SINGULAR MAPPINGS AND THEIR ZERO-FORMS

GOO ISHIKAWA AND STANISŁAW JANECZKO

ABSTRACT. We study the quotient complexes of the de Rham complex on singular mappings; the complex of algebraic restrictions, the complex of geometric restrictions and the residual complex. Vanishing theorem for algebraic, geometric and residual cohomologies on quasi-homogeneous map-germs was proved. The finite order and symplectic zero-forms were characterized on parametric singularities. In this context the singular parametric Lagrangian surfaces were investigated, with the classification list of $\mathscr{A}$-simple Lagrangian singularities of $\mathbb{R}^2$ into $\mathbb{R}^4$.

1. INTRODUCTION

We consider smooth or holomorphic map-germs $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^m, 0)$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The set of such map-germs is denoted by $\mathscr{E}_{n,m}$.

Let Λ^q_m denote the space of germs of q-forms of m-variables at zero. Note that $\Lambda^0_m = \mathscr{E}_m$ is the space of function-germs on $(\mathbb{F}^m, 0)$. The subspace Z^q_f of q-forms ω, with vanishing pullbacks (geometric restriction to the image of f) $f^*\omega = 0$ is called the space of *zero forms on* f ([6]). This is a module over smooth (or holomorphic) function-germs and its properties depend heavily on n, m, q and the singularity of f.

In this paper we study problems related to zero forms on map-germs from various viewpoints and provide some observations on them.

One of main problems in geometric singularity theory is the classification of the pairs (f, ω) such that ω is a zero-form on f. Two pairs (f, ω) and (f', ω') are equivalent if there exist diffeomorphisms σ on $(\mathbb{F}^n, 0)$ and τ on $(\mathbb{F}^m, 0)$ such that $f' = \tau \circ f \circ \sigma^{-1}$ and $\omega = \tau^*\omega'$. If ω is a symplectic form, then the problem is weakened to the classification and the characterization under the left-right equivalence of map-germs having zero-form which is symplectic ([5]). Regarding Darboux theorem for symplectic forms, the problem is reduced, for a fixed symplectic form ω, to classify map-germs under right-left equivalences (σ, τ) with $\tau^*\omega = \omega$, i.e. τ is a symplectomorphism. Such a classification problem is understood well by introducing the notion of algebraic restrictions of differential forms ([3]). In §2, we observe the related notions for the study of zero forms of map-germs.

By the condition $f^*\omega = 0$ that ω is a zero form on f is approximated by the nullity of finite jets $j^k(f^*\omega)(0) = 0$ of forms. In §3, we provide several observations on the "order of nullity" or "order of isotropness" for map-germs and differential forms.

Key words: Differential forms, Singularities, Geometric restriction, Algebraic restriction, Residual cohomology, Parametric curves and surfaces.

2010 *Mathematics Subject Classification*: Primary 53D05. Secondary 58K05, 57R42, 58A10.

The Darboux normal form for symplectic forms is linear, i.e. represented by its 0-jet. Then, for a fixed system of coordinates on $(\mathbb{F}^m, 0)$, with even m, it has a sense to ask the existence of linear symplectic zero forms for a given map-germ $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^m, 0)$ related to the original problem. In §4 we provide several basic observations on the problem on the existence of symplectic zero forms and in §5, in particular the case $n = 2, m = 4$. Moreover related to the results in §5, we provide some examples of map-germs with many symplectic zero forms in §6.

The authors would like to thank an anonymous referee for his/her valuable comment and suggestion.

2. Algebraic, geometric and residual cohomologies of map-germs

Let (Λ^*_m, d) be de Rham complex over $(\mathbb{F}^m, 0)$. Then (Z^*_f, d), the pair of the differential ideal of zero forms on f and the exterior differential d, is a sub-complex of (Λ^*_m, d). Moreover we consider the differential ideal AZ^*_f in Λ^*_m generated by

$$Z^0_f = \{h \in \Lambda^0_m \mid f^*h = 0\},$$

namely,

$$\begin{aligned} AZ^q_f &:= Z^0_f\Lambda^q_m + d(Z^0_f)\Lambda^{q-1}_m \\ &= \{\textstyle\sum_{i=1}^r h_i\alpha_i + \sum_{j=1}^s (dk_j)\wedge\beta_j \mid h_i \in Z^0_f, \alpha_i \in \Lambda^q_m, k_j \in Z^0_f, \beta_j \in \Lambda^{q-1}_m\}. \end{aligned}$$

Then $AZ^q_f \subset Z^q_f$ for any q. We call the forms in AZ^*_f *algebraically zero forms* on f. Then (AZ^*_f, d) is a sub-complex of (Z^*_f, d). This is the parametric version of algebraically zero forms on subsets of manifolds introduced in [3]. In fact we have

Lemma 2.1. *If $f : U(\subset \mathbb{F}^n) \to V(\subset \mathbb{F}^m)$ be a representative of f, and $Z = f(U)$, then the set of algebraically null forms on Z in $\Lambda^q(V)$ is equal to the set of forms $\gamma + d(\delta)$ with $\gamma \in \Lambda^q(V), \gamma(z) = 0 (z \in Z)$ and $\delta \in \Lambda^{q-1}(V), \delta(z) = 0 (z \in Z)$.*

Now, by setting $\mathscr{A}^*_f := \Lambda^*_m/AZ^*_f, \mathscr{G}^*_f := \Lambda^*_m/Z^*_f$ and $\mathscr{R}^*_f := Z^*_f/AZ^*_f$, we have the quotient complexes $(\mathscr{A}^*_f, \overline{d}), (\mathscr{G}^*_f, \overline{d})$ and $(\mathscr{R}^*_f, \overline{d})$, which we call the complex of algebraic restrictions, the complex of geometric restrictions and the residual complex on f respectively (see [6]). Then we have the exact sequences of complexes

$$\text{(i)}\ \ 0 \longrightarrow (AZ^*_f, d) \longrightarrow (\Lambda^*_m, d) \longrightarrow (\mathscr{A}^*_f, \overline{d}) \longrightarrow 0,$$

$$\text{(ii)}\ \ 0 \longrightarrow (Z^*_f, d) \longrightarrow (\Lambda^*_m, d) \longrightarrow (\mathscr{G}^*_f, \overline{d}) \longrightarrow 0,$$

$$\text{(iii)}\ \ 0 \longrightarrow (AZ^*_f, d) \longrightarrow (Z^*_f, d) \longrightarrow (\mathscr{R}^*_f, \overline{d}) \longrightarrow 0,$$

and

$$\text{(iv)}\ \ 0 \longrightarrow (\mathscr{R}^*_f, \overline{d}) \longrightarrow (\mathscr{A}^*_f, \overline{d}) \longrightarrow (\mathscr{G}^*_f, \overline{d}) \longrightarrow 0.$$

Note that $AZ^0_f = Z^0_f$ and therefore $\mathscr{R}^0_f = 0$.

Definition 2.2. We call the cohomology $H^\bullet(\mathscr{A}^*_f, \overline{d}), H^\bullet(\mathscr{G}^*_f, \overline{d})$ and $H^\bullet(\mathscr{R}^*_f, \overline{d})$ the *algebraic cohomology*, the *geometric cohomology* and the *residual cohomology* on f respectively.

These objects are invariant under the right-left equivalence of map-germs: If f is right-left equivalent to a germ g, then each cohomology of f and g are isomorphic. The algebraic and geometric cohomologies are studied in [2] for arbitrary subsets in manifolds. The homogeneity and quasi-homogeneity are important notions in singularity theory. See for the characterization problem of (quasi-)homogeneity the papers [8][1][9][10][11][12][13][14]. Here we intend to reformulate the results in [2] for map-germs and apply them to the study on zero forms, regarding the notion of homogeneity of map-germs in a generalized sense.

A map-germ $f = (f_1,\dots,f_m) : (\mathbb{F}^n,0) \to (\mathbb{F}^m,0)$ is called *weakly quasi-homogeneous* if there exist non-negative integers $\lambda_1,\dots,\lambda_m$ and $\mu_1,\dots,\mu_n$ such that

$$f(t^{\mu_1}x_1,\dots,t^{\mu_n}x_n) = (t^{\lambda_1}f_1(x_1,\dots,x_n),\dots,t^{\lambda_m}f_m(x_1,\dots,x_n)).$$

Suppose, by some permutations of coordinates, that $\lambda_i = 0 (1 \le i \le m_1), \lambda_i > 0 (m_1+1 \le i \le m)$ and that $\mu_i = 0 (1 \le i \le n_1), \mu_i > 0 (n_1+1 \le i \le n)$. Define the families of map-germs, for $t \ge 0$, $\varphi_t : (\mathbb{F}^n,0) \to (\mathbb{F}^n,0)$ and $\Phi_t : (\mathbb{F}^m,0) \to (\mathbb{F}^m,0)$ by

$$\varphi_t(x_1,\dots,x_n) = (t^{\mu_1}x_1,\dots,t^{\mu_n}x_n), \quad \Phi_t(y_1,\dots,y_m) = (t^{\lambda_1}y_1,\dots,t^{\lambda_m}y_m).$$

Then we have $f \circ \varphi_t = \Phi_t \circ f : (\mathbb{F}^n,0) \to (\mathbb{F}^m,0)$. Moreover φ_t (resp. Φ_t) defines a contraction of $(\mathbb{F}^n,0)$ to $(\mathbb{F}^{n_1}\times 0,0)$ (resp. a contraction of $(\mathbb{F}^m,0)$ to $(\mathbb{F}^{m_1}\times 0,0)$). Note that φ_t (resp. Φ_t) is smooth or holomorphic on $(x_1,\dots,x_n)$ (resp. on $(y_1,\dots,y_m)$) and is smooth on t. Define $f^1 : (\mathbb{F}^{n_1},0) \to (\mathbb{F}^{m_1},0)$, by $f^1 := p_1 \circ f \circ i_1$, called the *zero-weight part* of f, where

$$i_1(x_1,\dots,x_{n_1}) = (x_1,\dots,x_{n_1},0,\dots,0) \text{ and } p_1(y_1,\dots,y_m) = (y_1,\dots,y_{m_1}),$$

In general, we say that f is *contractible* to f^1 if there exist contractions φ_t of $(\mathbb{F}^n,0)$ to $(\mathbb{F}^{n_1}\times 0,0)$ and Φ_t from $(\mathbb{F}^m,0)$ to $(\mathbb{F}^{m_1}\times 0,0)$, smooth or holomorphic on $(x_1,\dots,x_n)$ and on $(y_1,\dots,y_m)$, smooth on t respectively, such that $f \circ \varphi_t = \Phi_t \circ f$ with

$$\varphi_t|_{\mathbb{F}^{n_1}\times 0} = \mathrm{id}_{\mathbb{F}^{n_1}\times 0}, \varphi_1 = \mathrm{id}_{\mathbb{F}^n}, \varphi_0(\mathbb{F}^n) \subset \mathbb{F}^{n_1}\times 0,$$

$$\Phi_t|_{\mathbb{F}^{m_1}\times 0} = \mathrm{id}_{\mathbb{F}^{m_1}\times 0}, \Phi_1 = \mathrm{id}_{\mathbb{F}^m}, \Phi_0(\mathbb{F}^m) \subset \mathbb{F}^{m_1}\times 0.$$

Since $f \circ \varphi_0 = \Phi_0 \circ f$, we see that $f|_{\mathbb{F}^{n_1}\times 0}$ is a mapping to $\mathbb{F}^{m_1}\times 0$, is identified with $f^1 = \Phi_0 \circ f \circ i_1$ using the above notations. Note that $\Phi_0 = p_1$ in the quasi-homogeneous case. Then, based on the ideas in [2] applied and modified to our parametric version, we have the following result:

Lemma 2.3. *If f is contractible to f^1, then $H^\bullet(AZ^*_f,d)$ and $H^\bullet(AZ^*_{f^1},d)$ (resp. $H^\bullet(Z^*_f,d)$ and $H^\bullet(Z^*_{f^1},d)$) are isomorphic. Moreover the algebraic (resp. geometric, residual) cohomology of f is isomorphic to the algebraic (resp. geometric, residual) cohomology of f^1.*

Proof: Let $j_1 : (\mathbb{F}^{m_1},0) \to (\mathbb{F}^m,0)$ be the inclusion defined by

$$j_1(y_1,\dots,y_{m_1}) = (y_1,\dots,y_{m_1},0,\dots,0).$$

Let $h \in Z^0_f$. Then $(f^1)^*(j_1^*h) = (j_1\circ f^1)^*h = (j_1\circ\Phi_0\circ f\circ i_1)^*h = (f\circ i_1)^*h = i_1^*(f^*h) = 0$. Therefore we have $j_1^*(AZ^q_f) \subset AZ^q_{f^1}$. Hence j_1 induces a morphism $(j_1)^*_{AZ} : H^q(AZ^*_f,d) \to$

$H^q(AZ^*_{f^1}, d)$. Similarly we have $j_1^*(Z^q_f) \subset Z^q_{f^1}$ and j_1 induces a morphism $(j_1)^*_Z : H^q(AZ^*_f, d) \to H^q(Z^*_{f^1}, d)$.

To show $(\overline{j}_1)^*_{AZ}$ is surjective, take any $\omega \in AZ^q_{f^1}$ with $d\omega = 0$. Consider $\Phi_0^*\omega$ where Φ_0 is regarded as a map-germ $(\mathbb{F}^m, 0) \to (\mathbb{F}^{m_1}, 0)$. Then $d(\Phi_0^*\omega) = \Phi_0^*(d\omega) = 0$. Now $\Phi_0^* : \Lambda^q_{m_1} \to \Lambda^q_m$ satisfies $\Phi_0^*(AZ^q_{f^1}) \subset AZ^q_f$. In fact, let $k \in Z^0_{f^1}$. Then $f^*(\Phi_0^*k) = (\Phi_0 \circ f)^*k = (f^1 \circ \varphi_0)^*k = \varphi^*(f^1)^*k = 0$. Thus $\Phi_0^*k \in AZ^q_f$. We have $d(\Phi_0^*\omega) \in AZ^{q+1}_f$ and $j_1^*(\Phi_0^*\omega) = (\Phi_0 \circ j_1)^*\omega = \omega$. Therefore $(\overline{j}_1)^*_{AZ}([\Phi_0^*\omega]) = [\omega]$, and we have that $(\overline{j}_1)^*_{AZ}$ is surjective. Similarly we have $\Phi_0^*(Z^q_{f^1}) \subset Z^q_f$ and $(\overline{j}_1)^*_Z$ is surjective.

Let us show $(\overline{j}_1)^*_{AZ}$ and $(\overline{j}_1)^*_Z$ are injective. Take $\omega \in AZ^{q+1}_f$ with $d\omega = 0$. Suppose $(\overline{j}_1)^*_{AZ}[\omega] = 0$, i.e. $j_1^*\omega = d\eta$ for some $\eta \in AZ^q_{f^1}$. We have $\Phi_1^*\omega - \Phi_0^*\omega = \int_0^1 (\frac{d}{dt}\Phi_t^*\omega)dt = \int_0^1 \Phi_t^*(L_{V_t}\omega)dt$, where $V_t = \frac{d\Phi_t}{dt}$ as a vector field along Φ_t. Since $L_{V_t}\omega = V_t \rfloor d\omega + d(V_t \rfloor \omega) = d(V_t \rfloor \omega)$ and $\Phi_1 = \mathrm{id}_{\mathbb{F}^m}$, we have

$$\omega = \Phi_0^*\omega + d\alpha, \quad \alpha = \int_0^1 (V_t \rfloor \omega)dt.$$

Since $\Phi_0^*\omega = \Phi_0^* j_1^*\omega = \Phi_0^*(d\eta) = d(\Phi_0^*\eta)$, we have $\omega = d(\Phi_0^*\eta + \alpha)$, with $\Phi_0^*\eta + \alpha \in AZ^q_f$. So $[\omega] = 0 \in H^q(AZ^*_f, d)$. Therefore $(\overline{j}_1)^*_{AZ}$ is injective. Thus we have that $(j_1)^*_{AZ} : H^q(AZ^*_f, d) \to H^q(AZ^*_{f^1}, d)$ is an isomorphism. Similarly we have $(\overline{j}_1)^*_Z$ is injective. Note that if $\omega \in Z^{q+1}_f$ then α defined as above belongs to Z^q_f, since $f \circ \varphi_t = \Phi_t \circ f$ and V_t is contained in the image of differential map of f. Thus we have that $(j_1)^*_Z : H^q(Z^*_f, d) \to H^q(Z^*_{f^1}, d)$ is an isomorphism.

Moreover we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (AZ^*_f, d) & \longrightarrow & (\Lambda^*_m, d) & \longrightarrow & (\mathscr{A}^*_f, \overline{d}) & \longrightarrow & 0, \\
 & & \downarrow{\scriptstyle (j^1)^*} & & \downarrow{\scriptstyle (j^1)^*} & & \downarrow{\scriptstyle (j^1)^*_{\mathscr{A}}} & & \\
0 & \longrightarrow & (AZ^*_{f^1}, d) & \longrightarrow & (\Lambda^*_{m_1}, d) & \longrightarrow & (\mathscr{A}^*_{f^1}, \overline{d}) & \longrightarrow & 0,
\end{array}$$

of complexes induced by j^1, related to the exact sequence (i), and the induced homomorphism $(\overline{j}_1)^*_{\mathscr{A}} : H^q(\mathscr{A}^*_f, \overline{d}) \to H^q(\mathscr{A}^*_{f^1}, \overline{d})$. Similarly we have the induced morphism $(j_1)^*_{\mathscr{G}} : \mathscr{G}^q_f \to \mathscr{G}^q_{f^1}$ and the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (Z^*_f, d) & \longrightarrow & (\Lambda^*_m, d) & \longrightarrow & (\mathscr{G}^*_f, \overline{d}) & \longrightarrow & 0, \\
 & & \downarrow{\scriptstyle (j^1)^*} & & \downarrow{\scriptstyle (j^1)^*} & & \downarrow{\scriptstyle (j^1)^*_{\mathscr{G}}} & & \\
0 & \longrightarrow & (Z^*_{f^1}, d) & \longrightarrow & (\Lambda^*_{m_1}, d) & \longrightarrow & (\mathscr{G}^*_{f^1}, \overline{d}) & \longrightarrow & 0,
\end{array}$$

related to the exact sequence (ii), and thus the induced homomorphism $(\overline{j}_1)^*_{\mathscr{G}} : H^q(\mathscr{G}^*_f, \overline{d}) \to H^q(\mathscr{G}^*_{f^1}, \overline{d})$.

Regarding the long exact sequences of cohomologies, by virtue of the fact that de Rham complexes are acyclic, the Poincaré lemma, we have the commutative diagram

$$\begin{array}{ccccccccc}
H^q(AZ^*_f,d) & \longrightarrow & H^q(\Lambda^*_m,d) & \longrightarrow & H^q(\mathscr{A}^*_f,\overline{d}) & \longrightarrow & H^{q+1}(AZ^*_f,d) & \longrightarrow & H^{q+1}(\Lambda^*_m,d), \\
{\scriptstyle (\bar{j}^1)^*_{AZ}}\downarrow & & {\scriptstyle (\bar{j}^1)^*_{\Lambda}}\downarrow & & {\scriptstyle (\bar{j}^1)^*_{\mathscr{A}}}\downarrow & & {\scriptstyle (\bar{j}^1)^*_{AZ}}\downarrow & & {\scriptstyle (\bar{j}^1)^*_{\Lambda}}\downarrow \\
H^q(AZ^*_{f^1},d) & \longrightarrow & H^q(\Lambda^*_{m_1},d) & \longrightarrow & H^q(\mathscr{A}^*_{f^1},\overline{d}) & \longrightarrow & H^{q+1}(AZ^*_{f^1},d) & \longrightarrow & H^{q+1}(\Lambda^*_{m^1},d),
\end{array}$$

with isomorphisms $(\bar{j}^1)^*_{AZ}$ and $(\bar{j}^1)^*_{\Lambda}$. Thus, by the five lemma, we have that $(\bar{j}_1)^*_{\mathscr{A}}$ is an isomorphism. Similarly we have that $(\bar{j}_1)^*_{\mathscr{G}}$ is an isomorphism. Finally by the exact sequence (iii) or (iv), we have that $H^q(\mathscr{R}^*_f,\overline{d})$ and $H^q(\mathscr{R}^*_{f^1},\overline{d})$ are isomorphic. □

A map-germ f is called *contractible* if there exists a sequence of map-germs $f^i : (\mathbb{F}^{n_i},0) \to (\mathbb{F}^{m_i},0), (1 \le i \le r)$ with $n_1 \ge n_2 \ge \cdots \ge n_r = 0, m_1 \ge m_2 \ge \cdots \ge m_r = 0$ such that f^{i-1} is contractible to $f^i, (1 \le i \le r)$ with $f^0 = f$.

Theorem 2.4. (Vanishing theorem [2]) *Let $f : (\mathbb{F}^n,0) \to (\mathbb{F}^m,0)$ be right-left equivalent to a contractible map-germ in the above sense. Then the algebraic and geometric complexes of f are acyclic, i.e.,*

$$H^q(\mathscr{A}^*_f,\overline{d}) = 0, (q \ne 0), \quad H^0(\mathscr{A}^*_f,\overline{d}) = \mathbb{R},$$

$$H^q(\mathscr{G}^*_f,\overline{d}) = 0, (q \ne 0), \quad H^0(\mathscr{G}^*_f,\overline{d}) = \mathbb{R}.$$

Furthermore we have that the residual cohomologies vanish:

$$H^q(\mathscr{R}^*_f,\overline{d}) = 0, \text{ for any } q,$$

Proof: First note that our cohomologies are invariant under the right-left equivalence of map-germs. Then by Lemma 2.3 and that $(\mathscr{A}^*_{f^r},\overline{d})$ is acyclic for $f^r : \mathbb{F}^0 \to \mathbb{F}^0$, we have $H^q(\mathscr{A}^*_f,\overline{d}) \cong H^q(\mathscr{A}^*_{f^1},\overline{d}) \cong H^q(\mathscr{A}^*_{f^2},\overline{d}) \cong \cdots \cong H^q(\mathscr{A}^*_{f^r},\overline{d})$, which is 0 if $q \ne 0$ and is isomorphic to $\mathbb{R}$ if $q = 0$. The proof for $(\mathscr{G}^*_{f^r},\overline{d})$ is similar. Finally by the long exact sequence of (iv), we have the result for $H^\bullet(\mathscr{R}^*_f,\overline{d})$. □

Since $\mathscr{R}^0_f = 0$ in general, we have

Corollary 2.5. *If $f : (\mathbb{F}^n,0) \to (\mathbb{F}^m,0)$ be contractible, then the sequence*

$$0 \longrightarrow \mathscr{R}^1_f \xrightarrow{\overline{d}} \mathscr{R}^2_f \xrightarrow{\overline{d}} \cdots \xrightarrow{\overline{d}} \mathscr{R}^{m-1}_f \xrightarrow{\overline{d}} \mathscr{R}^m_f \longrightarrow 0,$$

induced by the exterior differential is exact.

Remark 2.6. We call f *quasi-homogeneous in the generalized sense* if f is weakly quasi-homogeneous, the zero-weight part $f^1 : (\mathbb{F}^{n_1},0) \to (\mathbb{F}^{m_1},0)$ of f is weak quasi-homogeneous, the zero-weight part f^2 of f^1 is weak quasi-homogeneous, and so on, with $n_1 \ge n_2 \ge \cdots \ge n_r = 0, m_1 \ge m_2 \ge \cdots \ge m_r = 0$ for some r. Then f is contractible, and therefore we have the same results as in Theorem 2.4 for such an f.

Example 2.7. The formulations in this section coincide with those in [3], if f has a well-defined image Z as a set-germ of $(\mathbb{F}^m, 0)$, for instance, if $n \leq m$ and f is finite, i.e. $\dim_{\mathbb{F}}(\mathscr{E}_n / f^*\mathfrak{m}_m\mathscr{E}_n) < \infty$. However in general the image-germ of a map-germ is not necessarily well-defined, for instance, for the map-germ $\pi : (\mathbb{F}^2, 0) \to (\mathbb{F}^2, 0)$ defined by $\pi(x_1, x_2) = (x_1, x_1x_2)$, the germ of image is not well-defined.

3. Finite order zero-forms on parametric singularities

There is a natural stratification of Λ^q_m associated with an order of multiplicity of geometric restriction of differential forms. Let $\omega \in \Lambda^q_m$. We say that the order of vanishing of the germ ω is k if $(j^{(k-1)}\omega)(0) = 0$ and $(j^k\omega)(0) \neq 0$. By $\Lambda^q_{m,k}$ we denote the germs of q-forms of m-variables at zero having order of vanishing $\geq k$. (cf. [3, 6, 11]). Note that $\Lambda^q_{m,k} = \mathfrak{m}^k_m \Lambda^q_m$, where $\mathfrak{m}_m = \{h \in \Lambda^0_m = \mathscr{E}_m \mid h(0) = 0\}$.

Let $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^m, 0)$ be a smooth map-germ at zero. Now the finite order zero forms are defined as follows.

$$Z^q_{f,k} = \{\omega \in \Lambda^q_m : f^*\omega \in \Lambda^q_{n,k}\}.$$

And we have the sequence of ideals in Λ^q_m :

$$Z^q_f \subset \ldots \subset Z^q_{f,k+1} \subset Z^q_{f,k} \subset Z^q_{f,k-1} \subset \ldots \subset Z^q_{f,0} = \Lambda^q_m.$$

In C^∞ case $Z^q_{f,\infty}$ means that $f^*\omega$ has the zero Taylor expansion. The corresponding sequence

$$d^q_{f,k} = \dim \frac{Z^q_{f,k}}{Z^q_{f,k+1}}$$

defines the *invariant spectrum of the approximation*.

If $f : N \to \mathbb{R}^{2n}$ is a smooth mapping from a C^∞ manifold N, and we denote $Z = f(N)$, then there is a natural symplectic invariant of Z in the symplectic space $(\mathbb{R}^{2n}, \omega)$ called the index of isotropness of Z defined as a maximal order of vanishing of the two forms $\omega\mid_{TM}$ over all non-singular submanifolds M containing Z. If Z is contained in a non-singular Lagrangian submanifold, then the index of isotropness is ∞. This is a measure of maximal order of tangency between non-singular submanifolds containing Z and non-singular isotropic submanifolds of the same dimension (see [3]).

We define the *index of isotropness* for a map-germ $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^m, 0)$ by

$$\mathscr{I}(f) := \sup\{\mathrm{ord}(f^*\omega) \mid \omega : \text{symplectic forms on } (\mathbb{F}^m, 0)\}.$$

Then clearly we have

Lemma 3.1. *The index of isotropness $\mathscr{I}(f)$ is an invariant of the right-left equivalence class of f. Moreover $\mathscr{I}(f) = \infty$ if and only if f is isotropic for some symplectic form on $(\mathbb{F}^m, 0)$.*

4. SYMPLECTIC ZERO-FORMS ON PARAMETRIC SINGULARITIES

A smooth 2-form $\Omega \in \Lambda^2_m$ is called *linear* (for the system of coordinates $x_1, \dots, x_m$ of $\mathbb{R}^m$) if Ω is of the form $\sum_{i<j} a_{ij} dx_i \wedge dx_j$ for some $a_{ij} \in \mathbb{F}$. We denote by L^2_m the space of linear 2-forms on $(\mathbb{F}^m, 0)$ which is isomorphic to $\wedge^2(T^*_0\mathbb{F}^m)$. There is the evaluation map $\Lambda^2_m \to L^2_m, \omega \mapsto \omega(0) (= \omega|_{T_0\mathbb{F}^m})$, where $\omega(0)$ is regarded as a linear form. Then, for the given coordinates on $(\mathbb{F}^m, 0)$, we have the decomposition $\Lambda^2_m = L^2_m \oplus \mathfrak{m}_m \Lambda^2_m$, where $\mathfrak{m}_m \subset \Lambda^0_m$ is the maximal ideal of the $\mathbb{R}$-algebra $\Lambda^0_m = \mathscr{E}_m$, the algebra of all function-germs $(\mathbb{F}^m, 0) \to \mathbb{F}$.

Given $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^m, 0)$, let us set

$$\widetilde{L}Z^2_f := \{\omega(0) \mid \omega \in Z^2_f\}, \quad \widetilde{r}(f) := \max\{\mathrm{rank}(\omega(0)) \mid \omega \in Z^2_f\}.$$

Note that, if f and g are $\mathscr{A}$-equivalent, then $\widetilde{r}(f) = \widetilde{r}(g)$.

Moreover we set $LZ^2_f = L^2_{2n} \cap Z^2_f$, the space of linear 2-forms Ω satisfying $f^*\Omega = 0$, and set

$$r(f) := \max\{\mathrm{rank}(\Omega) \mid \Omega \in LZ^2_f\}.$$

Note that $LZ^2_f \subset \widetilde{L}Z^2_f$ and both $LZ^2_f, \widetilde{L}Z^2_f$ are linear subspaces of the space $L^2_{2n} \cong \wedge^2(T^*_0\mathbb{F}^{2n})$ of linear 2-forms on $\mathbb{F}^{2n}$. We set

$$R(f) := \max\{r(g) \mid g \sim_{\mathscr{L}} f\} = \max\{r(g) \mid g \sim_{\mathscr{A}} f\}.$$

Then we have that $0 \leq r(f) \leq R(f) \leq \widetilde{r}(f) \leq m$.

A differential 2-form $\omega \in \Lambda^2_m$ is called *symplectic* if ω is non-degenerate and closed.

Let $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^m, 0)$ be a map-germ and $\omega \in Z^2_f$ a symplectic zero form of f. Then, since ω is non-degenerate, we have $\omega \notin AZ^2_f$ and therefore $[\omega] \neq 0$ in $\mathscr{R}^2_f$. Moreover, since f is closed, $[\omega] \in \mathrm{Ker}(\overline{d} : \mathscr{R}^2_f \to \mathscr{R}^3_f)$. If f is contractible in the sense of §2, then by Corollary 2.5 there exists the unique $[\alpha] \in \mathscr{R}^1_f$ such that $\alpha \in \Lambda^1_n$, $f^*\alpha = 0$, and $[\omega] = \overline{d}[\alpha] = [d\alpha]$.

A linear 2-form $\Omega = \sum_{i<j} a_{ij} dx_i \wedge dx_j$ is symplectic if and only if Ω is non-degenerate i.e. $\det(a_{ij}) \neq 0$, where we set $a_{ji} = -a_{ij}$ for $i < j$ and $a_{ii} = 0$. Note that any linear symplectic form is transformed to the Darboux normal form $\sum_{i=1}^n dx_i \wedge dx_{n+i}$ by a linear transformation of $\mathbb{F}^{2n}$. If m is odd, then there are no symplectic forms on $(\mathbb{F}^m, 0)$. Let m be even and $m = 2n$. Let P denote the Pfaffian of the skew-symmetric matrix (a_{ij}). Note that P is a homogeneous polynomial of degree n of variables a_{ij}. Then the non-symplectic forms in L^2_{2n} form a hypersurface Σ defined by $P = 0$.

Let ω be a symplectic form on $\mathbb{F}^{2n}$. A map-germ $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^{2n}, 0)$ is called a (parametric) *Lagrangian map-germ for* ω, if $f^*\omega = 0$.

Then we propose the problem:

Characterize map-germs $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^{2n}, 0)$ such that Z^2_f contains a smooth (or holomorphic) symplectic form on $(\mathbb{F}^{2n}, 0)$. In other words, characterize possible singularities of parametric Lagrangian map-germs.

Then we naturally concern the condition that $\widetilde{r}(f) = 2n$, $R(f) = 2n$ or $r(f) = 2n$.

The followings are clear.

Lemma 4.1. *We have, for a map-germ $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^{2n}, 0)$,*

(1) *If $\widetilde{r}(f) < 2n$, then f is never Lagrangian, for any symplectic form on $(\mathbb{F}^{2n}, 0)$.*

(2) *$r(f) = 2n$ if and only if $LZ^2_f \setminus \Sigma \neq \emptyset$.*

(3) *If $r(f) = 2n$, then f is Lagrangian for a linear symplectic form on $(\mathbb{F}^{2n}, 0)$.*

(4) *If $R(f) = 2n$, then f is $\mathscr{L}$-equivalent to a Lagrangian map-germ for a linear symplectic form on $(\mathbb{F}^{2n}, 0)$.*

Note that $LZ^2_f \setminus \Sigma$ and $\widetilde{L}Z^2_f \setminus \Sigma$ are invariant under $\mathbb{R}^\times$-multiplication, and semi-algebraic. Therefore $P(LZ^2_f \setminus \Sigma), P(\widetilde{L}Z^2_f \setminus \Sigma)$ are defined as semi-algebraic sets in the projective space $P(L^2_{2n}) \cong P^{n(2n-1)-1}$. Moreover we have

Lemma 4.2. *If f and g are right-equivalent, then $Z^2_f = Z^2_g$, and $\widetilde{L}Z^2_f = \widetilde{L}Z^2_g$.*

We define

$$\widetilde{\ell}(f) := \dim P(\widetilde{L}Z^2_f \setminus \Sigma).$$

If f and g are $\mathscr{A}$-equivalent, then $\widetilde{\ell}(f) = \widetilde{\ell}(g)$.

We consider, given $f \in \mathscr{E}_{n,2n}$, the sets $P(LZ^2_g \setminus \Sigma) \subset P(L^2_{2n})$ for all germs $g \in \mathscr{E}_{n,2n}$ which are left equivalent to f. Then define

$$\ell(f) := \max\{\dim P(LZ^2_g \setminus \Sigma) \mid g \sim_{\mathscr{L}} f\}$$

where we define that the dimension of the empty set $\dim(\emptyset) = -1$. Then we see

$$\ell(f) = \max\{\dim P(LZ^2_g \setminus \Sigma) \mid g \sim_{\mathscr{A}} f\}.$$

In fact, the inequality $\leq$ is clear. Moreover, if g is $\mathscr{A}$-equivalent to f, then g is right equivalent to g' such that g' is left equivalent to f. Then by Lemma 4.2, we have $L^2_{g'} = L^2_g$, and therefore we have the required equality.

Now we have

$$-1 \leq \ell(f) \leq \widetilde{\ell}(f) \leq n(2n-1) - 1.$$

Then we obtain

Lemma 4.3. *For an $f \in \mathscr{E}_{n,2n}$, the following conditions are equivalent to each other:*

(i) *f is Lagrangian for some symplectic form on $(\mathbb{F}^{2n}, 0)$.*

(ii) *$R(f) = 2n$.*

(iii) *$\ell(f) \geq 0$.*

Proof: (i) $\Rightarrow$ (iii): Let f be Lagrangian for a symplectic form ω on $\mathbb{F}^{2n}$. By the Darboux theorem, there exists a diffeomorphism-germ $\tau : (\mathbb{F}^{2n}, 0) \to (\mathbb{F}^{2n}, 0)$ such that $\omega = \tau^*(\Omega)$

for the linear symplectic form $\Omega = \sum_{i=1}^{n} dx_i \wedge dx_{n+i}$, Darboux normal form. Set $g = \tau \circ f$. Then g is left equivalent to f and $g^*\Omega = f^*\omega = 0$. Therefore $\Omega \in LZ_g^2 \setminus \Sigma$, hence $\dim P(LZ_g^2 \setminus \Sigma) \neq \emptyset$, and $\ell(f) \geq 0$.

(iii) $\Rightarrow$ (ii) : By (iii), there exists $g \in \mathscr{E}_{n,2n}$ such that g is $\mathscr{L}$-equivalent to f and $r(g) = 2n$. Therefore we have (ii).

(ii) $\Rightarrow$ (i) : Suppose $R(f) = 2n$. Then there exists $g \in \mathscr{E}_{n,2n}$ such that g is left equivalent to f and a linear symplectic form Ω on $(\mathbb{F}^{2n}, 0)$ with $g^*\Omega = 0$. Since g is left equivalent to f, there exists a diffeomorphism-germ $\tau : (\mathbb{F}^{2n}, 0) \to (\mathbb{F}^{2n}, 0)$ such that $g = \tau \circ f$. Set $\omega = \tau^*\Omega$. Then ω is a symplectic form on $(\mathbb{F}^{2n}, 0)$ and $f^*\omega = f^*(\tau^*\Omega) = g^*\Omega = 0$. Therefore f is Lagrangian for some symplectic form on $(\mathbb{F}^{2n}, 0)$. □

Lemma 4.4. *If $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^{2n}, 0)$ satisfies the condition that $\{t \in (\mathbb{F}^n, 0) \mid \operatorname{rank}(f_* : T_t\mathbb{F}^n \to T_{f(t)}\mathbb{F}^{2n}) \geq 2\}$ is dense in $(\mathbb{F}^n, 0)$. Then $\widetilde{\ell}(f) \leq n(2n-1) - 2$ and therefore $\ell(f) \leq n(2n-1) - 2$.*

Proof: Suppose $\widetilde{\ell}(f) = n(2n-1) - 1$. Then $LZ_f^2 \setminus \Sigma$ contains a non-void open set U in L_{2n}^2. By the assumption, there exists a two-dimensional plane $\Pi \subset T_0\mathbb{F}^{2n}$ such that $\Omega|_\Pi = 0$ for any $\Omega \in U$. Then for any $1 \leq i < j \leq 2n$, $dx_i \wedge dx_j = 0$ on Π. Then we have a contradiction. Therefore $\widetilde{\ell}(f) \leq n(2n-1) - 2$. □

Now we remark a general result which is going to be applied to our case.

Let $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^m, 0)$ be a map-germ whose immersion locus is dense. Then *Nash limit set* $N(f)$ of f is the closure of the set of n-planes Π in $\mathrm{Gr}(n, \mathbb{F}^m)$, Grassmannian of n-planes in $\mathbb{F}^m = T_0\mathbb{F}^m$, such that there exists a sequence of immersive point $t(i) \in \mathbb{F}^n$ of f converging to 0 as $i \to \infty$ and $\Pi = \lim_{i \to \infty} f_*(T_{t(i)}\mathbb{F}^n)$.

Then we have

Lemma 4.5. *Let $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^m, 0), \omega \in Z_f^n$ and $\Pi \in N(f)$. Then $\omega(0)|_\Pi = 0$.*

Let $\mathrm{Gr}(n, \mathbb{F}^m) \hookrightarrow P(\Lambda^n(T_0\mathbb{F}^m))$ be Plücker embedding. Then we have

Lemma 4.6. *Let $\omega \in Z_f^n$. Then $\omega(0)$ vanishes on the projective linear hull of $N(f)$ in $P(\Lambda^n(T_0\mathbb{F}^m))$.*

5. Parametric Lagrangian surfaces

In particular, setting $n = 2$ and $\mathbb{F} = \mathbb{R}$, we consider smooth map-germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^4, 0)$ whose immersion locus is dense. Then $\ell(f) = -1, 0, 1, 2, 3$ or 4 by Lemma 4.4.

Let

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$$

be a skew symmetric 4×4-matrix. Then $\det(A) = P(A)^2$, where $P(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$. Then the hypersurface $\Sigma \subset L^2_4$ is defined by $P(A) = 0$.

Example 5.1. Let $f : (\mathbb{R}^2,0) \to (\mathbb{R}^4,0)$ be the immersion defined by $f(t_1,t_2) = (t_1,t_2,0,0)$. Then LZ^2_f is defined by $a_{12} = 0$ in $L^2_4 \cong \mathbb{R}^6$. Then $LZ^2_f \cap \Sigma$ is given by $a_{12} = 0, a_{13}a_{24} - a_{14}a_{23} = 0$. Thus $\dim P(LZ^2_f \setminus \Sigma) = 4$. Therefore we have $\ell(f) = 4$.

Example 5.2. (Open Whitney umbrella) Let $f \in \mathscr{E}_{2,4}$ be defined by

$$f(t_1,t_2) = (t_1,\ t_2^2,\ t_1t_2,\ t_2^{2k+1}).$$

If $k \geq 2$, then $\widetilde{r}(f) < 4$. Therefore f is never Lagrangian for any symplectic form. If $k = 1$, then f is called an *open Whitney umbrella* and we have that $r(f) = 4$ and that $\ell(f) = \widetilde{\ell}(f) = 0$. Thus, if $k = 1$, then f is Lagrangian for the linear symplectic form $\Omega = 3dx_1 \wedge dx_4 + 2dx_2 \wedge dx_3$, which is unique up to non-zero constant multiplication.

Moreover we determine the invariants $\ell(f)$ and $\widetilde{\ell}(f)$ for all simple singularities $f : (\mathbb{R}^2,0) \to (\mathbb{R}^4,0)$ ([7]). In fact we have:

Proposition 5.3. *A simple map-germ $f(\mathbb{R}^2,0) \to (\mathbb{R}^4,0)$ is Lagrangian for some symplectic form on $(\mathbb{R}^4,0)$ if and only if f is right-left equivalent to one of the following list (among the list in* [7]*):*

$$\begin{array}{lll} (t_1,t_2) & \mapsto & (t_1,\ t_2^2,\ t_1t_2,\ t_2^3) \qquad (I_1), \\ & & (t_1,\ t_2^2,\ t_2^3 + (\pm 1)^{j+1}t_1^jt_2,\ t_1^{2j-1}t_2), (j = 2,3,4,\dots) \qquad (III_{j,2j-1}), \\ & & (t_1,\ t_1t_2,\ t_3^3,\ t_1t_2^2 + t_2^4) \qquad (IV_1), \\ & & (t_1,\ t_2^2,\ t_1^2t_2 + t_2^3,\ t_1t_2^3) \qquad (VII_1), \\ & & (t_1,\ t_1t_2,\ t_2^3,\ t_2^4) \qquad (IX_1). \end{array}$$

In all of above cases, we have $\ell(f) = \widetilde{\ell}(f) = 0$.

Example 5.4. (Open swallowtail) Let $f : (\mathbb{R}^2,0) \to (\mathbb{R}^4,0)$ be the germ defined by

$$f(t_1,t_2) = (t_1,\ t_2^3 + t_1t_2,\ \tfrac{3}{4}t^4 + \tfrac{1}{2}t_1t_2^2,\ \tfrac{3}{5}t_1^5 + \tfrac{1}{3}t_1t_2^3),$$

which is called *open swallow-tail*. Then, by calculation, we see that $\ell(f) = \widetilde{\ell}(f) = 0$. In fact f is Lagrangian for the linear symplectic form $\Omega = 2dx_1 \wedge dx_4 - dx_2 \wedge dx_4$, which is unique up to non-zero constant multiplication.

6. LAGRANGIAN MAPPINGS FOR PLENTY OF SYMPLECTIC FORMS

A plane (2-dimensional linear subspace) $\Pi \subset L^2_4 = \mathbb{R}^6$ is called *elliptic* (resp. *hyperbolic*, *parabolic*) if $\Pi \cap \Sigma = \{0\}$ (resp. $\Pi \cap \Sigma$ consists of two lines, $\Pi \subset \Sigma$). Recall that Σ is the set of non-symplectic forms.

A projective line $P(\Pi)$ in $P(L^2_4) = P^5$ is called *elliptic* (resp. *hyperbolic*, *parabolic*) if Π is elliptic (resp. hyperbolic, parabolic).

Example 6.1. (Product of curves) Let $a,b : (\mathbb{R},0) \to (\mathbb{R}^2,0)$ be planer curve-germs. Then define $f : (\mathbb{R}^2,0) \to (\mathbb{R}^4,0)$ by $f(t_1,t_2) = (a(t_1),b(t_2))$. Then $\ell(f) \geq 1$. In fact there exist two-parameter linear symplectic forms

$$\Omega_{\lambda,\mu} = \lambda dx_1 \wedge dx_2 + \mu dx_3 \wedge dx_4,$$

$\lambda\mu \neq 0$, which satisfy $f^*(\Omega_{\lambda,\mu}) = 0$. In this case $P(LZ^2_f)$ contains a *hyperbolic* line.

For example, taking a and b are planar cusps, then we have the germ defined by

$$f(t_1,t_2) = (t_1^2,\ t_1^3,\ t_2^2,\ t_2^3),$$

which is called the *product of cusps*. Then Z^0_f is generated by $x_1^3 - x_2^2, x_3^3 - x_4^2$. Then $\mathscr{R}^1_f$ is described as the set of equivalence classes $[\alpha]$ of 1-forms of form

$$\alpha = (a(x_3,x_4)+x_1b(x_3,x_4))(-3x_2dx_1+2x_1dx_2)+(c(x_1,x_2)+x_3e(x_1,x_2))(-3x_4dx_3+2x_3dx_4),$$

where the function-germs a,b,c,e are regarded modulo Z^0_f. Since the product of cusps is quasi-homogeneous and therefore contractible, we conclude that any symplectic zero form ω of f is described as

$$\omega = d\{(a(x_3,x_4)+x_1b(x_3,x_4))(-3x_2dx_1+2x_1dx_2)+(c(x_1,x_2)+x_3e(x_1,x_2))(-3x_4dx_3+2x_3dx_4)\},$$

modulo AZ^2_f.

Note that the products of singular curves and regular curves were studied in [4].

Example 6.2. (Holomorphic curves, anti-holomorphic curves) Let $f : (\mathbb{R}^2,0) = (\mathbb{C},0) \to (\mathbb{C}^2,0) = (\mathbb{R}^4,0)$ be a holomorphic or anti-holomorphic map-germ regarded as an element in $\mathscr{E}_{2,4}$. Then $\ell(f) \geq 1$. In fact there exist two-parameter linear symplectic forms

$$\Omega_w = \mathrm{Re}(w dz_1 \wedge dz_2),$$

$w \in \mathbb{C} = \mathbb{R}^2, w \neq 0$, which satisfy $f^*(\Omega_w) = 0$. In this case $P(LZ^2_f)$ contains an *elliptic* line.

For example, we have, from $z \in \mathbb{C} \mapsto (z^2,z^3)$, the germ

$$f(t_1,t_2) = (t_1^2 - t_2^2,\ 2t_1t_2,\ t_1^3 - 3t_1t_2^2,\ 3t_1^2t_2 - t_2^3),$$

which is called *complex cusp*.

We are naturally led to the problem: *Classify singularities of $f : \mathbb{R}^2 \to \mathbb{R}^4$ with $\ell(f) \geq 1$, in particular for the cases with $\ell(f) = 2,3$.*

References

[1] I. Chen, K.-P. Lin, S. Yau, H. Zuo, *Coordinate-free characterization of homogeneous polynomials with isolated singularities,* Comm. Anal. Geom., **19-4** (2011), 667–704.

[2] W. Domitrz, S. Janeczko, M. Zhitomirskii, *Relative Poincaré lemma, contractibility, quasi-homogeneity and vector fields tangent to a singular variety,* Illinois J. Math., **48-3** (2004), 803–835.

[3] W. Domitrz, S. Janeczko, M. Zhitomirskii, *Symplectic singularities of varieties: The method of algebraic restrictions,* J. reine angew Math. **618** (2008), 197–235.

[4] A. B. Givental', *Singular Lagrangian manifolds and their Lagrangian maps,* Journal of Soviet Mathematics, **52–4** (1990), 3246–3278.

[5] G. Ishikawa, S. Janeczko, *Symplectic invariants of parametric singularities,* in Proceedings of the Conference on Geometry in honour of Shing-Tung Yau's 60th Birthday, April 6–8, 2009, Warsaw, Poland, in *Advances in Geometric Analysis,* Advanced Lectures in Mathematics series **21**, International Press (2012), pp. 259–280.

[6] G. Ishikawa, S. Janeczko, *Residual algebraic restrictions of differential forms,* Methods and Applications of Analysis, **24–1** (2017), 045–062.

[7] C. Klotz, O. Pop, J.H. Rieger, *Real double-points of deformations of $\mathscr{A}$-simple map-germs from $\mathbb{R}^n$ to $\mathbb{R}^{2n}$,* Math. Proc. Camb. Phil. Soc. **142** (2007), 341–363.

[8] K. Saito, *Quasihomogene isolierte Singularitäten von Hyperflächen,* Invent. Math. **14** (1971), 123–142.

[9] S. Yau, H. Zuo, *Lower estimate of Milnor number and characterization of isolated homogeneous hypersurface singularities,* Pacific J. Math., **260-1** (2012), 245–255.

[10] S. Yau, H. Zuo, *Characterization of isolated complete intersection singularities with $\mathbb{C}^*$-action of dimension ≥ 2 by means of geometric genus and irregularity,* Comm. Anal. Geom., **21–3** (2013), 509–526.

[11] S. Yau, H. Zuo, *Complete characterization of isolated homogeneous hypersurface singularities,* Pacific Journal of Mathematics, **273–1** (2015), 213–224.

[12] S. Yau, H. Zuo, *A sharp upper estimate conjecture for the Yau number of weighted homogeneous isolated hypersurface singularity,* Pure Appl.Math. Q., **12-1** (2016), 165–181.

[13] S. Yau, H. Zuo, *Nine characterizations of weighted homogeneous isolated hypersurface singularities,* Methods Appl. Anal., **24-1** (2017), 155–168.

[14] S. Yau, H. Zuo, *A sharp lower bound for the geometric genus and Zariski multiplicity question,* Math. Z. **289** no. 3-4 (2018), 1299–1310.

FACULTY OF SCIENCES, DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, JAPAN.

E-mail address: `ishikawa@math.sci.hokudai.ac.jp`

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, UL. SNIADECKICH 8, 00-956 WARSZAWA, POLAND.

WARSAW UNIVERSITY OF TECHNOLOGY, FACULTY OF MATHEMATICS AND INFORMATION SCIENCE, PLAC POLITECHNIKI 1, 00-661 WARSZAWA, POLAND.

E-mail address: `janeczko@impan.pl`