curves* $f : (\mathbf{K}, 0) \to (\mathbf{K}^2, 0)$, *symplectic moduli appear from* $\mathcal{A}_e$*-codim* $= 5$ *on* (E_{12})*; while differential moduli appear from* $\mathcal{A}_e$*-codim* $= 8$ *on* (N_{20}).

We can say that symplectic moduli appear *earlier* than differential moduli.

For a detailed symplectic classification of planar-mono-germs see [38][40].

5 Symplectic-isotropic codimension

Let κ be a germ of 2-form on $(\mathbf{K}^m, S)$, S being finite. Then we denote by $\mathcal{O}^{\kappa}_{m,2n}$ the set of map-germs $f : (\mathbf{K}^m, S) \to (\mathbf{K}^{2n}, 0)$ with the geometric restriction $f^*\omega = \kappa$.

A deformation f_t of $f_0 = f \in \mathcal{O}^{\kappa}_{m,2n}$ is called *isotropic* if $f_t \in \mathcal{O}^{\kappa}_{m,2n}$, i.e. $f_t^*\omega = f^*\omega$ $(= \kappa)$. Then we set

$$\text{sp-cod}(f) = \dim_{\mathbf{C}} VI_f/[f_*(V_{S,\kappa}) + (VH_{2n}) \circ f],$$

and call it the *symplectic codimension* (or the *symplectic-isotropic codimension*) of $f : (\mathbf{C}^m, S) \to (\mathbf{C}^{2n}, 0)$, where $VI_f = \{v : (\mathbf{K}^n, S) \to T\mathbf{K}^{2n} \mid v^*\dot{\omega} = 0, \pi \circ v = f\}$. Here we set

$$V_{S,\kappa} = \{\xi \in V_S \mid L_\xi \kappa = 0\},$$

the space of vector fields which leave κ invariant. Note that $V_{S,\kappa} = V_S$ if $\kappa = 0$.

Example 5.1 A germ f is coisotropic if and only if $f^*\omega = g^*\eta$ for some $g : (\mathbf{K}^m, S) \to (\mathbf{K}^{2k}, 0)$. A coisotropic map-germ $f : (\mathbf{K}^m, S) \to (\mathbf{K}^{2n}, 0)$ is a *coisotropic map-germ with regular reduction* if g can be taken to be a submersion. Then $\kappa = g^*\eta$ is of constant rank and the *coisotropic deformation* of f is investigated by studying the space $\mathcal{O}^{\kappa}_{m,2n}$. The *characteristic foliation* $\mathcal{F}_f$ is generated by the kernel field defined by $f^*\omega = g^*\eta$. Then any vector field in $V_{S,\kappa}$ preserves $\mathcal{F}_f$.

Now, for an isotropic $f : (\mathbf{K}^n, S) \to (\mathbf{K}^{2n}, 0)$, we define

$$\text{diff-cod}(f) = \dim_{\mathbf{K}} \frac{VI_f}{f_*(V_S) + (V_{2n} \circ f) \cap VI_f},$$

while

$$\text{sp-cod}(f) = \dim_{\mathbf{K}} \frac{VI_f}{f_*(V_S) + VH_{2n} \circ f},$$

and

$$\mathcal{A}_e\text{-cod}(f) = \dim_{\mathbf{K}} \frac{V_f}{f_*(V_S) + V_{2n} \circ f}.$$

Moreover we set

$$\text{sd}(f) = \text{sp-cod}(f) - \text{diff-cod}(f) \quad (\geq 0),$$

the *symplectic defect* or *symplectic multiplicity* of f.

Note that, for $n = 1$, we have $VI_f = V_f$: any infinitesimal deformation is isotropic.

We define subspaces $\mathcal{O}_S \supseteq \mathcal{R}_f \supseteq \mathcal{G}_f \supseteq f^*\mathcal{O}_{2n}$ by

$$\begin{aligned} \mathcal{R}_f &= \{e \in \mathcal{O}_S \mid de \in \mathcal{O}_S \cdot f^*(\Lambda^1_{2n})\}, \\ \mathcal{G}_f &= \{e \in \mathcal{O}_S \mid de \in f^*(\Lambda^1_{2n})\}, \end{aligned}$$

where de is the exterior differential of the function e, Λ^1_{2n} is the space of 1-forms on $(\mathbf{K}^{2n}, 0)$. Then we have algebraic formulae for symplectic invariants.

Theorem 5.2 ([41]) *Let $n \geq 2$. Let $f : (\mathbf{K}^n, S) \to (\mathbf{K}^{2n}, 0)$ be isotropic. If f is a normalization of its image and the codimension of non-immersive locus $\mathrm{cod}_{\mathbf{C}}\Sigma(f) \geq 2$, then*

$$\begin{aligned}\text{sp-cod}(f) &= \dim_{\mathbf{K}} \frac{\mathcal{R}_f}{f^*\mathcal{O}_{2n}} - r + 1,\\ \text{diff-cod}(f) &= \dim_{\mathbf{K}} \frac{\mathcal{R}_f}{\mathcal{G}_f},\\ \text{sd}(f) &= \dim_{\mathbf{K}} \frac{\mathcal{G}_f}{f^*\mathcal{O}_{2n}} - r + 1,\end{aligned}$$

where $r = \#S$.

Remark 5.3 The mono-germ case of Theorem 5.2 is proved in [37].

Since $\mathcal{R}_f, \mathcal{G}_f$ are defined independently of the symplectic structure, we have:

Corollary 5.4 *For isotropic map-germs $f : (\mathbf{K}^n, S) \to (\mathbf{K}^{2n}, 0)$, sp-cod$(f)$ and diff-cod(f) are differential invariants. Namely, if f, f' are diffeomorphic, then sp-cod(f) = sp-cod(f') and diff-cod(f) = diff-cod(f').*

5.1 Symplectic codimension and double points

In what follows we suppose $\mathbf{K} = \mathbf{C}$.

We recall the Artin-Nagata formula (Mumford's formula) [7]: For an $\mathcal{A}$-finite map-germ $f : X = (\mathbf{C}^n, S) \to Y = (\mathbf{C}^{2n}, 0)$, *the number of double points* is given by $\delta(f) = \frac{1}{2}\dim_{\mathbf{C}} \epsilon$, where $\epsilon = \mathrm{Ker}(\mathcal{O}_{X\times_Y X} \to \mathcal{O}_X)$ is the kernel of the induced morphism from the diagonal map $X \to X \times_Y X$ to the fiber product of f. For a map-germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^{2n}, 0)$, we have as in [24]:

$$\delta(f) = \dim_{\mathbf{C}} \frac{\langle x_1 - \tilde{x}_1, \dots, x_n - \tilde{x}_n\rangle_{\mathcal{O}_{2n}}}{\langle f_1(x) - f_1(\tilde{x}), \dots, f_{2n}(x) - f_{2n}(\tilde{x})\rangle_{\mathcal{O}_{2n}}}.$$

Also we have $\delta(f) = \frac{1}{2}\dim_{\mathbf{C}}\ \mathcal{O}_X \otimes_{f^*\mathcal{O}_Y} (\mathcal{O}_X/f^*\mathcal{O}_Y)$. See also [47].

For $n \geq 2$, the inequality $\mathcal{A}_e$-cod$(f) \leq \delta(f) - r + 1$ does not hold in general.

Example 5.5 ([7]): Let $f : (\mathbf{C}^2, S) \to (\mathbf{C}^4, 0)$ be an immersion whose image consists of three planes intersecting transversely to each other at $0 \in \mathbf{C}^4$. Then $\mathcal{A}_e$-cod$(f) = 2$, $\delta(f) = 3$, $\#S = r = 3$,

Originally, the above Mumford example is for $\delta(f) \neq \dim_{\mathbf{C}} \mathcal{O}_n/f^*(\mathcal{O}_{2n})$. In fact, $\dim_{\mathbf{C}} \mathcal{O}_n/f^*(\mathcal{O}_{2n}) =$ 4 for that example.

On the other hand, Gaffney [24] showed the following: For an $\mathcal{A}$-finite map-germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^{2n}, 0)$,

$$\begin{aligned}\delta(f) &= \frac{1}{2}[\ \mathrm{Segre}_{2n}\langle f_1(x) - f_1(\tilde{x}), \dots, f_{2n}(x) - f_{2n}(\tilde{x})\rangle_{\mathcal{O}_{2n}}\\ &\qquad -\mathrm{Whitney}(\pi \circ f : (\mathbf{C}^n, 0) \to (\mathbf{C}^{2n-1}, 0))\ ],\end{aligned}$$

half of [*the Segre number of the ideal defining the double points in* $\mathcal{O}_{2n} = \mathcal{O}_{\mathbf{C}^n \times \mathbf{C}^n}$ *minus the number of Whitney umbrellas of a generic projection* $\pi : \mathbf{C}^n \to \mathbf{C}^{2n-1}$ *composed with* f].

Now we consider symplectic-isotropic singularities: If an isotropic map-germ $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ is of corank one and is stable among isotropic perturbations under symplectomorphisms, then f is symplectomorphic to an *open umbrella*, which can be explicitly represented as a polynomial normal form, and projects to the Whitney umbrella (Theorem 6.3, [33]). Note that, though the result was stated in the real C^∞ case, even in the holomorphic and local case, similar results follow.

If an isotropic map-germ $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ is of corank ≤ 1 and sp-cod$(f) < \infty$, then f can be perturbed to a symplectically stable isotropic mapping $\tilde{f}$ whose singularities consist of "open umbrellas"(singularities of codimension 2) and transverse self-intersection points (double points). The number of transverse self-intersection points of the perturbation $\tilde{f}$ does not depend on symplectically stable perturbations. It is called *the number of isotropic double points* of f and denoted by $\delta_I = \delta_I(f)$.

We set

$$B_\varepsilon = \{x \in \mathbf{C}^{2n} \mid |x| < \varepsilon\}.$$

Then we have

Proposition 5.6 *Let* $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ *be a multi-germ of an isotropic mapping of corank* ≤ 1 *and* sp-cod$(f) < \infty$. *Then a representative* $f : f^{-1}(B_\varepsilon) \to \mathbf{C}^{2n}$ *can be perturbed to a symplectically stable isotropic mapping* $\tilde{f} : \tilde{f}^{-1}(B_\varepsilon) \to \mathbf{C}^{2n}$ *whose singularities consist of open umbrellas and transverse double points. The number of double points is independent of the perturbation, provided* $\varepsilon > 0$ *is sufficiently small.*

Example 5.7 Let $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$, S be a set of transverse double points, $\#S = r = 2$. Then $\dim_{\mathbf{C}} \mathcal{R}_f / f^*\mathcal{O}_{2n} = 1$.

Example 5.8 For an open umbrella $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^4, 0)$ (Theorem 6.3), f is of corank one and its singular locus is of codimension 2. Moreover we have $\mathcal{A}_e$-cod$(f) = 1$, $\delta(f) = 1$. The open umbrella is symplectically stable under isotropic deformations. Therefore we have diff-cod$(f) =$ sp-cod$(f) = 0$ and $\delta_I(f) = 0$.

We have the following inequalities.

Theorem 5.9 ([41]) *For an isotropic map-germ* $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ *of corank one and with* sp-cod$(f) < \infty$, *we have*

$$\dim_{\mathbf{C}} \frac{\mathcal{R}_f}{f^*\mathcal{O}_{2n}} \;\geq\; \delta_I(f).$$

Therefore we have

$$\text{diff-cod}(f) \;\leq\; \text{sp-cod}(f) \;\geq\; \delta_I(f) - r + 1.$$

Again we remark that, in the inequality sp-cod$(f) \geq \delta_I(f) - r + 1$, equality holds in the case $n = 1$, but not in general for $n \geq 2$. Therefore, setting

$$i(f) = \text{sp-cod}(f) - (\delta_I(f) - r + 1),$$

it is natural to ask for the interpretation of $i(f)$ in symplectic terms. We remark that the numbers $\delta(f) - r + 1$ and $\delta_I(f) - r + 1$ have a clear topological meaning.

Proposition 5.10 *For $\mathcal{A}$-finite $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$, the disentanglement (the image of a stable perturbation) is homotopically equivalent to the bouquet of $\delta(f) - r + 1$ circles. For an isotropic $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ of corank ≤ 1 with* sp-cod$(f) < \infty$ *the isotropic disentanglement (the image of an isotropically stable perturbation) is homotopically equivalent to the bouquet of $\delta_I(f) - r + 1$ circles.*

Remark 5.11 Any open umbrella $V \subset (\mathbf{C}^{2n}, 0)$ has local trivial topology: $(\mathbf{C}^{2n}, V, 0)$ is homeomorphic to $(\mathbf{C}^{2n}, \mathbf{C}^n, 0)$.

5.2 Symplectic invariants of surfaces

First we observe

Lemma 5.12 *For an isotropic map-germ $f : (\mathbf{C}^2, S) \to (\mathbf{C}^4, 0)$ of corank ≤ 1,* sp-cod$(f) < \infty$ *if and only if $\mathcal{A}_e$-*cod$(f) < \infty$.

Remark 5.13 The similar result to Lemma 5.12 for $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ with $n \geq 3$ never hold. In fact the three dimensional open umbrella $f : (\mathbf{C}^3, 0) \to (\mathbf{C}^6, 0)$ has 1-dimensional singular locus, therefore $\mathcal{A}_e$-cod$(f) = \infty$, while sp-cod$(f) = 0$ ([33]). Note that map-germs $(\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ with $\mathcal{A}_e$-cod$(f) < \infty$ must be immersive off S.

For an isotropic $f : (\mathbf{C}^2, S) \to (\mathbf{C}^4, 0)$ of corank ≤ 1, we can define "the number of open umbrellas" $u_I = u_I(f)$, in addition to $\delta_I = \delta_I(f)$. Then the sum of the number of open umbrellas $u_I(f)$ and the number of isotropic double points $\delta_I(f)$ is equal to the number of double points $\delta(f)$:

$$\delta_I(f) + u_I(f) = \delta(f),$$

because $\delta = 1$ for each open umbrella. Moreover, by the isotropic nature of f, we have

Lemma 5.14 *Let $f : (\mathbf{C}^2, S) \to (\mathbf{C}^4, 0)$ be an isotropic map-germ of corank ≤ 1. Here* $\mathrm{corank}(f) = \max_{s \in S} \mathrm{corank}_s(f)$. *Then,*

$$u_I(f) = \mathrm{Whitney}(\pi \circ f),$$

the number of Whitney umbrellas of a generic projection $\pi : \mathbf{C}^4 \to \mathbf{C}^3$ composed with f.

Therefore we have, by Gaffney's formula,

Proposition 5.15 *For an isotropic map-germ $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^4, 0)$ with* sp-cod$(f) < \infty$, *we have*

$$\mathrm{Segre}_4 = 2\delta_I + 3u_I.$$

Example 5.16 Consider the isotropic map-germ

$$f_{\mathrm{ou}} := (x_1, x_2, p_1, p_2) = \left(t^2,\ u,\ ut,\ \frac{2}{3}t^3\right) : (\mathbf{C}^2, 0) \to (\mathbf{C}^4, 0)$$

(see also Theorem 6.3 below). Then we have $\mathcal{R}_f = \mathcal{G}_f = f^*\mathcal{O}_4$. Moreover we have sp-cod$(f_{\mathrm{ou}}) = 0$, sd$(f_{\mathrm{ou}}) = 0$, $\delta_I = 0$, $u_I = 1$, $\delta = 1$, $\mathrm{Segre}_4 = 3$.

Example 5.17 (multiple open umbrella): The isotropic map-germ

$$f^{\pm}_{\text{mou}}(t,u) := (t^2,\ u,\ t^3 \pm u^2 t,\ \frac{4}{3}ut^3) : (\mathbf{C}^2, 0) \to (\mathbf{C}^4, 0),$$

has isolated singularity at 0 and is quasi-homogeneous for the weights $w(t) = 1$ and $w(u) = 2$. By Corollary 4.5, $\text{sd}(f^{\pm}_{\text{mou}}) = 0$. In fact, we have $\mathcal{R}_f \supsetneq \mathcal{G}_f = f^*\mathcal{O}_{2n}$ and $\text{sp-cod}(f^{\pm}_{\text{mou}}) = 1$. Moreover $f^{\pm}_{\text{mou}}$ is isotropically perturbed into two open umbrellas and one double point, and therefore $\delta_I = 1$, $u_I = 2$, $\delta = 3$, $\text{Segre}_4 = 8$.

Remark 5.18 An algebraic formula for the number u_I of open umbrellas is known ([35]): For $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^4, 0)$, we have

$$u_I = \dim_{\mathbf{C}} \frac{\mathcal{O}_2}{J_f},$$

where J_f is the ideal generated by the 2-minors of the Jacobi matrix of f.

6 Residual algebraic restrictions

6.1 Classification problem of symplectic forms over subsets

A real or complex symplectic manifold (M, ω) has the local model $(\mathbf{K}^{2n}, \omega)$, by Darboux theorem,

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

$\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$.

A diffeomorphism $\psi : (M_1, \omega_1) \to (M_2, \omega_2)$ between symplectic manifolds is called a *symplectomorphism* if $\psi^*\omega_2 = \omega_1$.

Let N_1 and N_2 be germs of subsets of (M_1, ω_1) and (M_2, ω_2) respectively. We call them *symplectomorphic* if $\psi(N_1) = N_2$ for a germ of symplectomorphism $\psi : (M_1, \omega_1) \to (M_2, \omega_2)$.

Let S be s smooth submanifold of a symplectic manifold (M, ω). Then we consider the geometric restriction $\omega|_S := \omega|_{TS}$. By the following theorem of Darboux-Givental, the geometric restriction is a complete symplectic invariant.

Theorem 6.1 (Darboux-Givental) *Germs of submanifolds S_1, S_2 of symplectic manifolds (M_1, ω_1), (M_2, ω_2) are symplectomorphic if and only if the germs of geometric restrictions $\omega_1|_{TS_1}$ and $\omega_2|_{TS_2}$ are diffeomorphic, i.e. there is a diffeomorphism germ $\Psi : S_1 \to S_2$ such that $\Psi^*(\omega_2|_{TS_2}) = \omega_1|_{TS_1}$.*

Corollary 6.2 *All non-singular hypersurface-germ in $(\mathbf{K}^{2n}, \omega)$ are symplectomorphic. All coisotropic (resp. isotropic, symplectic) submanifold-germs of fixed dimension in $\mathbf{K}^{2n}$ are symplectomorphic.*

In the papers [17][18], the notion of algebraic restrictions of differential forms is introduced and established its basic properties. The spaces of algebraic restrictions of contact forms and symplectic forms are effectively applied to contact and symplectic classifications of singularities [82][15][16][20][21].

The difference of geometric restrictions and algebraic restrictions are compared with the following general situation: A "variety" Z in a manifold M is regarded as the image of a mapping (parametrization) $f : N \to M$, $f(N) = Z$, while Z is regarded as a zero-set of a mapping (a system of defining equations) $F : M \to \mathbf{R}^p, F^{-1}(0) = Z$. If f and F satisfy certain conditions respectively, then the space of geometric restrictions is described in terms of f and the space of algebraic restrictions is described in terms of F.

Of course it is a fundamental but a difficult problem to give a general method choosing f and F as above from an arbitrary subset $Z \subset M$. Nevertheless we give the general framework of the theory and provide several useful observations for general Z to be effective in concrete calculations of residual modules for important examples which are shown also in this paper.

A pair (f, ω) is called *isotropic* if $f^*\omega = 0$. Then f is called *isotropic* with respect to ω. If $m = 1$, then any pair (f, ω) is isotropic. Moreover if $f : \mathbf{R}^n \to \mathbf{R}^{2n}$ then we call (f, ω) *Lagrangian*.

In the case $m = n = 2$, we have

Theorem 6.3 ([33]) *Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^4, \omega)$ be isotropic. Suppose f is diffeomorphic to*

$$f_{\mathrm{ou}}(t, u) = (ut, t^2, \frac{2}{3}t^3, u) = (p_1, q_1, p_2, q_2).$$

Then, for any symplectic form ω, the pair (f, ω) is symplectomorphic to $(f_{\mathrm{ou}}, \omega_{\mathrm{st}})$. (Darboux-type theorem). *Moreover, for any n, there exists a class of open umbrellas, characterised by the symplectic structural stability, and for them, Darboux type theorem holds.*

We refer to a generalization of Darboux-Givental case to singular case.

Theorem 6.4 ([18]) *For any quasi-homogeneous $N, N' \subset \mathbf{R}^{2n}$ and for any symplectic forms ω, ω' on $\mathbf{R}^{2n}$, the pairs (N, ω) and (N', ω') are symplectomorphic if and only if their algebraic restrictions $[\omega]_N$ and $[\omega']_{N'}$ are diffeomorphic.*

Corollary 6.5 *Algebraic restrictions of symplectic forms to an open umbrella are diffeomorphic to each other at least in analytic category.*

6.2 Geometric and algebraic restrictions

Let N, M be manifolds. Given $f \in C^\infty(N, M)$, two differential forms ω, ω' are called *geometrically equivalent* for f, if $f^*\omega = f^*\omega'$, i.e. if they have the same geometric restriction for f.

A finer equivalence relation on Λ^p_M is introduced in [82][17][18]. Two differential p-forms ω, ω' are called *algebraically equivalent* for f, if they have the same algebraic restriction for f, namely, if there exist a p form α and a $(p-1)$ form β on M such that $\alpha(x) = 0, \beta(x) = 0$ for any $x \in f(N)$ and

$$\omega - \omega' = \alpha + d\beta,$$

where d means the exterior differential. If ω, ω' are algebraically equivalent, then they are geometrically equivalent. The converse does not hold in general.

Let $\Lambda^\bullet(M) = \sum_k \Lambda^k(M)$ denote the totality of differential forms on a C^∞ manifold M and $(\Lambda^\bullet(M), d)$ de Rham complex on M. Here $\bullet$ indicates the natural graduation. We set

$\Lambda^k(M) = 0$ if $k < 0$ or $\dim(M) < k$. Given a subset $Z \subset M$, then the notion of algebraic restrictions of differential forms is introduced in [17]. Let $\Lambda^\bullet_Z(M)$ denote the subspace of $\Lambda^\bullet(M)$ consisting of differential forms vanishing on Z. Note that $\Lambda^\bullet_Z(M)$ is not necessarily d-closed. Let $\mathcal{A}^\bullet(Z, M)$ denote the differential ideal of $(\Lambda^\bullet(M), d)$ generated by $\Lambda^\bullet_Z(M)$:

$$\mathcal{A}^k(Z, M) = \{\alpha + d\beta \mid \alpha \in \Lambda^k_Z(M), \beta \in \Lambda^{k-1}_Z(M)\}.$$

For an $\omega \in \Lambda^\bullet(M)$, the residue class $[\omega]^a_Z \in \Lambda^\bullet(M)/\mathcal{A}^\bullet(Z, M)$ is called the *algebraic restriction* of ω to Z.

We introduce the notion of *geometric restriction* for any subset Z in a C^∞ manifold M as follows: Define

$$\mathcal{G}^\bullet(Z, M) := \{\omega \in \Lambda^\bullet(M) \mid f^*\omega = 0 \text{ for any } f : N \to M \text{ with } f(N) \subset Z\}.$$

Note that $\mathcal{G}^0(Z, M) = \mathcal{A}^0(Z, M) = \{h \in \Lambda^0(M) \mid h|_Z = 0\}$.

For an $\omega \in \Lambda^\bullet(M)$, the residue class $[\omega]^g_Z \in \Lambda^\bullet(M)/\mathcal{G}^\bullet(Z, M)$ is called the *geometric restriction* of ω to Z.

Accordingly we introduce the vector space

$$\mathcal{A}^\bullet(Z) := \Lambda^\bullet(M)/\mathcal{A}^\bullet(Z, M)$$

of algebraic restrictions to Z, and the vector space

$$\mathcal{G}^\bullet(Z) := \Lambda^\bullet(M)/\mathcal{G}^\bullet(Z, M).$$

of geometric restrictions to Z.

Lemma 6.6 *For any subset Z in a C^∞ manifold M, we have*
(1) *$\mathcal{G}^\bullet(Z, M)$ is d-closed.*
(2) *$\mathcal{G}^\bullet(Z, M) \supset \mathcal{A}^\bullet(Z, M)$.*

Consider the ideal $I(Z) := \{h \in \Lambda^0(M) \mid h|_Z = 0\}$ in the $\mathbf{R}$-algebra $\Lambda^0(M)$. Then we have

Lemma 6.7 *$\mathcal{A}^\bullet(Z, M)$ is the d-closed graded ideal in $\Lambda^\bullet(M)$ generated by $I(Z)$, i.e.*

$$\mathcal{A}^\bullet(Z, M) = \{\sum_i h_i\alpha_i + \sum_j (dh_j)\beta_j \mid \alpha_i, \beta_j \in \Lambda^\bullet(M)\}.$$

Now we introduce the quotient module

$$\mathcal{R}^\bullet(Z) := \mathcal{G}^\bullet(Z, M)/\mathcal{A}^\bullet(Z, M) \; (\subset \mathcal{A}^\bullet(Z)\,),$$

which consists of algebraic restrictions with null geometric restrictions to Z.

Definition 6.8 *The $\mathcal{R}^\bullet(Z)$ is called the space of residual algebraic restrictions.*

Then there arises the natural exact sequence

$$0 \to \mathcal{R}^\bullet(Z) \to \mathcal{A}^\bullet(Z) \to \mathcal{G}^\bullet(Z) \to 0.$$

The space $\mathcal{A}^\bullet(Z)$ of algebraic restrictions of differential forms to Z has the natural module structure over de Rham exterior algebra $\Lambda^\bullet(M)$, which is defined by

$$\beta \wedge [\alpha]_Z^a := [\beta \wedge \alpha]_Z^a,$$

with the differential

$$\overline{d} : \mathcal{A}^\bullet(Z) \to \mathcal{A}^{\bullet+1}(Z)$$

defined by $\overline{d}[\alpha]_Z^a := [d\alpha]_Z^a$ and satisfying

$$\overline{d}(\beta \wedge [\alpha]_Z^a) = d\beta \wedge [\alpha]_Z^a + (-1)^k \beta \wedge \overline{d}[\alpha]_Z^a,$$

whenever $\beta \in \Lambda^k(M)$.

Also the space $\mathcal{G}^\bullet(Z)$ (resp. $\mathcal{R}^\bullet(Z)$) has the natural module structure over the de Rham exterior algebra $\Lambda^\bullet(M)$ as well.

Remark 6.9 The non-zero algebraic restrictions of symplectic forms to a curve in a symplectic space was called "ghosts" according to [3]. The symplectic forms has null geometric restrictions on parametric curves. Since we are regarding all algebraic restrictions with null geometric restrictions of differential forms, we may call our residues "pure ghosts".

6.3 Symplectic classification of map-germs

Let ω be a symplectic form on $\mathbf{R}^{2n}$, and $f : (\mathbf{R}^m, a) \to \mathbf{R}^{2n}$ a C^∞ map-germ. We consider the classification problem of the pair (f, ω) fixing m and n: The pair (f, ω) is called *symplectomorphic* to another pair (f', ω') if there exist a diffeomorphism-germ $\sigma : (\mathbf{R}^m, a) \to (\mathbf{R}^m, a')$ and a symplectomorphism-germ $\tau : (\mathbf{R}^{2n}, f(a)) \to (\mathbf{R}^{2n}, f'(a'))$, $\tau^*\omega' = \omega$, such that $f' \circ \sigma = \tau \circ f$, namely that the diagram

$$\begin{array}{ccc} (\mathbf{R}^m, a) & \xrightarrow{f} & (\mathbf{R}^{2n}, f(a)),\ \omega \\ \sigma \downarrow & & \downarrow \tau \\ (\mathbf{R}^m, a') & \xrightarrow{f'} & (\mathbf{R}^{2n}, f'(a')),\ \omega' \end{array}$$

commutes.

If the above condition is satisfied just for a diffeomorphism-germ τ, (not necessarily a symplectomorphism-germ), then we say that f and f' are *diffeomorphic*.

First we mention again Darboux-Givental theorem in the form:

Theorem 6.10 (Darboux-Givental [6]) *For any immersion-germs $f, f' : \mathbf{R}^m \to \mathbf{R}^{2n}$ and for any symplectic forms ω, ω' on $\mathbf{R}^{2n}$, (f, ω) and (f', ω') are symplectomorphic if and only if two forms $f^*\omega$ and $f'^*\omega'$ are diffeomorphic; for some diffeomorphism-germ σ on $\mathbf{R}^m$, $\sigma^*(f'^*\omega') = f^*\omega$.*

Thus in the non-singular case (the case of immersion-germs), the classification problem is reduced to that of pull-back forms to the sources. Note that the pull-backs of symplectic forms are not arbitrary. In particular we have

Corollary 6.11 All non-singular hypersurface-germs in $\mathbf{R}^{2n}$ are symplectomorphic. All coisotropic (resp. isotropic) submanifold-germs of fixed dimension in $\mathbf{R}^{2n}$ are symplectomorphic.

Note that all immersion-germs (on a fixed dimensional source) are diffeomorphic in our sense. In the singular case, however, even if f and f' are diffeomorphic and $f^*\omega$ and $f'^*\omega'$ are diffeomorphic, (f, ω) and (f', ω') are not necessarily symplectomorphic.

In fact, in the case $m = n = 1$ (planar curves), we have given both symplectic and differential exact classifications of differentially uni-modal planar curve singularities, and clarified the difference of differential and symplectic classifications ([36][38][40]). For the classification of curves ($m = 1, n \geq 2$), see [3][4][48][17][18][15]. See §3.

Example 6.12 Let $f_\lambda(u,t) := (t^5 + ut^3 + \lambda u^2 t,\ t^2,\ \frac{2}{5}t^5 + \frac{4}{3}\lambda ut^3,\ u) = (p_1, q_1, p_2, q_2)$, $\lambda \neq \frac{21}{100}$. Then the family f_λ of isotropic map-germs with respect to $\omega_{\rm st}$ is trivialised by diffeomorphisms, but λ gives the "symplectic moduli".

There is the notion of *symplectic codimension* sp-codim(f, ω) also for an isotropic pair (f, ω). The number sp-codim(f, ω) is characterised as the minimal number of symplectically versal unfoldings of f.

Theorem 6.13 ([37]) sp-codim(f, ω) *is a diffeomorphism invariant for isotropic nomalizations* $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n}, \omega)$*: If f and f' are diffeomorphic, then* sp-codim$(f, \omega) =$ sp-codim(f', ω') *for any symplectic forms ω, ω' with $f^*\omega = 0, f'^*\omega' = 0$.*

In the complex analytic case, if codim$\Sigma(f) \geqq 2$, then

$$\text{sp-codim}(f, \omega) = \dim_{\mathbf{C}} \mathcal{R}_f / f^*\mathcal{O}_{2n},$$

where

$$\mathcal{R}_f := \{h \in \mathcal{O}_n \mid dh \in \mathcal{O}_n \cdot df\}.$$

In the case $n = 1$, we have

$$\text{sp-codim}(f, \omega) = \dim_{\mathbf{C}} \mathcal{O}_1 / f^*\mathcal{O}_2.$$

Moreover the difference of differential/symplectic classification is given by

$$\text{gh}(f, \omega) := \dim_{\mathbf{C}} \mathcal{G}_f / f^*\mathcal{O}_{2n},$$

symplectic defect or *ghost number,* where

$$\mathcal{G}_f := \{h \in \mathcal{O}_n \mid dh \in f^*\Lambda^1_{2n}\} = \{h \in \mathcal{O}_n \mid dh \in f^*\mathcal{O}_{2n} \cdot df\}.$$

Remark that

$$\mathcal{R}_f \supseteq \mathcal{G}_f \supseteq f^*\mathcal{O}_{2n}, \qquad f^* : \mathcal{O}_n \leftarrow \mathcal{O}_{2n}.$$

Example 6.14 For the open umbrella

$$f_{\rm ou} = (ut, t^2, \frac{2}{3}t^3, u) : (\mathbf{R}^2, 0) \to (\mathbf{R}^4, 0),$$

we have that

$$dh(t,u) \in \langle d(t^2), du, d(ut), d(\frac{2}{3}t^3)\rangle_{\mathcal{O}_2} = \langle tdt, du, udt\rangle_{\mathcal{O}_2}$$

if and only if $h = a(t^2, t^3, ut, u)$ for some C^∞ function a. Therefore $\mathcal{R}_f = \mathcal{G}_f = f^*\mathcal{O}_4$.

Proposition 6.15 *Let $f : \mathbf{R}^2 \to \mathbf{R}^4$ be isotropic map-germ of corank ≤ 1 for a symplectic form ω. If* sp-codim$(f, \omega) \leq 1$ *then (f, ω) is symplectomorphic to $(f_{\mathrm{ou}}, \omega_{\mathrm{st}})$ the open umbrella, or to $(f^{\pm}_{\mathrm{mou}}, \omega_{\mathrm{st}})$ the multiple open umbrella, where $f^{\pm}_{\mathrm{mou}}(t, u) := (t^3 \pm u^2 t, t^2, \frac{4}{3}ut^3, u)$.*

Moreover $(f^{+}_{\mathrm{mou}}, \omega_{\mathrm{st}})$ is not symplectomorphic to $(f^{-}_{\mathrm{mou}}, \omega_{\mathrm{st}})$. In fact f^{+}_{mou} and f^{-}_{mou} are not diffeomorphic.

Remark 6.16 For the multiple open umbrella, $\mathcal{R}_f \supsetneq \mathcal{G}_f = f^*\mathcal{O}_{2n}$: There is no ghost in this case. For the map-germs f_λ in Example 6.12, we have that sp-codim$(f_\lambda, \omega_{\mathrm{st}}) = 2$, and that $\mathcal{R}_{f_\lambda} \supsetneq \mathcal{G}_{f_\lambda} \supsetneq {f_\lambda}^*\mathcal{O}_{2n}$.

6.4 Symplectic classification of Whitney umbrellas

Now we consider the symplectic classification of generic map-germs $f : \mathbf{R}^3 \to \mathbf{R}^4$ as a typical example of our classification problem.

As for the differential classification, it is known that a generic map-germ $f : \mathbf{R}^3 \to \mathbf{R}^4$ is diffeomorphic to an immersion or to a Whitney umbrella. A map-germ $f : (\mathbf{R}^3, a) \to \mathbf{R}^4$ is called a *Whitney umbrella* if f is diffeomorphic to the map-germ $(\mathbf{R}^3, 0) \to (\mathbf{R}^4, 0)$ given by $(u, v, w) \mapsto (p_1, q_1, p_2, q_2) = (uv, u^2, w, v)$.

The double point locus $D(f)$ (resp. singular point locus $S(f)$) of the (normalized) Whitney umbrella, designated also as f, is given by $\{v = 0\}$ (resp. $\{u = v = 0\}$). In fact, the points $(\pm u, 0, w)$ are mapped to the same point by f. Thus we have the *canonical stratification* of $\mathbf{R}^3$ associated to f : $\mathbf{R}^3 \supset D(f) \supset S(f)$. Moreover note that the kernel field $K(f)$ of the differential $f_* : T\mathbf{R}^3 \to T\mathbf{R}^4$ along $S(f) = \{u = v = 0\}$ is given by $K(f)(0, 0, w) = \frac{\partial}{\partial u}$.

On the other hand, for a generic symplectic form ω, the pullback $f^*\omega$ on $\mathbf{R}^3$ is of rank 2. Then the kernel field of $f^*\omega$ is called the *characteristic filed* of (f, ω) and we have the *characteristic foliation* $\mathcal{F} = \mathcal{F}_{(f,\omega)}$ on $\mathbf{R}^3$. The relative position of the characteristic foliation of (f, ω) and the canonical stratification of f is clearly a symplectically invariant character of (f, ω).

For example, the standard symplectic form $\omega_{\mathrm{st}} = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, pulled back by f,

$$f^*\omega_{\mathrm{st}} = d(uv) \wedge d(u^2) + dw \wedge dv = d(w - \frac{2}{3}u^3) \wedge dv$$

is of rank 2. In this example, the characteristic foliation is given by $w - \frac{2}{3}u^3 =$ const., $v =$ const.. Therefore each characteristic curve is contained in the singular locus $S(f) = \{v = 0\}$, and that situation is never generic.

Note that the kernel field $K(f)$ of the differential f_* coincides with the characteristic field along $S(f)$. Hence each characteristic curve is necessarily tangent to the locus $D(f)$ of double points along $S(f)$.

Generically, each characteristic curve contacts with the double point locus $D(f)$ in the second order along $S(f)$ except isolated points of $S(f)$, and, in the third order at those isolated points.

Define $g : \mathbf{R}^3 \to \mathbf{R}^2$ as the symplectic reduction determined by the characteristic foliation of f (which is determined up to left equivalence). Consider the map $g|_{D(f)} : D(f) \to \mathbf{R}^2$. If each characteristic curve contacts with the double point locus $D(f)$ in the second order along $S(f)$, then $g|_{D(f)}$ has a fold singularity along $S(f)$ and it is a two-to-one mapping off $S(f)$, which induces an involution $\tau(f) : D(f) \to D(f)$ on the surface $D(f)$. Moreover, $f|_{D(f)} : D(f) \to \mathbf{R}^3$

is also two-to-one off $S(f)$. It also induces an involution $\eta(f) : D(f) \to D(f)$ on $D(f)$. So we have a *pair of involutions* $(\tau(f), \eta(f))$ on the surface $D(f)$. If a characteristic curve contacts with $D(f)$ in the third order at a point $S(f)$, then $g|_{D(f)} : D(f) \to \mathbf{R}^2$ has a more degenerate singularity than the fold singularity.

Similar situation appeared in the classification of glancing hypersurfaces due to Melrose [57][58]. See also [2][73].

Consider (not a mono-germ but) a bi-germ $f = f_1 \coprod f_2 : (\mathbf{R}^3, 0) \coprod (\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^4, 0)$ and the standard symplectic form $\omega_{\rm st}$ on $(\mathbf{R}^4, 0)$. Suppose f_1 and f_2 are transversal immersion-germs. Then the self-intersection forms a smooth surface S in $(\mathbf{R}^4, 0)$. Consider the characteristic foliations $\mathcal{F}_1$ on $M_1 = f_1(\mathbf{R}^3, 0)$ and $\mathcal{F}_2$ on $M_2 = f_2(\mathbf{R}^3, 0)$. Then the relative position of $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to S is a symplectically invariant character. If $\mathcal{F}_1$ is transversal to S in M_1, then $\mathcal{F}_2$ is transversal to S in M_2. Then the pair is symplectomorphic to the standard one: $M_1 = \{p_1 = 0\}$ and $M_2 = \{q_1 = 0\}$.

M_1 and M_2 are said to be *glancing* at a point in S if the both characteristic curves through the point are tangent to S in the second order [57]. Generically M_1 and M_2 are glancing along a smooth curve in S and at isolated points the tangency becomes of higher order.

Melrose [57] showed that any glancing pair is C^∞ symplectomorphic to the pair $\{p_1 = p_2^2\}$ and $\{q_2 = 0\}$. On the other hand, in [68], Oshima gave a counter example to the uniqueness result for the analytic classification. (A counter example to Sato's conjecture [67]). In fact it is known that the analytic symplectic classification of glancing pairs has a functional moduli.

Actually we announce the following result:

Theorem 6.17 *For a generic pair (f, ω) of a C^∞ mapping $f : \mathbf{R}^3 \to \mathbf{R}^4$ and a C^∞ symplectic form ω, at any singular point $a \in \mathbf{R}^3$ of f, (f, ω) is symplectomorphic to the normal form*

$$\omega_1 = dp_1 \wedge dq_1 + dp_2 \wedge d(q_2 - q_1),$$

or to

$$\omega_2 = dp_1 \wedge dq_1 + dp_2 \wedge d(q_2 - q_1 p_2 - \varphi(q_1^2)),$$

for a functional moduli φ, $(\varphi(0) = \varphi'(0) = 0)$ with the normal form $(u, v, w) \mapsto (p_1, q_1, p_2, q_2) = (uv, u^2, w, v)$.

Note that, for the normal forms in Theorem 6.17, the pull-back form turns out to be

$$d(w - \frac{2}{3}u^3) \wedge d(v - u^2), \text{ or } d(w - \frac{2}{3}u^3) \wedge d(v - u^2 w + \frac{2}{5}u^5 - \varphi(u^2)),$$

Remark 6.18 There appears a difference between C^∞ and analytic classification in Theorem 6.17 arising from the conjugate classification of map-germs $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ with 3-jets of type $(u, w) \to (u, w + u^3)$: In the sense of Voronin [77], the $\mathcal{B}_3$-classification problem arises. In fact the composition $\eta(f) \circ \tau(f) : D(f) \to D(f)$ is of this form. Remark that the symplectic classification of swallowtails corresponds to the $\mathcal{B}_5$-classification problem.

Remark 6.19 The above classification is also regarded as the classification of *coisotropic pairs*. A pair (f, ω) of a map-germ $f : (\mathbf{R}^m, 0) \to \mathbf{R}^{2n}$ and a symplectic form-germ ω on $\mathbf{R}^{2n}$, $(m \geq n)$, is called *coisotropic* if f lifts to an isotropic map-germ $\widetilde{f} : (\mathbf{R}^m, 0) \to (\mathbf{R}^{2m}, 0) = (\mathbf{R}^{2n}, 0) \times (\mathbf{R}^{2(m-n)}, 0)$ with a symplectic form $\pi_1^*\omega - \pi_2^*\mu$ on $(\mathbf{R}^{2m}, 0)$.

Any coisotropic immersion $(\mathbf{R}^m, 0) \to (\mathbf{R}^{2n}, 0)$ for any symplectic form, in the ordinary sense, ω lifts to a Lagrangian immersion into $\mathbf{R}^{2m}$, so coisotropic in the above sense.

Then we define the *symplectic codimension* of coisotropic pair (f, ω) by

$$\text{sp-codim}(f, \omega) := \dim_{\mathbf{R}} \mathcal{R}_f / (f^* \mathcal{O}_{2n} + g^* \mathcal{O}_{2(m-n)}).$$

For the normal forms of Theorem 6.17 we have

$$\text{sp-codim}(f, \omega_1) = 0$$

and

$$\text{sp-codim}(f, \omega_2) = \infty.$$

6.5 Geometric and algebraic restrictions via a mapping

Let $f : N \to M$ be a C^∞ mapping from a C^∞ manifold N.

Let $\omega \in \Lambda^\bullet(M)$ a differential form on M. Then we call the pull-back $f^*\omega$ the *geometric restriction of* ω *by* f. Then, regarding the morphism $f^* : \Lambda^\bullet(M) \to \Lambda^\bullet(N)$, we consider the subspace consisting of differential forms with null geometric restrictions by f:

$$(\mathrm{Ker} f^*)^\bullet := \{\omega \in \Lambda^\bullet(M) \mid f^*\omega = 0\}.$$

Then we have

$$(\mathrm{Ker} f^*)^\bullet \supset \mathcal{G}^\bullet(f(N), M).$$

The space of geometric restrictions by f of differential forms, which is identified with

$$\mathcal{G}^\bullet(f) := \Lambda^\bullet(M)/(\mathrm{Ker} f^*)^\bullet,$$

has the natural module structure over the de Rham exterior algebra $\Lambda^\bullet(M)$.

In the case $Z = f(N)$, we describe $\mathcal{A}^k(Z, M)$ in terms of mapping f.

First we introduce the space

$$\Lambda^k(f) = \{\beta : N \to \wedge^k(T^*M) \mid \beta \text{ covers } f \text{ via the projection } \pi : \wedge^k(T^*M) \to M\},$$

the space of differential k-forms along f, and a morphism $\omega f : \Lambda^\bullet(M) \to \Lambda^\bullet(f)$ defined by $\alpha \mapsto \alpha \circ f$. Here $\wedge^k(T^*M)$ is the exterior product of the cotangent bundle T^*M. The notion ωf is used, based on the classical Mather's notation. As for Mather's notation, we define also a morphism $t^*f : \Lambda^\bullet(f) \to \Lambda^\bullet(N)$, by

$$(t^*f(\beta))(x) = \wedge^k(f_{*x})^*(\beta(x))$$

where $\wedge^k(f_{*x})^* : \wedge^k T^*_{f(x)}M \to \wedge^k T^*_x N$ is the wedge of the dual linear map of the differential map $f_{*x} : T_x N \to T_{f(x)}M$.

We have the commutative diagram for $k \geq 1$,

$$\begin{array}{ccccc} \Lambda^{k-1}(M) & \xrightarrow{\omega^{k-1}f} & \Lambda^{k-1}(f) & \xrightarrow{t^{*k-1}f} & \Lambda^{k-1}(N) \\ d\downarrow & & & & \downarrow d \\ \Lambda^k(M) & \xrightarrow{\omega^k f} & \Lambda^k(f) & \xrightarrow{t^{*k}f} & \Lambda^k(N). \end{array}$$

Note that $t^{*0}f$ gives the identification of $\Lambda^0(f)$ and $\Lambda^0(N)$, which is the space of sections of the trivial line bundle.

The following is clear by the definition of $\mathcal{A}(f(N), M)$:

Lemma 6.20 *Let $f : N \to M$ be a C^∞ mapping. Then we have, for any $k \geq 0$,*

$$\text{Ker } \omega^k f + d(\text{Ker } \omega^{k-1} f) = \mathcal{A}^k(f(N), M).$$

We study the quotient space

$$\mathcal{R}^k(f) := (\text{Ker } f^*)^k / \mathcal{A}^k(f(N), M) = (\text{Ker } f^*)^k / (\text{Ker } \omega^k f + d(\text{Ker } \omega^{k-1} f)),$$

the space of algebraic restrictions to the image of f with null geometric restrictions by f.

The above constructions are localised naturally, i.e. they are formulated in terms of sheaves. The following is clear:

Lemma 6.21 *If $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ is an immersion-germ, then*

$$\mathcal{R}^k(f) = \text{Ker } f^{*k} / (\text{Ker } \omega^k f + d(\text{Ker } \omega^{k-1} f)) = 0,$$

for $k \geq 0$.

Moreover we have

Proposition 6.22 *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$, $2n \leq m$, be a finitely determined map-germ, Z the germ of the image of f. Then the $\mathbf{R}$-vector space $\mathcal{R}^\bullet(Z) = \mathcal{R}^\bullet(f)$ is of finite dimension.*

Let $Z \subset (\mathbf{R}^m, 0)$ be a subset-germ in $\mathbf{R}^m$ at 0. The *embedding dimension* of Z is defined as the minimum of the dimensions of submanifold-germs $S \subset (\mathbf{R}^m, 0)$ with $Z \subset S$.

Lemma 6.23 *Suppose the embedding dimension of $Z \subset (\mathbf{R}^m, 0)$ is equal to r. Let $S \subset (\mathbf{R}^m, 0)$ be a submanifold-germ of dimension r with $Z \subset S$. Let $h : (\mathbf{R}^m, 0) \to \mathbf{R}$ be a function-germ vanishing on Z. Then we have $dh|_{T_0 S} = 0$. Therefore the tangent space $T_0 S$ to a submanifold-germ S of $(\mathbf{R}^m, 0)$ of dimension r containing Z is uniquely determined. In fact $T_0 S$ coincides with the Zariski tangent space $(T^a Z)_0^\circ$ of Z at 0 in $\mathbf{R}^m$.*

Lemma 6.24 *For any $k = 1, 2, \dots, r$, and any k-form α in $\mathcal{A}^k(Z, M)$, α vanishes on $\wedge^k (T^a Z)_0^\circ$.*

Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ be a germ of a proper mapping. Then the germ of the image of f is well-defined as a subset-germ in $(\mathbf{R}^m, 0)$. Therefore the embedding dimension of f is defined via the image of f.

Proposition 6.25 *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ be a proper map-germ. Suppose the embedding dimension of f is equal to $r > n$. Let $S \subset (\mathbf{R}^m, 0)$ be a minimal dimensional submanifold-germ containing the image of f with $\dim S = r$. Then, for any $k = 1, 2, \dots, r$, any k-form in* $\text{Ker } \omega^k f + d(\text{Ker } \omega^{k-1} f)$ *vanishes at $T_0 S$. In particular we have $\mathcal{R}^r(f) \neq 0$.*

Example 6.26 Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^3, 0)$, $f(t) = (\frac{1}{3!}t^3, \frac{1}{4!}t^4, \frac{1}{5!}t^5)$. Then the embedded dimension of f is equal to 3. Then, in fact, the geometric restriction $[dx_1 \wedge dx_2 \wedge dx_3]^g = 0$ and the algebraic restriction $[dx_1 \wedge dx_2 \wedge dx_3]^a \neq 0$. Therefore the residue of volume form $[dx_1 \wedge dx_2 \wedge dx_3]^r \neq 0$ in $\mathcal{R}^3(f)$.

6.6 Residues of planar curves

Classification of $\mathcal{R}^2(Z)$-residues for the planar curves $Z = \{x \in \mathbf{R}^{2n} \mid H(x_1, x_2) = 0, x_i = 0, i \geq 3\}$, with H being one of the singularities A_k, D_k, E_6, E_7, E_8 were done in [18].

Example 6.27 Let us consider the case of A_k-singularity. $Z = A_k = \{x \in \mathbf{R}^{2n} \mid H = x_1^{k+1} - x_2^2 = 0 = x_{\geq 3} = 0\}, (k \geq 1)$. In this case an algebraic restriction of any any two-form to A_k can be represented by a two form on the plane $f(x_1, x_2)dx_1 \wedge dx_2$. As we have the conditions: $[dH]^a_{A_k} = 0$, $[dH \wedge dx_1]^a_{A_k} = 0$ and $[dH \wedge dx_2]^a_{A_k} = 0$, then $f dx_1 \wedge dx_2 \in \mathcal{A}^2(A_2, \mathbf{R}^{2m})$ iff f belongs to the Jacobi ideal of H, $f \in \Delta H = \langle x_2, x_1^k \rangle$. Thus $\mathcal{R}^2(A_k) = \{\sum_{i=0}^{k-1} c_i x_1^i dx_1 \wedge dx_2, c_i \in \mathbf{R}\}$.

If we apply the group of symmetries of the curve $Z = \{H = 0\} \subset \mathbf{R}^2$ to the algebraic restrictions

$$\mathcal{R}^2(Z) = \Lambda^2(\mathbf{R}^2)/\mathcal{A}(Z, \mathbf{R}^2), \quad \mathcal{G}^2(Z, \mathbf{R}^2) = \Lambda^2(\mathbf{R}^2)$$

then we get the classification of residues $[f(x_1, x_2)dx_1 \wedge dx_2]^a_Z$ given in the table 4 (see, [18], p. 214).

Remark 6.28 The classification of Table 4 can be applied to symplectic classification of planar curves diffeomorphic to A, D, E classes. If $n \geq 1$ then the residues $[f_i(x_1, x_2)dx_1 \wedge dx_2]^a_Z$ can be realized by the symplectic forms $\omega_i = f_i dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \ldots + dx_{2n-1} \wedge d_{2n}$. Each form ω_i can be brought, by a local diffeomorphism Φ_i, to the fixed Darboux form $\omega_0 = dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n$. Then any singular curve in the symplectic space $(\mathbf{R}^{2n}, \omega_0)$ which is diffeomorphic to Z, say A_k-singularity is symplectomorphic to one and only one of the curves $A_k^i = \Phi_i^{-1}(A_k)$.

Theorem 6.29 ([18]) *Fix a function $H = H(x_1, x_2)$ as in Table 4. Any curve in the symplectic space $(\mathbf{R}^{2n}, \omega_0)$, $n \geq 2$, which is diffeomorphic to the curve $Z : H(x_1, x_2) = x_{\geq 3} = 0$ can be reduced by a symplectomorphism to one and only one of the normal forms*

$$Z^i = \{(p, q) \in \mathbf{R}^{2n} \mid H(p_1, p_2) = 0 = q_1 - \int_0^{p_2} f_i(p_1, t)dt = q_{\geq 2} = p_{geq3} = 0\}, i = 0, \ldots, \mu,$$

where f_i are the functions in Table 4 and μ is the multiplicity of H. The parameters b, b_1, b_2 are the symplectic moduli. The codimension of the symplectic singularity class defined by the normal form Z^i in the class of all curves diffeomorphic to Z is equal to i.

We study on $\mathcal{R}^1(Z)$ for a germ of planar curve Z in $\mathbf{R}^2$, as a special case of arguments discussed in the previous section. We also treat $\mathcal{R}^1(f)$ on the parametric case $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$.

Now our idea to treat parametric planar curves is to fix a symplectic form (an area form) Ω on $\mathbf{R}^2$, say,

$$\Omega = dx_1 \wedge dx_2,$$

and apply the classification established in [36][40]. We conclude the paper by showing several examples from our previous classification result. Though the classification was performed in complex analytic case in [40], we can give the real classification by adding necessary $\pm$ to the lists.

Remark 6.30 In the case of A_2, $g : t \mapsto (t^2, t^3)$, $dim_{\mathbf{R}}\mathcal{R}^1(Z) = 2$, $\mathcal{R}^1(Z) = \mathcal{R}^1(g)$ and it is generated over $\mathbf{R}$ by the classes of Euler form: $E^\sharp = -3x_2 dx_1 + 2x_1 dx_2$ and $x_1 E^\sharp$.

$H(x_1, x_2)$	$f_i(x_1, x_2),\ i = 0, 1, \ldots, \mu$
$A_k : x_1^{k+1} - x_2^2$ $k \geq 1$	$f_0 = 1$ $f_i = x_1^i,\ i = 1, \ldots, k-1$ $f_k = 0$
$D_k : x_1^2 x_2 - x_2^{k-1}$ $k \geq 4$	$f_0 = 1$ $f_i = bx_1 + x_2^i,\ \ i = 1, \ldots, k-4$ $f_{k-3} = (\pm 1)^k x_1 + bx_2^{k-3},$ $f_{k-2} = x_2^{k-3},\ f_{k-1} = x_2^{k-2},\ f_k = 0$
$E_6 : x_1^3 - x_2^4$	$f_0 = 1,\ f_1 = \pm x_2 + bx_1,\ f_2 = x_1 + bx_2^2,$ $f_3 = x_2^2 + bx_1x_2,\ f_4 = \pm x_1x_2,\ f_5 = x_1x_2^2,\ f_6 = 0$
$E_7 : x_1^3 - x_1x_2^3$	$f_0 = 1,\ f_1 = x_2 + bx_1,\ f_2 = \pm x_1 + bx_2^2,$ $f_3 = x_2^2 + bx_1x_2,\ f_4 = \pm x_1x_2 + bx_2^3,$ $F_5 = x_2^3,\ f_6 = x_2^4,\ f_7 = 0$
$E_8 : x_1^3 - x_2^5$	$f_0 = \pm 1,\ f_1 = x_2 + bx_1,\ f_2 = x_1 + b_1x_2^2 + b_2x_2^3$ $f_3 = \pm x_2^2 + bx_1x_2,\ f_4 = \pm x_1x_2 + bx_2^3,$ $f_5 = x_2^3 + bx_1x_2^2,\ f_6 = x_1x_2^2,\ f_7 = \pm x_1x_2^3,\ f_8 = 0$

Table 4: Classification of residues for A, D, E singularities

6.7 Residues for hypersurfaces

Let $F : (\mathbf{R}^m, 0) \to (\mathbf{R}, 0)$ be a non-zero analytic function-germ and consider the set-germ $Z \subset (\mathbf{R}^m, 0)$. Suppose the ideal $I_Z := \Lambda_Z(\mathbf{R}^m, 0) \subset \mathcal{O}_m := \Lambda^0(\mathbf{R}^m, 0)$ of function-germs vanishing on Z is generated by F. Then we have on the residues of top degree:

Proposition 6.31 *$\mathcal{R}^m(Z) \cong \mathcal{O}_m/\langle F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_m}\rangle_{\mathcal{O}_m}$. In particular $\dim_{\mathbf{R}} \mathcal{R}^m(Z)$ is given by the Turina number of F at 0.*

Proof: Let α be any m-form on $M = (\mathbf{R}^m, 0)$. Then $\alpha \in \mathcal{G}^k(Z, M)$. We have that $\alpha \in \mathcal{A}^k(Z, M)$ if and only if there exist an m-form β and an $(m-1)$-form γ such that $\alpha = F\beta + d(F\gamma)$. Take the volume form $\omega = dx_1 \wedge \cdots \wedge dx_m$. There exists a unique $h \in \mathcal{O}_m$ with $\alpha = h\omega$. Then $\alpha \in \mathcal{A}^k(Z, M)$ if and only if $h \in \langle F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_m}\rangle_{\mathcal{O}_m}$. Thus we have the result. $\square$

For the residue of degree 1 of hypersurface, we have:

Proposition 6.32 *Let $F : M = (\mathbf{R}^m, 0) \to (\mathbf{R}, 0)$ be a C^∞ function-germ. Let Z denote the germ of zero-locus of F in $(\mathbf{R}^m, 0)$. Suppose the ideal $I(T^g Z)$ of function-germs on $(T\mathbf{R}^m, (0, 0))$ vanishing on the geometric tangent bundle $T^g Z \subset T\mathbf{R}^m$ is generated by*

$$\sum_{i=1}^{m} v_i \frac{\partial F}{\partial x_i}(x) \text{ and } F(x).$$

Here (x, v) denote the system of coordinate functions on $T\mathbf{R}^m$. Then we have $\mathcal{R}^1(Z) = 0$.

By the similar proof of 6.32, we have:

Proposition 6.33 *Let $F_1, \dots, F_r : M = (\mathbf{R}^m, 0) \to (\mathbf{R}, 0)$ be a C^∞ function-germ. Let Z denote the germ of zero-locus of $F = (F_1, \dots, F_r) : M \to (\mathbf{R}^r, 0)$ in M. Suppose the ideal $I(T^g Z)$ of function-germs on $(T\mathbf{R}^m, (0, 0))$ vanishing on the geometric tangent bundle $T^g Z \subset T\mathbf{R}^m$ is generated by*

$$\sum_{i=1}^{m} v_i \frac{\partial F_j}{\partial x_i}(x), 1 \le j \le r, \text{ and } F_j(x), 1 \le j \le r.$$

Then we have $\mathcal{R}^1(Z) = 0$.

6.8 Residues for Lagrangian varieties

Now we suppose M is a symplectic manifold of dimension $2n$ with a symplectic form Ω. A subset $Z \subset M$ is called a *Lagrangian variety* if the geometric restriction $[\Omega]^g_Z = 0$ and the maximal rank of the geometric tangent bundle $T^g Z \subset TM$ is equal to n.

We describe $\mathcal{R}^1(Z)$ in terms of vector fields via the symplectic duality. The space of vector fields $V(M)$ over M corresponds to the space of 1-forms $\Lambda^1(M)$ by

$$X \mapsto X^\sharp := i_X\Omega \in \Lambda^1(M), \ (X \in V(M)).$$

The inverse of the correspondence is written, for any $\alpha \in \Lambda^1(M)$, by $\alpha \mapsto \alpha^\flat \in V(M)$. If $\alpha = dH$ for some $H \in \Lambda^0(M)$, then $X_H := (dH)^\flat$ is the Hamiltonian vector field with the Hamiltonian H.

If $M = \mathbf{R}^{2n}$ with the symplectic coordinates (x, p), then a vector filed $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + \sum_{j=1}^n b_j \frac{\partial}{\partial p_j}$ corresponds to the 1-form $\omega = -\sum_{j=1}^n b_j dx_j + \sum_{i=1}^n a_i dp_i$.

The tangent bundle TM is identified with the cotangent bundle T^*M. Therefore to any subset $S \subset TM$, there corresponds a subset $S^\flat \subset T^*M$.

Similarly the space of 2-vector fields $V^2(M)$ corresponds to the space of functions $\Lambda^0(M)$ by

$$X \wedge Y \mapsto i_Y i_X \Omega \in \Lambda^0(M), \ (X \wedge Y \in V^2(M)).$$

The space of 0-vector fields (i.e. functions) $V^0(M)$ corresponds to $\Lambda^2(M)$ simply by

$$h \mapsto h\Omega \in \Lambda^2(M), \ (h \in V^0(M)).$$

Let $Z \subset M$ be a Lagrangian variety. Let $\omega \in \Lambda^1(M)$. Then $\omega \in \mathcal{G}^1(Z, M)$ if and only if the corresponding vector field $X = \omega^\flat$ (satisfying $i_X\Omega = \omega$) is tangent to Z_{reg}. In fact $0 = \omega(T_pZ) = i_X\Omega(T_pZ) = \Omega(X, T_pZ)$, so $X(p) \in T_pZ$, for any regular point $p \in Z_{\mathrm{reg}}$. A vector field X over M is called *logarithmic* if it is tangent to Z_{reg}. The 1-form ω belongs to $\mathcal{A}^1(Z, M)$ if and only if $\omega^\flat = X + X_H$ for a vector field X vanishing on Z and the Hamiltonian vector field X_H of a Hamiltonian function H vanishing on Z. In fact $\omega \in \mathcal{A}^1(Z, M)$ if and only if there exist a 1-form α vanishing on Z and a function H vanishing on Z such that $\omega = \alpha + dH$. Then $\omega^\flat = \alpha^\flat + (dH)^\flat$ and $\alpha^\flat$ vanishes on Z if and only if α vanishes on Z. Moreover we have $(dH)^\flat = \pm X_H$ (depending on the convention). Thus we have:

Proposition 6.34 *The residue $\mathcal{R}^1(Z)$ of first order is isomorphic as $\Lambda^0(M)$-modules to the space of logarithmic vector fields modulo Hamiltonian vector fields restricted to $TM|_Z$.*

In the recent paper [43], we have studed more on algebraic, geometric and residual restrictions via a mapping and related cohomology theories of de Rham type. In particular we have developed analogous theory to [17][18] and have given the vanishing theorem on the residual cohomology for "contractible" map-germs.

References

[1] V.I. Arnold, *Lagrangian manifolds with singularities, asymptotic rays and the open swallowtail,* Funct. Anal. Appl., **15** (1981), 235–246.

[2] V.I. Arnold, *Singularities of Caustics and Wave Fronts,* Mathematics and its applications (Soviet series), **62**, Kluwer Academic Publishers., Dordrecht, (1990).

[3] V. I. Arnol'd, *First step of local symplectic algebra,* in Differential topology, infinite-dimensional Lie algebras, and applications, Amer. Math. Soc. Transl. Ser. 2, **194(44)** (1999), 1–8.

[4] V.I. Arnold, *Simple singularities of curves,* Proc. Steklov Inst. Math., **226–3** (1999), 20–28.

[5] V.I. Arnold, V.V. Goryunov, O.V. Lyashko, V.A. Vasil'ev, *Singularity Theory II: Classification and Applications,* Encyclopaedia of Math. Sci., Vol. 39, Dynamical System VIII, Springer-Verlag (1993).

[6] V.I. Arnold, A.B. Givental, *Symplectic geometry,* in Dynamical systems, IV, 1–138, Encyclopaedia Math. Sci., 4, Springer, Berlin, (2001).

[7] M. Artin, M. Nagata, *Residual intersections in Cohen-Macauley rings,* J. Math. Kyoto Univ., **12–2** (1972), 307–323.

[8] B. Banos, *On symplectic classification of effective 3-forms and Monge-Ampère equations,* Diff. Geom. Appl., **19** (2003), 147–166.

[9] E. Bierstone, P.D. Milman, *Relations among analytic functions I,* Ann. Inst. Fourier (Grenoble) **37-1** (1987), 187–239.

[10] J.W. Bruce, T. Gaffney, *Simple singularities of mappings* $\mathbf{C},0 \to \mathbf{C}^2,0$, J. London Math. Soc., **26** (1982), 465–474.

[11] J.W. Bruce, J.M. West, *Functions on a crosscap,* Math. Proc. Camb. Phil. Soc., **123** (1998), 19–39.

[12] J. Damon, *The unfolding and determinacy theorems for subgroups of* A *and* K*: a survey*, Singularities, vol. 1, Proc. Symp. Pure Math., **40–1** (1983), pp. 233–254.

[13] J. Damon, *The unfolding and determinacy theorems for subgroups of* A *and* K, Memoirs Amer. Math. Soc., vol.**50**, No. **306**, Amer. Math. Soc. (1984).

[14] J. Damon, D. Mond, $\mathcal{A}$*-codimension and the vanishing topology of discriminant,* Invent. Math., **106** (1991), 217–242.

[15] W. Domitrz, *Local symplectic algebra and simple symplectic singularities of curves,* Fundamenta Mathematicae **204-1** (2009) 57–86.

[16] W. Domitrz, *Zero-dimensional symplectic isolated complete intersection singularities,* Journal of Singularities **6** (2012), 19–26.

[17] W. Domitrz, S. Janeczko, M. Zhitomirskii, *Relative Pincaré lemma, contractibility, quasi-homogeneity and vector fields tangent to a singular variety,* Illinois J. Math. **48–3** (2004), 803–835.

[18] W. Domitrz, S. Janeczko, M. Zhitomirskii, *Symplectic singularities of varieties: the method of algebraic restrictions,* J. für die Reine und Angewandte Mathematik, **618** (2008),197-235.

[19] W. Domitrz, J. H. Rieger, *Volume preserving subgroups of* $\mathcal{A}$ *and* $\mathcal{K}$ *and singularities in unimodular geometry*, Math. Ann. **345** (2009), 783–817.

[20] W. Domitrz, Ż. Trębska, *Symplectic* T_7, T_8 *singularities and Lagrangian tangency orders,* Proc. Edinb. Math. Soc. **55–3** (2012), 657–683.

[21] W. Domitrz, Ż. Trębska, *Symplectic* S_μ *singularities,* Real and complex singularities, Contemp. Math., **569**, Amer. Math. Soc., Providence, RI, (2012), pp. 45–65.

[22] S. Ebey, *The classification of singular points of algebraic curves,* Trans. Amer. Math. Soc., **118** (1965), 454–471.

[23] W. Fulton, *Intersection Theory,* Erbegnisse der Mathematik und ihrer Grenzgebiete 3.Folge, Band 2, Springer-Verlag Berlin Heidelberg (1984).

[24] T. Gaffney, $\mathcal{L}^0$*-equivalence of maps,* Math. Proc. Camb. Phil. Soc., **128** (2000), 479–496.

[25] M.D. Garay, *An isochore versal deformation theorem,* Topology, **43** (2004), 1081–1088.

[26] A.B. Givental, *Singular Lagrangian manifolds and their Lagrangian mappings*, Itogi Nauki Tekh., Ser. Sovrem. Prob. Mat., (Contemporary Problems of Mathematics) **33**, VINITI, 1988, pp. 55–112. English transl., J. Soviet Math. **52** (1990), no. 4, 3246–3278.

[27] G.-M. Greuel, C. Lossen, E. Shustin, *Introduction to Singularities and Deformations,* Springer-Verlag, (2007).

[28] A. Hefez, M.E. Hernandes, *Classification of algebroid plane curves with semigroup* $\langle 6, 9, 19\rangle^{\#}$, Comm. in Algebra, **31–8** (2003), 3847–3861.

[29] M.E. Hernandes, M.E. Rodrigues Hernandes, M.A.S. Ruas, $\mathcal{A}_e$*-codimension of germs of analytic curves,* manuscripta math., **124** (2007), 237–246.

[30] H. Hofer, E. Zhender, *Symplectic Invariants and Hamiltonian Dynamics*, Basel, Boston, 1994

[31] G. Ishikawa, *Families of functions dominated by distributions of* $\mathcal{C}$*-classes of mappings,* Ann. Inst. Fourier, **33-2** (1983), 199-217.

[32] G. Ishikawa, *Parametrized Legendre and Lagrange varieties,* Kodai Math. J. **17–3** (1994), 442–451.

[33] G. Ishikawa, *Symplectic and Lagrange stabilities of open Whitney umbrellas,* Invent. math., **126-2** (1996), 215–234.

[34] G. Ishikawa, *Determinacy, transversality and Lagrange stability,* Banach Center Publ. **50** (1999), 123–135.

[35] G. Ishikawa, *Perturbations of Caustics and Fronts,* in "Geometry and Topology of Caustics — Caustics '02", Banach Center Publ. **62** (2004), 101–116.

[36] G. Ishikawa, S. Janeczko, *Symplectic bifurcations of plane curves and isotropic liftings,* Quarterly Journal of Mathematics, Oxford, 54 (2003), 73–102.

[37] G. Ishikawa, S. Janeczko, *Symplectic singularities of isotropic mappings,* in *Geometric Singularity Theory,* Banach Center Publications **65**, ed. by Heisuke Hironaka, Stanislaw Janeczko, Stanislaw Lojasiewicz. (2004), pp. 85-106.

[38] G. Ishikawa, S. Janeczko, *The complex symplectic moduli spaces of uni-modal parametric plane curve singularities,* Preprint, Institute of Mathematics, Polish Academy of Sciences, **664** (January 2006).

[39] G. Ishikawa, S. Janeczko, *Bifurcations in symplectic space*, Banach Center Publ. **82** (2008),111 - 124.

[40] G. Ishikawa, S. Janeczko, *Symplectic classification of parametric complex plane curves,* Annales Polonici Mathematici, **99–3** (2010), 263–284.

[41] G. Ishikawa, S. Janeczko, *Symplectic invariants of parametric singularities,* in Advances in Geometric Analysis, Advanced Lectures in Mathematics series **21**, ed. by Stanislaw Janeczko, Jun Li, Duong H. Phong, Higher Education Press and International Press, Beijing-Boston (2012), pp. 259–280.

[42] G. Ishikawa, S. Janeczko, *Residual algebraic restrictions of differential forms,* Methods and Applications of Analysis, **24-1** (2017), 045–062.

[43] G. Ishikawa, S. Janeczko, *Singular mappings and their zero-forms,* preprint 2019 (submitted).

[44] S. Janeczko, *Generating families for images of Lagrangian submanifolds and open swallowtails,* Math. Proc. Cambridge Philos. Soc, **100–1** (1986), 91–107.

[45] T. de Jong, D. van Straten, *Disentanglement,* in Singularity Theory and its Applications, Lecture Notes in Math., **1462** (1991), Springer-Verlag, pp. 199–211.

[46] J.B. Keller, *Rays, waves and asymptotics,* Bull. Amer. Math. Soc. 84 (1978), 727-749.

[47] C. Klotz, O. Pop, J.H. Rieger, *Real double points of deformations of $\mathcal{A}$-simple map-germs from $\mathbf{R}^n$ to $\mathbf{R}^{2n}$,* Math. Proc. Camb. Phil. Soc., **142** (2007), 341–363.

[48] P.A. Kolgushkin, *Classification of simple multigerms of curves in a space endowed with a symplectic structure,* St. Petersburg Math. J., **15–1** (2004), 103–126.

[49] J.L. Lagrange, *Mémoire sur la th6orie des variations des 616ments des planets*, Mdmoires de la classe des sciences mathdmatiques et physiques de rinstitut de France, (1808), pp. 1-72.

[50] S.K. Lando, *Normal forms of the degrees of a volume form,* Funct. Anal. Appl. **19–2** (1984), 146–148.

[51] O.A. Laudal, G. Pfister, *Local moduli and singularities,* Lecture Notes in Math., **1310** Springer-Verlag, Berlin (1988).

[52] Lê Dũng Tráng, *Sur un critère d'équisingularité,* C. R. Acad. Sci. Paris Sér. A, **272** (1971), 138–140.

[53] I. Luengo, G. Pfister, *Normal forms and moduli spaces of curve singularities with semigroup $\langle 2p, 2q, 2pq + d\rangle$,* Comp. Math., **76** (1990), 247–264.

[54] B. Malgrange, *Ideals of Differentiable Functions,* Oxford Univ. Press, 1966.

[55] V.P. Maslov, *Thdorie des Perturbations et Methodes Asymptotiques,* Dunod, Gauthier-Villars, Paris, 1972.

[56] J.N. Mather, *Stability of C^∞ mappings III: Finitely determined map-germs,* Publ. Math. I.H.E.S., **35** (1968), 127–156.

[57] R. Melrose, *Equivalence of glancing hypersurfaces,* Invent Math., **37** (1976), 165–191.

[58] R. Melrose, *Equivalence of glancing hypersurfaces II,* Math. Ann., **255–2** (1981), 159–198.

[59] M. Mikosz, *Secondary characteristic classes for the isotropic Grassmannian,* Banach Center Publications, **50** (1999), 195–204.

[60] J. Milnor, *Singular Points of Complex Hypersurfaces,* Annals of Math. Studies, **61**, Princeton Univ. Press, 1986.

[61] D. Mond, *Some remarks on the geometry and classification of germs of maps from surfaces to* 3*-space,* Topology, **26** (1987), 361–383.

[62] D. Mond, *Vanishing cycles for analytic maps,* Singularities Theory and Applications, Warwick 1989. Lecture Notes in Math., **1462**, Springer, Heidelberg (1991), pp. 221–234.

[63] D. Mond, *Looking at bend wires, $\mathcal{A}_e$-codimension and the vanishing topology of parametrized curve singularities,* Math. Proc. Camb. Phil. Soc., **117** (1995), 213–222.

[64] R. Montgomery, M. Zhitomirskii, *Geometric approach to Goursat flags,* Ann. Inst. H. Poincaré, **18–4** (2001), 459–493.

[65] R. Montgomery, M. Zhitomirskii, *Points and curves in the monster tower,* Memoirs of the Amer. Math. Soc. **203(956)** (2010).

[66] P. Mormul, *Real moduli in local classification of Goursat flags,* Hokkaido Math. J., **34–1** (2005), 1–35.

[67] T. Oshima, *On singularities of differential forms,* (in Japanese), Sûrikaiseki-kenkyûsho Kôkyûroku, **227** (1974), 97–108.

[68] T. Oshima, *Local equivalence of differential forms and their deformations,* Sûrikaiseki-kenkyûsho Kôkyûroku, **266** (1976), 108–129.

[69] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields,* J. Fac. Sci. Univ. Tokyo, **27–2** (1980), 265–281.

[70] R. Thom, *Structural Stability and Morphogenesis, An Outline of the General Theory of Models,* Benjamin/Cummings, New York, 1975.

[71] J.-Cl. Tougeron, *Idéaux de fonctions différentiables,* Ergebnisse der Mathematik, **71** Springer-Verlag, (1972).

[72] A.N. Varchenko, *Local classification of volume forms in the presence of a hypersurface,* Funct. Anal. Appl. **19–4** (1984), 269–276.

[73] S.M. Voronin, *The Darboux-Whitney theorem and related questions,* in Nonlinear Stokes Phenomena, S. Il'yashenko ed., Advanced in Soviet Mathematics, **14** (1993), pp. 139–233.

[74] C.T.C. Wall, *Finite determinacy of smooth map-germs,* Bull. London Math. Soc. **13** (1981), 481–539.

[75] C.T.C. Wall, *Singular Points of Plane Curves,* Cambridge Univ. Press (2004).

[76] A. Weinstein, *Lectures on symplectic manifolds,* CBMS Regional Conf. Ser. in Math. 29 (1977).

[77] S.M. Voronin, *The Darboux-Whitney theorem and related questions,* in Nonlinear Stokes Phenomena, S. Il'yashenko ed., Advanced in Soviet Mathematics, **14** (1993), pp. 139–233.

[78] V. M. Zakalyukin, *Simple coisotropic caustics,* Journal of Mathematical Sciences, **124-5** (2004), 5310–5320.

[79] O. Zariski, *Le problème des modules pour les branches planes,* Cours donné au Centre de mathématiques de l'École Polytechnique, 1973, (ed. F. Kmety, M. Merle, with an appendix of B. Tessier), Hermann, Paris (1987).

[80] M. Zhitomirskii, *Typical singularities of differential* 1*-forms and Pfaff systems,* Amer. Math. Soc., Providence, R.I., 1992.

[81] M. Zhitomirskii, *Germs of integral curves in contact 3-space, plane and space curves,* Isaac Newton Inst. preprint NI00043-SGT, December 2000.

[82] M. Zhitomirskii, *Relative Darboux theorem for singular manifolds and local contact algebra,* Canad. J. Math. **57-6** (2005), 1314–1340.

[83] S. Yau, H. Zuo, *Complete characterization of isolated homogeneous hypersurface singularities,* Pacific Journal of Mathematics, **273–1** (2015), 213–224.

[84] S.-T. Yau, S. Nadis, *The Shape of Inner Space*, Basic Books, New York, 2010

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