

# Openings of differentiable map-germs and unfoldings

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*Dedicated to Professor Satoshi Koike for his 60th birthday*

## Abstract

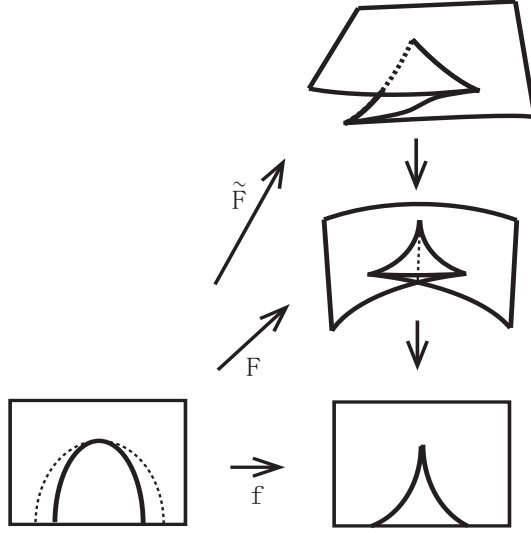
The algebraic notion of *openings* of a map-germ is introduced in this paper. An opening separates the self-intersections of the original map-germ, preserving its singularities. The notion of openings is different from the notion of unfoldings. Openings do not unfold the singularities. For example, the swallowtail is an opening of the Whitney cusp map-germ from plane to plane and the open swallowtail is a versal opening of them. Openings of map-germs appear as typical singularities in several problems of geometry and its applications. The notion of openings has close relations to isotropic map-germs in a symplectic space and integral map-germs in a contact space. We describe the openings of Morin singularities, namely, stable unfoldings of map-germs of corank one. The relation of unfoldings and openings are discussed. Moreover we provide a method to construct versal openings of map-germs and give versal openings of stable map-germs  $(\mathbf{R}^4, 0) \rightarrow (\mathbf{R}^4, 0)$ . Lastly the relation of lowerable vector fields and openings is discussed.

## 1 Introduction

There is a sequence of well-known singularities of map-germs: The *Whitney cusp*  $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ ,  $f(x, u) = (x^3 + ux, u)$ , the *swallowtail*  $F : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$ ,  $F(x, u) = (f(x, u), x^4 + \frac{2}{3}ux^2)$ , and the *open swallowtail*  $\tilde{F} : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^4, 0)$ ,  $\tilde{F}(x, u) = (F(x, u), x^5 + \frac{4}{9}ux^3)$ .

They have the same singular locus and the same kernel field of the differential along the singular locus, while the self-intersections are resolved.

What is the algebraic structure behind them? One of answers to the above question is presented in this paper. In fact we observe, for the swal-



lowtail  $F(x, u) = (x^3 + ux, u, x^4 + \frac{2}{3}ux^2)$ , we see that

$$d(x^4 + \frac{2}{3}ux^2) = \frac{4}{3}x d(x^3 + ux) - \frac{4}{9}x^2 du \in \langle d(x^3 + ux), du \rangle_{\mathcal{E}_2}.$$

For the open swallowtail  $\tilde{F}(x, u) = (x^3 + ux, u, x^4 + \frac{2}{3}ux^2, x^5 + \frac{4}{9}ux^3)$ , we have that

$$d(x^5 + \frac{4}{9}ux^3) = \frac{5}{3}x^2 d(x^3 + ux) - \frac{10}{9}x^3 du \in \langle d(x^3 + ux), du \rangle_{\mathcal{E}_2}.$$

Here  $d$  means the exterior differential and  $\mathcal{E}_2$  denotes the  $\mathbf{R}$ -algebra of  $C^\infty$  function-germs on  $(\mathbf{R}^2, 0)$ .

As the key construction, we introduce the notion of *opening* of multi-germs of mappings. To do this, first we summarise the auxiliary notions in this paper.

Let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  be a multi-germ of a  $C^\infty$  map with  $n \leq m$ . Here  $A$  is a finite subset of  $\mathbf{R}^n$ ,  $b \in \mathbf{R}^m$  and  $f(A) = \{b\}$ . We define the *Jacobi module*  $\mathcal{J}_f$  of  $f$  by

$$\mathcal{J}_f = \left\{ \sum_{j=1}^m p_j df_j \mid p_j \in \mathcal{E}_{\mathbf{R}^n, A} (1 \leq j \leq m) \right\} \subset \Omega_{\mathbf{R}^n, A}^1$$

in the space  $\Omega_{\mathbf{R}^n, A}^1$  of 1-form-germs on  $(\mathbf{R}^n, A)$ . Note that  $\mathcal{J}_f$  is just the first order component of the graded differential ideal  $\mathcal{J}_f^\bullet$  in  $\Omega_{\mathbf{R}^n, A}^\bullet$  generated

by  $df_1, \dots, df_m$ . Then the singular locus, the non-immersive locus, of  $f$  is given by

$$\Sigma_f = \{x \in (\mathbf{R}^n, A) \mid \text{rank } \mathcal{J}_f(x) < n\}.$$

Also we consider the *kernel field*  $\text{Ker}(f_* : T\mathbf{R}^n \rightarrow T\mathbf{R}^m)$  of the differential of  $f$ , along  $\Sigma_f$ .

For another map-germ  $f' : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^{m'}, b')$ ,  $n \leq m'$ , if  $\mathcal{J}_{f'} = \mathcal{J}_f$ , then  $\Sigma_{f'} = \Sigma_f$  and  $\text{Ker}(f'_*) = \text{Ker}(f_*)$ .

Then define the *ramification module*  $\mathcal{R}_f$  of  $f$  by

$$\mathcal{R}_f = \{h \in \mathcal{E}_{\mathbf{R}^n, A} \mid dh \in \mathcal{J}_f\},$$

(cf. [12][15]).

For  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$ ,  $f' : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^{m'}, b')$ , easily we see that  $\mathcal{J}_{f'} = \mathcal{J}_f$  if and only if  $\mathcal{R}_{f'} = \mathcal{R}_f$  (Lemma 2.1).

**Definition 1.1** Given  $h_1, \dots, h_r \in \mathcal{R}_f$ , the map-germ  $F : (\mathbf{R}^n, A) \rightarrow \mathbf{R}^m \times \mathbf{R}^r = \mathbf{R}^{m+r}$  defined by

$$F = (f_1, \dots, f_m, h_1, \dots, h_r)$$

is called an *opening* of  $f$ , while  $f$  is called a *closing* of  $F$ .

Then, for any opening  $F$  of  $f$ , we have  $\mathcal{R}_F = \mathcal{R}_f$ ,  $\mathcal{J}_F = \mathcal{J}_f$ ,  $\Sigma_F = \Sigma_f$  and  $\text{Ker}(F_*) = \text{Ker}(f_*)$ .

For example, the swallowtail is an opening of the Whitney cusp. The open swallowtail is an opening of the swallowtail and of the Whitney cusp.

Note that an opening of an opening of  $f$  is an opening of  $f$ .

**Definition 1.2** An opening  $F = (f, h_1, \dots, h_r)$  of  $f$  is called a *versal opening* (resp. a *mini-versal opening*) of  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$ , if  $1, h_1, \dots, h_r$  form a (minimal) system of generators of  $\mathcal{R}_f$  as an  $\mathcal{E}_{\mathbf{R}^m, b}$ -module via  $f^* : \mathcal{E}_{\mathbf{R}^m, b} \rightarrow \mathcal{E}_{\mathbf{R}^n, A}$ .

Note that a versal opening of an opening of  $f$  is a versal opening of  $f$ . An opening of a versal opening of  $f$  is a versal opening of  $f$ .

A mini-versal opening  $F : (\mathbf{R}^n, A) \rightarrow \mathbf{R}^{m+r}$  of  $f$  is unique up to left-equivalence and a versal opening  $G : (\mathbf{R}^n, A) \rightarrow \mathbf{R}^{m+s}$  of  $f$  is left-equivalent to a mini-versal opening composed with an immersion  $(\mathbf{R}^n, A) \rightarrow \mathbf{R}^{m+r} \hookrightarrow \mathbf{R}^{m+s}$  (Proposition 2.14).

Openings of map-germs appear as typical singularities in several problems of geometry and its applications. The openings naturally appear in the

classification problem of “tangential singularities” ([5][16][17]). Open swallowtails, open folded umbrellas, etc. appear as tangent varieties ([1]). We have applied opening constructions to solve the “stable” classification problem of tangent varieties to generic submanifolds in [17]. Moreover openings are related to singularities of isotropic mappings in symplectic spaces.

Let  $T^*\mathbf{R}^n = \mathbf{R}^{2m}$  be the  $2m$ -dimensional symplectic space with the symplectic form  $\omega = \sum_{i=1}^m dp_i \wedge dx_i$ , and  $f : (\mathbf{R}^n, A) \rightarrow \mathbf{R}^{2m}$  a multi-germ of isotropic mapping. Since  $f^*\omega = 0$ , we have that  $\sum_{i=1}^m (p_i \circ f)d(x_i \circ f)$  is closed, so it is exact and there exists  $e \in \mathcal{E}_{\mathbf{R}^n, A}$  such that

$$de = \sum_{i=1}^m (p_i \circ f)d(x_i \circ f).$$

Define  $g : (\mathbf{R}^n, A) \rightarrow \mathbf{R}^m$  by  $g(x) = (x_1 \circ f(x), \dots, x_m \circ f(x))$ . Then  $e \in \mathcal{R}_g$ . Conversely, given  $e \in \mathcal{R}_g$ , we have  $de = \sum_{i=1}^m a_i dg_i$  for some functions  $a_1, \dots, a_m$ , and we obtain an isotropic multi-germ  $f : (\mathbf{R}^n, A) \rightarrow \mathbf{R}^{2m}$  by  $p_i \circ f = a_i, x_i \circ f = g_i, (1 \leq i \leq m)$ . The opening  $(g, e) : (\mathbf{R}^n, A) \rightarrow \mathbf{R}^{m+1}$  is the *frontal* germ associated to  $f$ . For example the frontal of a open Whitney-Morin umbrella is the folded umbrella (see Proposition 4.1). The open folded umbrella appears also as a “frontal-symplectic singularity” [18]. Several geometric applications of the theory of openings are given in [16][17].

**Remark 1.3** (Relations with known notions.) Pellikaan (his thesis and [25], p.358), de Jong and van Straten ([19], p.185) introduced the notion of *primitive ideal*

$$\int I = \{h \in \mathcal{O}_n \mid h \in I, \frac{\partial h}{\partial x_i} \in I, 1 \leq i \leq n\}$$

of an ideal  $I$  of the ring  $\mathcal{O}_n$  of holomorphic function-germs on  $(\mathbf{C}^n, 0)$ . It is motivated for the deformation theory of non-isolated singularities. We can introduce the analogous notion to it in  $C^\infty$  case and also the notion of *primitive ring* by

$$\int' I = \{h \in \mathcal{E}_n \mid \frac{\partial h}{\partial x_i} \in I, 1 \leq i \leq n\}$$

It is related to the notion of ramification module as follows: Let  $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^m, 0), n \leq m$  be a map-germ. We take as  $I$  the ideal  $J_f$  generated by  $n$ -minors of the Jacobi matrix of  $f$ , which is associated with the singular locus of  $f$ . If  $n = 1$  (the case of curves), then we have that  $\int' J_f = \mathcal{R}_f$ . However, they are different in general in the case  $n \geq 2$ . In fact  $f^*\mathcal{E}_m \not\subseteq \int' J_f$  and

$\int' J_f$  does not have the natural  $\mathcal{E}_m$ -module structure via  $f^*$  in general, while  $\mathcal{R}_f$  does. For example, let  $f(x_1, x_2) = (x_1, x_2^2), f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ . Then  $\int' J_f = \mathbf{R} + x_2^2 \mathcal{E}_2$ , and  $\mathcal{R}_f = \mathcal{E}_1 + x_2^2 \mathcal{E}_2 = f^* \mathcal{E}_2 + x_2^3 f^* \mathcal{E}_2$  where  $\mathcal{E}_1$  means the ring of functions of  $x_1$ .

Mond [24] introduced a notion  $R_0(f)$ , which is more closely related to  $\mathcal{R}_f$ , also from the motivation of deformation theory on map-germs in complex analytic category. Its  $C^\infty$  analogue is given by

$$R_0(f) = \{h \in \mathcal{E}_n \mid df_{i_1} \wedge \cdots \wedge df_{i_{n-1}} \wedge dh \in J_f dx_1 \wedge \cdots \wedge dx_n, \\ 1 \leq i_1 < \cdots < i_{n-1} \leq m\}.$$

Then clearly we have  $\mathcal{R}_f \subseteq R_0(f)$ . We have that, for two map-germ  $f, f'$ ,  $\mathcal{R}_f = \mathcal{R}_{f'}$  implies  $R_0(f) = R_0(f')$ . If  $n = 1$ , then  $\mathcal{R}_f = R_0(f) = \int' J_f$ . Moreover we have that, if  $n = m$  and the singular locus of  $f$  is nowhere dense, then  $\mathcal{R}_f = R_0(f)$ , using Cramer's rule in linear algebra. We conjecture that the equality  $\mathcal{R}_f = R_0(f)$  holds also in the case  $n < m$  under a rather mild condition. Note that, in Propositions 4.1 and 4.3 of [24], they were already given the related results to the results in the present paper (Lemma 2.2, Corollary 2.10, Proposition 2.16 and Proposition 5.2).

In §2, we give a detailed exposition on ramification modules and openings of multi-germs.

In §3, the relation of unfoldings and openings are discussed. We treat the problem to find a versal opening of an unfolding of a given map-germ. Then the notion of *extendability* of an unfolding is introduced. If the given map-germ is of corank one, then any unfolding is extendable and, in particular its versal opening is obtained from that of its stable unfolding. Then, in §4, we give the explicit presentation on versal openings of stable unfoldings of map-germs of corank 1, namely, versal openings of Morin maps.

In §5, we remark the existence of versal openings in finite analytic case. In §6, we give a direct method to find versal openings for several examples and show the existence of the versal opening for any stable map-germ  $(\mathbf{R}^4, A) \rightarrow (\mathbf{R}^4, b)$ , explicitly. In §7, the relation of lowerable vector fields of map-germs and openings is discussed.

In this paper we often abbreviate  $\mathcal{E}_{(\mathbf{R}^n, A)}$  by  $\mathcal{E}_A$ , the  $\mathbf{R}$ -algebra of  $C^\infty$  function-germs on  $(\mathbf{R}^n, A)$ . If  $A = \{a_1, \dots, a_s\}$ , then we denote by  $\mathfrak{m}_i$  the maximal ideal consisting of  $h \in \mathcal{E}_A$  with  $h(a_i) = 0$ . We set  $\mathfrak{m}_A = \bigcap_{i=1}^s \mathfrak{m}_i$ . If  $A$  consists of the origin, then we use  $\mathcal{E}_n, \mathfrak{m}_n$  instead of  $\mathcal{E}_A, \mathfrak{m}_A$  respectively.

All manifolds and mappings we treat in this paper are assumed to be of class  $C^\infty$ , unless otherwise stated.

## 2 Ramification modules and openings

In Introduction, we have introduced the notion of openings based on that of Jacobi modules and ramification modules.

**Lemma 2.1** *For map-germs  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$ ,  $f' : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^{m'}, b')$ , we have that  $\mathcal{J}_{f'} = \mathcal{J}_f$  if and only if  $\mathcal{R}_{f'} = \mathcal{R}_f$ .*

*Proof:* It is clear that  $\mathcal{J}_{f'} = \mathcal{J}_f$  implies  $\mathcal{R}_{f'} = \mathcal{R}_f$ . Conversely suppose  $\mathcal{R}_{f'} = \mathcal{R}_f$ . Then any component  $f'_j$  of  $f'$  belongs to  $\mathcal{R}_{f'} = \mathcal{R}_f$ , hence  $df'_j \in \mathcal{J}_f$ . Therefore  $\mathcal{J}_{f'} \subseteq \mathcal{J}_f$ . By the symmetry we have  $\mathcal{J}_{f'} = \mathcal{J}_f$ .  $\square$

**Lemma 2.2** *Let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  be a map-germ. Then we have*

- (1)  $f^*\mathcal{E}_b \subseteq \mathcal{R}_f \subseteq \mathcal{E}_A$ .
- (2)  $\mathcal{R}_f$  is an  $\mathcal{E}_b$ -submodule via  $f^* : \mathcal{E}_b \rightarrow \mathcal{E}_A$  of  $\mathcal{E}_A$ .
- (3)  $\mathcal{R}_f$  is  $C^\infty$ -subring of  $\mathcal{E}_A$ .
- (4) If  $\tau : (\mathbf{R}^m, b) \rightarrow (\mathbf{R}^m, b')$  is a diffeomorphism-germ, then  $\mathcal{R}_{\tau \circ f} = \mathcal{R}_f$ . If  $\sigma : (\mathbf{R}^n, A') \rightarrow (\mathbf{R}^n, A)$  is a diffeomorphism-germ, then  $\mathcal{R}_{f \circ \sigma} = \sigma^*(\mathcal{R}_f)$ .

*Proof:* Assertions (1) and (2) follow from the fact that, if  $h \in \mathcal{R}_f$  and  $dh = \sum_{j=1}^m p_j df_j$ , then

$$d\{(k \circ f)h\} = \sum_{j=1}^m \{(k \circ f)p_j + h(\partial k / \partial y_j)\} df_j.$$

Assertion (3) follows from the fact that, if  $h_1, \dots, h_r \in \mathcal{R}_f$  and if  $\tau : \mathbf{R}^r \rightarrow \mathbf{R}$  is a  $C^\infty$  function, then

$$d\{\tau(h_1, \dots, h_r)\} = \sum_{i=1}^r \frac{\partial \tau}{\partial y_i}(h_1, \dots, h_r) dh_i \in \mathcal{J}_f.$$

Assertion (4) follows from the fact that  $\mathcal{J}_{\tau \circ f} = \mathcal{J}_f$  and  $\mathcal{J}_{f \circ \sigma} = \sigma^*(\mathcal{J}_f)$ .  $\square$

**Lemma 2.3** *Let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  be a map-germ with  $A = \{a_1, \dots, a_s\}$ . We denote by  $f_i : (\mathbf{R}^n, a_i) \rightarrow (\mathbf{R}^m, b)$  the restriction of  $f$  to  $(\mathbf{R}^n, a_i)$ . Then  $\mathcal{R}_f \cong \prod_{i=1}^s \mathcal{R}_{f_i}$  as  $\mathcal{E}_b$ -module.*

*Proof:* We have the isomorphism  $\varphi : \mathcal{R}_f \rightarrow \prod_{i=1}^s \mathcal{R}_{f_i}$  defined by  $\varphi(h) = (h|_{(\mathbf{R}^n, a_i)})_{i=1}^s$ .  $\square$

**Remark 2.4** Any multi-germ  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$ ,  $\#(A) = s$ , is right-equivalent to a map-germ of form  $\prod_s f_i : \prod_s (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^m, b)$  from the disjoint union of  $s$ -copies of  $(\mathbf{R}^n, 0)$ .

A map-germ  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  is called *finite* if  $\mathcal{E}_A$  is a finite  $\mathcal{E}_b$ -module. The condition is equivalent to that  $\dim_{\mathbf{R}} \mathcal{E}_A / (f^* \mathfrak{m}_b) \mathcal{E}_A < \infty$  by the preparation theorem (see for example [4]). Moreover  $f$  is finite if and only if  $\mathcal{K}$ -finite and  $n \leq m$  ([26]).

**Proposition 2.5** *If  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  is finite and of corank at most one. Then we have*

- (1)  $\mathcal{R}_f$  is a finite  $\mathcal{E}_b$ -module. Therefore there exists a versal opening of  $f$ .
- (2)  $1, h_1, \dots, h_r \in \mathcal{R}_f$  generate  $\mathcal{R}_f$  as  $\mathcal{E}_b$ -module if and only if they generate the vector space  $\mathcal{R}_f / (f^* \mathfrak{m}_b) \mathcal{R}_f$  over  $\mathbf{R}$ .

*Proof:* In the case  $A$  consists of a point, the assertions are proved in Theorem 1.3 of [13] and Corollary 2.4 of [15]. For a general finite set  $A$ , the assertions are reduced to the case that  $A$  consists of a point by Lemma 2.3.  $\square$

**Remark 2.6** In §5, we define the analytic counterpart  $\mathcal{R}_f^\omega$  of the notion of ramification modules for an analytic map-germ  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, 0)$ . Then it is essentially obvious that, if  $f$  is a finite map-germ, then  $\mathcal{R}_f^\omega$  is a finite module over the ring  $\mathcal{O}_{(\mathbf{R}^n, A)}$  of analytic function-germs. In fact  $\mathcal{O}_{(\mathbf{R}^n, A)}$  is a Noetherian ring and it is a finite  $\mathcal{O}_{(\mathbf{R}^m, 0)}$ -module via  $f^*$ . Moreover  $\mathcal{R}_f^\omega$  is an  $\mathcal{O}_{(\mathbf{R}^m, 0)}$ -submodule of  $\mathcal{O}_{(\mathbf{R}^n, A)}$ . Since every submodule of a finite module over a Noetherian ring is finite, we have that  $\mathcal{R}_f^\omega$  is a finite module over the ring  $\mathcal{O}_{(\mathbf{R}^n, A)}$ .

Furthermore, in §5, we show that  $\mathcal{R}_f$  is a finite  $\mathcal{E}_b$ -module if  $f$  is finite and analytic (Proposition 5.2).

For a map-germ  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b), n \leq m$ , we have defined in Introduction the notions of openings and versal openings of  $f$ .

**Example 2.7** (1) Let  $h : (\mathbf{R}, 0) \rightarrow (\mathbf{R}, 0), h(x) = x^2$ . Then  $\mathcal{R}_h = \langle 1, x^3 \rangle_{f^*(\mathcal{E}_1)}$ . The map-germ  $H : (\mathbf{R}, 0) \rightarrow (\mathbf{R}^2, 0), H(x) := (x^2, x^3)$ , the simple cusp map, is the mini-versal opening of  $h$ .

(2) Let  $g : (\mathbf{R}, 0) \rightarrow (\mathbf{R}, 0), g(x) = x^3$ . Then  $\mathcal{R}_g = \langle 1, x^4, x^5 \rangle_{f^*(\mathcal{E}_1)}$ . The map-germ  $G : (\mathbf{R}, 0) \rightarrow (\mathbf{R}^3, 0), G(x) := (x^3, x^4, x^5)$  is the mini-versal opening of  $g$ .

(3) Let  $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0), f(x, u) = (x^3 + ux, u)$ , an unfolding of  $g$ . Then  $\mathcal{R}_f = \langle 1, x^4 + \frac{2}{23}ux^2, x^5 + \frac{4}{9}ux^3 \rangle_{f^*(\mathcal{E}_2)}$ . The map-germ  $F : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^4, 0)$  defined by  $F(x, u) := (x^3 + ux, u, x^4 + \frac{2}{23}ux^2, x^5 + \frac{4}{9}ux^3)$ , the open swallowtail is the mini-versal opening of Whitney cusp  $f$ .

(4) Let consider the multi-germ  $k : (\mathbf{R}, A) \rightarrow (\mathbf{R}^2, 0)$ ,  $A = \{0, 1\}$  defined by  $k(t) = (t, 0)$  near  $t = 0$  and  $k(t) = (0, t - 1)$  near  $t = 1$ . Then  $\mathcal{R}_k$  is generated by  $1 \in \mathcal{E}_A$  and  $s \in \mathcal{E}_A$  defined by  $s(t) = 1$  near  $t = 0$  and  $s(t) = 0$  near  $t = 1$ , over  $k^*\mathcal{E}_2$ . Then  $K = (k, s) : (\mathbf{R}, A) \rightarrow \mathbf{R}^3$  is the versal opening of  $k$ . In fact  $K$  resolves the self-intersection by the over and underpasses.

**Example 2.8** Here we add several additional illustrative examples. Let us consider the following five map-germs:  $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ ,  $g : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$ ,  $h : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$ ,  $k : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ ,  $\ell : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$  defined by

$$\begin{aligned} f(x, t) &= (x, t^2), & g(x, t) &= (x, xt, t^2), & h(x, t) &= (x^2, xt, t^2), \\ k(x, t) &= (x^2, t^2), & \ell(x, t) &= (x^2 - t^2, xt). \end{aligned}$$

Then we have

$$\mathcal{R}_k \subsetneq \mathcal{R}_h \subsetneq \mathcal{R}_g, \quad \mathcal{R}_\ell \subsetneq \mathcal{R}_h, \quad \mathcal{R}_f \subsetneq \mathcal{R}_g.$$

In fact

$$\begin{aligned} \mathcal{J}_f &= \langle dx, tdt \rangle_{\mathcal{E}_2}, & \mathcal{J}_g &= \langle dx, xdt, tdt \rangle_{\mathcal{E}_2}, & \mathcal{J}_h &= \langle xdx, xdt + tdx, tdt \rangle_{\mathcal{E}_2}, \\ \mathcal{J}_k &= \langle xdx, tdx \rangle_{\mathcal{E}_2}, & \mathcal{J}_\ell &= \langle xdx - tdt, tdx + xdt \rangle_{\mathcal{E}_2}. \end{aligned}$$

Then we see that  $\mathcal{R}_f$  is minimally generated by  $1, t^3$  over  $f^*\mathcal{E}_2$ ,  $\mathcal{R}_g$  is minimally generated by  $1, t^3$  over  $g^*\mathcal{E}_3$ , and  $\mathcal{R}_h$  is minimally generated by  $1, x^3, x^2t, xt^2, t^3$  over  $h^*\mathcal{E}_3$ . Moreover we have that  $\mathcal{R}_k$  is minimally generated by  $1, x^3, t^3, x^3t^3$  over  $k^*\mathcal{E}_2$  and that  $\mathcal{R}_\ell$  is minimally generated by  $1, x^3 - 3xt^2, 3x^2t - t^3, x^2(x^2 + t^2)^2$  over  $\ell^*\mathcal{E}_2$ .

**Remark 2.9** (continued with Remark 2.4). Let  $\coprod_s f_i : \coprod_s (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^m, b)$  be a multi-germ of map. Suppose  $F_i = (f_i; h_{i1}, \dots, h_{ir_i}) : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^{m+r_i}, b \times 0)$  be a versal opening of  $f_i$ . Then, setting  $r = \max_{1 \leq i \leq s} r_i$  and  $h_{ij} = 0$  if  $j > r_i$ , then  $F = \coprod_s (f_i, h_{i1}, \dots, h_{ir_i}, \dots) : \coprod_s (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^{m+r}, b \times 0)$  is a versal opening of  $\coprod_s f_i$ .

By Proposition 2.5, we have

**Corollary 2.10** *Let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  be finite and of corank at most one. Then there exists a versal opening of  $f$ .*



**Remark 2.11** The existence of a versal opening for a finite  $C^\infty$  map-germ of arbitrary corank is still open. However, using Proposition 5.2 together with the analytic result, we can show that Corollary 2.10 for finitely  $\mathcal{A}$ -determined map-germs  $(\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  with  $n \leq m$  without the corank condition.

Moreover we have the following:

**Corollary 2.12** *Let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  be finite and of corank at most one. Then an opening  $F = (f, h_1, \dots, h_r)$  of  $f$  is a mini-versal opening of  $f$ , namely,  $1, h_1, \dots, h_r \in \mathcal{R}_f$  form a minimal system of generators of  $\mathcal{R}_f$  as  $\mathcal{E}_b$ -module if and only if they form a basis of  $\mathbf{R}$ -vector space  $\mathcal{R}_f / (f^* \mathfrak{m}_b) \mathcal{R}_f$ .*

The following is useful for the classification problem of map-germs in a geometric context ([16][17]).

**Proposition 2.13** *Let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$ ,  $n \leq m$  be a  $C^\infty$  map-germ.*  
(1) *For any versal opening  $F : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^{m+r}, F(A))$  of  $f$  and for any opening  $G : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^{m+s}, G(A))$ , there exists an affine bundle map  $\Psi : (\mathbf{R}^{m+r}, F(A)) \rightarrow (\mathbf{R}^{m+s}, G(A))$  over  $(\mathbf{R}^m, b)$  such that  $G = \Psi \circ F$ .*  
(2) *For any mini-versal openings  $F : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^{m+r}, F(A))$  and  $F' : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^{m+r}, F'(A))$  of  $f$ , there exists an affine bundle isomorphism  $\Phi : (\mathbf{R}^{m+r}, F(A)) \rightarrow (\mathbf{R}^{m+r}, F'(A))$  over  $(\mathbf{R}^m, b)$  such that  $F' = \Phi \circ F$ . In particular, the diffeomorphism class of mini-versal opening of  $f$  is unique.*  
(3) *Any versal openings  $F'' : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^{m+s}, F''(A))$  of  $f$  is diffeomorphic to  $(F, 0)$  for a mini-versal opening  $F$  of  $f$ .*

*Proof:* (1) Let  $F = (f, h_1, \dots, h_r)$  and  $G = (f, k_1, \dots, k_s)$ . Since  $k_j \in \mathcal{R}_f$ , there exist  $c_j^0, c_j^1, \dots, c_j^r \in \mathcal{E}_b$  such that  $k_j = c_j^0 \circ f + (c_j^1 \circ f)h_1 + \dots + (c_j^r \circ f)h_r$ . Then it suffices to set  $\Psi(y, z) = (y, (c_j^0(y) + c_j^1(y)z_1 + \dots + c_j^r(y)z_r)_{1 \leq j \leq s})$ . (2) By (1) there exists an affine bundle map  $\Psi$  with  $F' = \Psi \circ F$ . From the minimality, we have that the matrix  $(c_j^i(b))$  is regular. (See Remark 2.12). Therefore  $\Psi$  is a diffeomorphism-germ. (3) Let  $F = \Psi \circ F''$  for some affine bundle map  $\Psi$ . Then the matrix  $(c_j^i(b))$  is of rank  $r$ . Therefore  $F''$  is diffeomorphic to  $(F, k_1, \dots, k_{s-r})$  for some  $k_j \in \mathcal{R}_f$ . Write each  $k_j = K_j \circ F$  for some  $K_j \in \mathcal{E}_{F(a)}$ . Then we set  $\Xi(y, z, w) = (y, z, w - K \circ F)$ . Then  $\Xi$  is a local diffeomorphism on  $\mathbf{R}^{m+r+(s-r)}$  and  $\Xi \circ (F, k_1, \dots, k_{s-r}) = (F, 0)$ .  $\square$

Two map-germs  $F : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^p, B)$  and  $G : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^q, C)$  is called  $\mathcal{L}$ -equivalent, or, left-equivalent, if there exists a diffeomorphism-germ  $\Psi : (\mathbf{R}^p, B) \rightarrow (\mathbf{R}^q, C)$  such that  $G = \Psi \circ F$ .

Then, by Proposition 2.13, we have:

**Corollary 2.14** *Let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  be a  $C^\infty$  map-germ ( $n \leq m$ ). Then a mini-versal opening of  $f$  is unique up to  $\mathcal{L}$ -equivalence. A versal opening of  $f$  is  $\mathcal{L}$ -equivalent to a mini-versal opening composed with an immersion.*

We introduce further the following notion:

**Definition 2.15** An opening  $F = (f, h_1, \dots, h_s)$  of a map-germ

$$f = (f_1, \dots, f_m) : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$$

is called  $\mathcal{L}$ -minimal if  $f_1, \dots, f_m, h_1, \dots, h_r$  minimally generate the  $C^\infty$ -ring  $\mathcal{R}_f$  over  $\mathbf{R}$ , in other words, if  $\mathcal{R}_f = (f, h)^*(\mathcal{E}_{m+r})$  and  $(f, h)$  in minimal with this property.

Then we have a similar uniqueness result for  $\mathcal{L}$ -equivalence of  $\mathcal{L}$ -minimal openings.

In Example 2.8, we have seen that  $1, x^3, t^3, x^3t^3$  minimally generate  $\mathcal{R}_k$  as  $k^*\mathcal{E}_2$ -module. However  $1, x^3, t^3$  already minimally generate  $\mathcal{R}_k$  as  $k^*\mathcal{E}_2$ - $C^\infty$ -ring. Therefore  $K = (x^2, y^2, x^3, t^3, x^3t^3)$  is a mini-versal opening of  $k = (x^2, y^2) : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$  and  $K' = (x^2, y^2, x^3, t^3)$  is a  $\mathcal{L}$ -minimal opening of  $k$ .

Lastly we show injectivity of versal openings:

**Proposition 2.16** *Let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  be a finite map-germ. Suppose  $F : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^{m+r}, F(A))$  is a versal opening of  $f$ . Then  $F$  has an injective representative.*

**Remark 2.17** The corresponding result to Proposition 2.16 in the analytic case is already proved in an earlier paper (Corollary 1.2 of [13]). Proposition 5.2 proved in §5 together with the analytic result, we can show that Proposition 2.16 for finitely  $\mathcal{A}$ -determined map-germs  $(\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  with  $n \leq m$ . Note that the proof of Corollary 1.2 in [13] heavily depends, via the finite coherence theorem in analytic geometry, on that the map-germ is finite and analytic. It seems very difficult to find a unified proof of Proposition 2.16 for  $C^\infty$  case with a similar vein to the analytic case, because of the

lack of the notion of “coherence” in the general  $C^\infty$  case. Here we provide an alternative proof applicable to any finite  $C^\infty$  map-germs, which can be applied to finite real analytic map-germs, of course. The proof is similar one to the proof for a similar result given in [9].

*Proof of Proposition 2.16:* Let  $A = \{a_1, \dots, a_s\}$ . If each restriction  $F|_{(\mathbf{R}^n, a_i)}$  for  $i = 1, 2, \dots, s$  is injective, then  $F$  is injective. In fact, for each  $i$  and for each  $v \in \mathbf{R}^m$ , we can take a map-germ  $F^{i,v} : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, F_i(A))$  which coincides with  $F$  near  $a_j$ , ( $j \neq i$ ) and  $F + v$  near  $a_i$ . Since any  $F^{i,v}$  is an opening of  $f$ , and  $F$  is a versal opening, different branches of  $F$  can not intersect each other. (See Proposition 2.13 (1)).

Now suppose that  $f$  is a mono-germ  $f : (\mathbf{R}^n, a) \rightarrow (\mathbf{R}^m, b)$ . Assume the versal opening map-germ  $F = (f, h) : (\mathbf{R}^n, a) \rightarrow (\mathbf{R}^{m+r}, F(a))$  of  $f$  has no injective representative. Then there must be a sequence of points  $b_i$  in  $\mathbf{R}^{m+r}$  which tends to  $F(a)$  when  $i \rightarrow \infty$ , and  $a_i^1, a_i^2, a_i^1 \neq a_i^2$  in  $\mathbf{R}^n$  which tend to  $a$  respectively when  $i \rightarrow \infty$ , such that  $F(a_i^1) = F(a_i^2) = b_i$ . We may suppose  $b_i \neq b_j$  if  $i \neq j$ . Take a  $C^\infty$  function  $h$  on  $\mathbf{R}^m$  such that the support is a disjoint union of small balls centred at  $b_i$ ,  $i = 1, 2, \dots$ . Note that such function must be infinitely flat at  $b$ , that is,  $j^\infty h(b) = 0$ . Take the  $C^\infty$  function  $k = F^*h$  on  $\mathbf{R}^n$ . Since  $F$  is finite, the support of  $k$  is a disjoint union of closed neighbourhoods of  $a_i^1$  and those of  $a_i^2$ , after shrinking the neighbourhood of  $\{b_i\}$  on which  $h$  is non-zero if necessary. Take the function  $k'$  on  $\mathbf{R}^n$  which coincides with  $k$  except on the closed neighbourhoods of  $a_i^2$ , and is identically zero there. Then we see that  $k'$  is  $C^\infty$ ,  $k'$  belongs to  $\mathcal{R}_F = \mathcal{R}_f$  and  $k'(a_i^1) \neq k'(a_i^2)$  for any  $i = 1, 2, \dots$ .

In fact,  $k'$  is  $C^\infty$  on  $\mathbf{R}^n \setminus \{0\}$  and it extends to an infinitely flat function on  $\mathbf{R}^n$  at  $a$ . Moreover we have  $dk = \sum_{i=1}^m F^*(\partial h / \partial y_i) df_i + \sum_{j=1}^r F^*(\partial h / \partial z_j) dh_j$ , where  $y_1, \dots, y_m, z_1, \dots, z_r$  are the coordinates of  $\mathbf{R}^{m+r}$ . Take the function  $a_i$  (resp.  $b_j$ ) on  $\mathbf{R}^n$  which coincides with  $F^*(\partial h / \partial y_i)$  (resp.  $F^*(\partial h / \partial z_j)$ ) except on the closed neighbourhoods of  $a_i^2$ , and is identically zero there. Then  $a_i$  and  $b_j$  are  $C^\infty$  and, then we have  $dk' = \sum_{i=1}^m a_i df_i + \sum_{j=1}^r b_j dh_j \in \mathcal{J}_F = \mathcal{J}_f$ . Therefore we have  $k' \in \mathcal{R}_f$ .

Consider the opening  $(f, k') : (\mathbf{R}^n, a) \rightarrow (\mathbf{R}^{m+1}, (b, 0))$  of  $f$ . Since  $F$  is a versal opening of  $f$ , we must have  $k' = \tau \circ F$  for a function-germ  $\tau : (\mathbf{R}^n, a) \rightarrow \mathbf{R}$ . See Proposition 2.13 (1), or, the definition of versal openings (Definition 1.2). Then we must have  $k'(a_i^1) = \tau(F(a_i^1)) = \tau(F(a_i^2)) = k'(a_i^2)$  for a sufficiently large  $i$ . This leads a contradiction. Therefore we have that  $F$  is injective.  $\square$

**Remark 2.18** The quotient vector space  $\mathcal{E}_{\mathbf{R}^n, A}/\mathcal{R}_f$  is not necessarily finite-dimensional over  $\mathbf{R}$  for a finite map-germ  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$ . For example, let  $f$  be the fold map  $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$  defined by  $f(x, t) = (x, t^2)$ . Then the cuspidal-edge  $F : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$  defined by  $F(x, t) = (x, t^2, t^3)$  is a versal opening of  $f$  (Example 2.8). Then  $\mathcal{R}_f = \mathcal{R}_F = F^*\mathcal{E}_{\mathbf{R}^3, 0}$  and  $t, xt, x^2t, \dots, x^\ell t$  are linearly independent in  $\mathcal{E}_{\mathbf{R}^2, 0}/\mathcal{R}_f$  over  $\mathbf{R}$  for any  $\ell \geq 0$ . Therefore  $\dim_{\mathbf{R}} \mathcal{E}_{\mathbf{R}^2, 0}/\mathcal{R}_f = \infty$ .

**Remark 2.19** By Proposition 2.16, the versal opening  $F = (f, h_1, \dots, h_r)$  of a finite map-germ  $f$  is injective. However  $F$  is not necessarily finitely  $\mathcal{A}$ -determined. For example, the cuspidal edge is a versal opening of the fold map and it is not finitely  $\mathcal{A}$ -determined (Example 2.8 and Remark 2.18). In fact a higher order perturbation of  $F$  destroys the singular locus of  $F$ . On the other hand, whether an opening  $F$  of  $f$  is versal or not depends only on finite jets of  $h_1, \dots, h_r$ . This follows from the preparation theorem (cf. Proposition 2.13).

**Remark 2.20** Related to the above Proposition 2.16, T. Gaffney suggested a relation of the notion of openings and that of weak normalisations [6], which is left to be our open problem.

### 3 Unfoldings and openings

We recall the notion of unfolding of map-germs ([21]).

Let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  be a map-germ. An unfolding of  $f$  is a map-germ  $F : (\mathbf{R}^{n+\ell}, A \times 0) \rightarrow (\mathbf{R}^{m+\ell}, (b, 0))$  of form  $F(x, u) = (F_1(x, u), u)$  and  $F_1(x, 0) = f(x)$ , for  $(x, u) \in (\mathbf{R}^{n+\ell}, A \times 0)$ .

For another unfolding  $G : (\mathbf{R}^{n+\ell}, A \times 0) \rightarrow (\mathbf{R}^{m+\ell}, (b, 0))$ ,  $F$  and  $G$  are called *isomorphic* if there exist an unfolding  $\Sigma : (\mathbf{R}^{n+\ell}, A \times 0) \rightarrow (\mathbf{R}^{n+\ell}, A \times 0)$  of the identity map on  $(\mathbf{R}^n, A)$  and an unfolding  $T : (\mathbf{R}^{m+\ell}, (b, 0)) \rightarrow (\mathbf{R}^{m+\ell}, (b, 0))$  of the identity map on  $(\mathbf{R}^m, b)$  such that  $G \circ \Sigma = T \circ F$ .

**Proposition 3.1** (Unfoldings and openings) *Let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  be a  $C^\infty$  map-germ and  $F : (\mathbf{R}^{n+\ell}, A \times 0) \rightarrow (\mathbf{R}^{m+\ell}, (b, 0))$  be an unfolding of  $f$ . Let  $i : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^{n+\ell}, A \times 0)$  be the inclusion,  $i(x) = (x, 0)$ . Then we have:*

- (1)  $i^*\mathcal{R}_F \subset \mathcal{R}_f$ .
- (2) *If  $f$  is of corank  $\leq 1$  with  $n \leq m$ , then  $i^*\mathcal{R}_F = \mathcal{R}_f$ . If  $1, H_1, \dots, H_r$  generate  $\mathcal{R}_F$  via  $F^*$ , then  $1, i^*H_1, \dots, i^*H_r$  generate  $\mathcal{R}_f$  via  $f^*$ .*

*Proof:* For the mono-germ case the assertions are proved in Proposition 1.6 of [13], Lemma 2.4 of [14]. Here we present the proof for the general case: (1) is clear. (2) Let  $H \in \mathcal{R}_F$ . Then  $dH \in \mathcal{J}_F$ . Hence  $d(i^*H) = i^*(dH) \in i^*\mathcal{J}_F \subset \mathcal{J}_f$ . Therefore  $i^*H \in \mathcal{R}_f$ . Let  $f$  be of corank at most one. Suppose  $h \in \mathcal{R}_f$ . Then  $dh = \sum_{j=1}^m a_j df_j$  for some  $a_j \in \mathcal{E}_a$ . There exist  $A_j, B_k \in \mathcal{E}_{(a,0)}$  such that  $i^*A_j = a_j$  and the 1-form  $\sum_{j=1}^m A_j d(F_1)_j + \sum_{k=1}^\ell B_k d\lambda_k$  is closed (cf. Lemma 2.5 of [15]). Then there exists an  $H \in \mathcal{E}_{(a,0)}$  such that  $dH = \sum_{j=1}^m A_j d(F_1)_j + \sum_{k=1}^\ell B_k d\lambda_k \in \mathcal{J}_F$  and  $d(i^*H) = i^*(dH) = dh$ . Then there exists  $c \in \mathbf{R}$  such that  $h = i^*H + c = i^*(H + c)$ , and  $H + c \in \mathcal{R}_F$ . Therefore  $h \in i^*\mathcal{R}_F$ . Since  $i^*$  is a homomorphism over  $j^* : \mathcal{E}_{(b,0)} \rightarrow \mathcal{E}_b$ , where  $j : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^{m+\ell}, 0)$  is the inclusion  $j(y) = (y, 0)$ , we have the consequence.  $\square$

An unfolding  $F : (\mathbf{R}^{n+\ell}, A \times 0) \rightarrow (\mathbf{R}^{m+\ell}, (b, 0))$  of a map-germ  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  is called *extendable* if  $i^*\mathcal{R}_F = \mathcal{R}_f$  for the inclusion  $i : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^{n+\ell}, A \times 0)$ . By Proposition 3.1, we have:

**Corollary 3.2** *If corank of  $f$  is at most one, then any unfolding of  $f$  is extendable.*

In §6, we will see that there exist non-extendable unfoldings for map-germs of corank  $\geq 2$ . Therefore the opening constructions do not behave well under unfoldings in general.

## 4 Openings of stable maps of corank one

We will give the explicit versal opening in the case of corank one. As is seen in Remark 2.9, it is sufficient to treat the case of mono-germs, namely, germs  $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^m, 0)$  of corank one. Moreover, by Corollary 3.2, it is sufficient to treat the case that  $f$  is stable, namely,  $f$  is a Morin map.

Let  $k \geq 0, m \geq 0$ . To present the normal forms of Morin maps, consider variables  $t, \lambda = (\lambda_1, \dots, \lambda_{k-1}), \mu = (\mu_{ij})_{1 \leq i \leq m, 1 \leq j \leq k}$  and polynomials

$$F(t, \lambda) = t^{k+1} + \sum_{i=1}^{k-1} \lambda_i t^i, \quad G_i(t, \mu) = \sum_{j=1}^k \mu_{ij} t^j, \quad (1 \leq i \leq m).$$

Let  $f : (\mathbf{R}^{k+km}, 0) \rightarrow (\mathbf{R}^{m+k+km}, 0)$  be a Morin map defined by

$$f(t, \lambda, \mu) := (F(t, \lambda), G(t, \mu), \lambda, \mu),$$

for the above polynomials  $F$  and  $G$ .

For  $\ell \geq 0$ , we denote by  $F_{(\ell)}, G_{i(\ell)}$  the polynomials

$$F_{(\ell)}(t, \lambda) = \int_0^t s^\ell F(s, \lambda) ds, \quad G_{i(\ell)}(t, \mu) = \int_0^t s^\ell G_i(s, \mu) ds.$$

Then we have:

**Proposition 4.1** (Theorem 3 of [12]) *The ramification module  $\mathcal{R}_f$  of the Morin map  $f$  is minimally generated over  $f^*\mathcal{E}_{m+k+km}$  by the  $1+k+(k-1)m$  elements*

$$1, F_{(1)}, \dots, F_{(k)}, G_{1(1)}, \dots, G_{1(k-1)}, \dots, G_{m(1)}, \dots, G_{m(k-1)}.$$

The map-germ  $\mathbf{F} : (\mathbf{R}^{k+mk}, 0) \rightarrow (\mathbf{R}^{m+k+km} \times \mathbf{R}^{k+(k-1)m}, 0) = (\mathbf{R}^{2(k+km)}, 0)$  defined by

$$\mathbf{F} = (f, F_{(1)}, \dots, F_{(k)}, G_{1(1)}, \dots, G_{1(k-1)}, \dots, G_{m(1)}, \dots, G_{m(k-1)})$$

is a mini-versal opening of  $f$ .

*Proof:* The first half is proved in [12]. The second half follows from the definition.  $\square$

**Remark 4.2** It is shown in [12] moreover that  $\mathbf{F}$  is an isotropic map for a symplectic structure on  $\mathbf{R}^{2(k+km)}$ .

In particular we have:

**Lemma 4.3** *Let  $\ell$  be a positive integer and  $F = (F_1(t, u), u) : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$  an unfolding of  $f : (\mathbf{R}, 0) \rightarrow (\mathbf{R}, 0), f(t) = F_1(t, 0) = t^\ell$ . Suppose  $H_1, \dots, H_r \in \mathcal{R}_F \cap \mathfrak{m}_n$ . Then  $1, H_1, \dots, H_r$  generate  $\mathcal{R}_F$  via  $F^*$  if and only if  $i^*H_1, \dots, i^*H_r$  generate  $\mathfrak{m}_1^{\ell+1}/\mathfrak{m}_1^{2\ell}$ . In particular  $F_{1(1)}, \dots, F_{1(\ell-1)}$  form a system of generators of  $\mathcal{R}_F$  via  $F^*$  over  $\mathcal{E}_n$ .*

*Proof:* It is easy to show that  $\mathcal{R}_f = \mathbf{R} + \mathfrak{m}_1^\ell$ . By Proposition 2.5 (2),  $1, H_1, \dots, H_r$  generate  $\mathcal{R}_F$  as  $\mathcal{E}_n$ -module via  $F^*$  if and only if they generate  $\mathcal{R}_F/F^*(\mathfrak{m}_n)\mathcal{R}_F$  over  $\mathbf{R}$ . Since

$$\mathcal{R}_F/F^*(\mathfrak{m}_n)\mathcal{R}_F \cong (\mathbf{R} + \mathfrak{m}_1^\ell)/(f^*\mathfrak{m}_1)(\mathbf{R} + \mathfrak{m}_1^\ell) \cong \mathfrak{m}_1^{\ell+1}/\mathfrak{m}_1^{2\ell}$$

we have the consequence.  $\square$

## 5 Versal openings of analytic map-germs

In this section we discuss the case  $f$  is analytic.

First we recall the complex analytic case briefly from [13].

Let  $(X, A)$  be a germ of complex analytic space at a finite set  $A$  with the structure sheaf  $\mathcal{O}_{X,A}$ , and  $f = (f_1, \dots, f_m) : (X, A) \rightarrow (\mathbf{C}^m, b)$  a finite analytic map-germ. In the graded differential  $\mathcal{O}_{X,A}$ -algebra (de Rham algebra)  $\Omega_{X,A}$  on  $(X, A)$ , consider the graded differential ideal  $\mathcal{I}_f$  generated by  $df_1, \dots, df_m$ . Then the differential  $d$  on  $\Omega_{X,A}$  induces the  $f^*\mathcal{O}_{\mathbf{C}^m}$ -homomorphism  $d : \Omega_{X,A}/\mathcal{I}_f \rightarrow \Omega_{X,A}/\mathcal{I}_f$  and then  $\mathcal{O}_{\mathbf{C}^m,b}$ -homomorphism  $d : f_*(\Omega_{X,A}/\mathcal{I}_f) \rightarrow f_*(\Omega_{X,A}/\mathcal{I}_f)$ . Then we consider the  $i$ -th cohomology  $\mathcal{H}^i(f_*(\Omega_{X,A}/\mathcal{I}_f); d)$  for the complex  $(f_*(\Omega_{X,A}/\mathcal{I}_f), d)$ . It is evident that  $\mathcal{H}^i(f_*(\Omega_{X,A}/\mathcal{I}_f); d)$  is a finite  $\mathcal{O}_{\mathbf{C}^m,b}$ -module (cf. Remark 2.6). In fact, we have moreover:

**Proposition 5.1** (Proposition 1.1 of [13])  *$\mathcal{H}^i(f_*(\Omega_{X,A}/\mathcal{I}_f); d)$  is a coherent  $\mathcal{O}_{\mathbf{C}^m,b}$ -module ( $i = 0, 1, 2, \dots$ ).*

We remark that the stalk of  $\mathcal{H}^0(f_*(\Omega_{X,A}/\mathcal{I}_f); d)$  at  $A$  is the complex analytic counterpart of  $\mathcal{R}_f$  in the real  $C^\infty$  case. We write it  $\mathcal{R}_f^{\text{hol}}$  to distinguish with the real  $C^\infty$  case. By Proposition 5.1, in particular, that  $\mathcal{R}_f^{\text{hol}}$  is a finite  $\mathcal{O}_{\mathbf{C}^m}$ -module.

Now let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  be a finite real analytic map-germ. We denote by  $\mathcal{O}_{\mathbf{R}^n,A}$  (resp.  $\mathcal{O}_{\mathbf{R}^m,b}$ ) the germ of sheaf of analytic functions on  $(\mathbf{R}^n, A)$  (resp.  $(\mathbf{R}^m, b)$ ). Then, besides with  $\mathcal{R}_f$ , we consider the sheaf

$$\mathcal{R}_f^\omega := \{h \in \mathcal{O}_{\mathbf{R}^n,A} \mid dh \in \langle df_1, \dots, df_m \rangle_{\mathcal{O}_{\mathbf{R}^n,A}}\}$$

and the direct image  $f_*(\mathcal{R}_f^\omega)$  as  $\mathcal{O}_{\mathbf{R}^m,b}$ -module. Then we see that, in particular,  $f_*(\mathcal{R}_f^\omega)$  is a finite  $\mathcal{O}_{\mathbf{R}^m,b}$ -module by Proposition 5.1 in the case  $X = (\mathbf{C}^n, A)$ . Thus we have that  $f_*(\mathcal{R}_f^\omega)$  is generated over  $\mathcal{O}_{\mathbf{R}^m,b}$  by some  $1, h_1, \dots, h_r \in \mathcal{R}_f^\omega$ . Moreover it turns out that  $F = (f, h_1, \dots, h_r) : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^{m+r}, b \times h(A))$  is injective (See [13]).

Then we show the following:

**Proposition 5.2** *Let  $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^m, 0)$  be a finite analytic map-germ. Suppose  $1, h_1, \dots, h_r$  generate  $\mathcal{R}_f^\omega$  over  $\mathcal{O}_{\mathbf{R}^m,0}$  via  $f^*$ . Then  $1, h_1, \dots, h_r$  generate  $\mathcal{R}_f$  over  $\mathcal{E}_{\mathbf{R}^m,0}$  via  $f^*$ .*

*Proof:* First we may suppose  $h_i(0) = 0, (1 \leq i \leq r)$ . The opening

$$F = (f_1, \dots, f_m, h_1, \dots, h_r) : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^{m+r}, 0)$$

of  $f$  is injective by [13].

Let  $\mathcal{F}_p$  stand for the  $\mathbf{R}$ -algebra of formal functions on  $(\mathbf{R}^n, p)$ . ( $\mathcal{F}_p = \mathcal{E}_{\mathbf{R}^n, p} / \mathfrak{m}_{\mathbf{R}^n, p}^\infty$ , and it is the completion of  $\mathcal{O}_{\mathbf{R}^n, p}$  as well.) Define a sheaf  $\tilde{\mathcal{F}}_{\mathbf{R}^n}$  on  $\mathbf{R}^n$  by  $\tilde{\mathcal{F}}_{\mathbf{R}^n}(U) = \prod_{p \in U} \mathcal{F}_p$ , for any open subset  $U$  of  $\mathbf{R}^n$ . Denote by  $\tilde{\mathcal{F}}_n$  the stalk of  $\tilde{\mathcal{F}}_{\mathbf{R}^n}$  at 0. For the definition see also Ch. III §4, pp. 45–46 of [20]. Then  $\tilde{\mathcal{F}}_n$  is faithfully flat over  $\mathcal{O}_n$  (Ch. III §4, p. 47, Corollary 4.13 of [20]). Define the formal counterpart

$$\tilde{\mathcal{R}}_f = \{(\hat{g}_p)_{p \in (\mathbf{R}^n, 0)} \in \tilde{\mathcal{F}}_n \mid d\hat{g}_p \in \langle d\hat{f}_{1,p}, \dots, d\hat{f}_{n,p} \rangle_{\mathcal{F}_p}, p \in (\mathbf{R}^n, 0)\}$$

of  $\mathcal{R}_f$ . Here  $\hat{g}_p$  means the formal function at a point  $p$  near 0 defined by  $g \in \mathcal{E}_{\mathbf{R}^n, p}$ . Then, for our  $F = (f, h_1, \dots, h_r)$ , we have that  $1, h_1, \dots, h_r$  generate  $\tilde{\mathcal{R}}_f$  over  $\tilde{\mathcal{F}}_n$ .

For a map-germ  $F : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^{m+r}, 0)$ , we say that a function-germ  $g \in \mathcal{E}_n$  *formally belongs* to  $F^*\mathcal{E}_{m+r}$  if, for any  $q \in (\mathbf{R}^{m+r}, 0)$ , there exists a formal function  $\hat{k} \in \mathcal{F}_q$  such that, for any  $p \in F^{-1}(q)$ ,  $\hat{g}_p = \hat{F}^*\hat{k}$ . Then, it is known the following Glaeser's type theorem on characterisation of composite differentiable functions: If  $g \in \mathcal{E}_n$  formally belongs to  $F^*\mathcal{E}_{m+r}$ , then  $g$  belongs to  $F^*(\mathcal{E}_{m+r})$ . In fact the theorem follows, for instance, from Theorem D and Theorem C (3) of [2]. (We apply Theorem D of [2] to the case that  $\phi = F$  is a finite map-germ, and  $X = (\mathbf{R}^n, 0), Y = (\mathbf{R}^{m+r}, 0), s = 1, p = q = r = 1, A = 1, B = 0$  for the notations in [2].) See also Theorem 11.8 of [3].

Since  $F$  is injective in our case, we have that any element of  $\mathcal{R}_f$  formally belongs to  $F^*\mathcal{E}_{m+r}$ , therefore we have  $\mathcal{R}_f \subseteq F^*\mathcal{E}_{m+r}$ . Since  $F$  is an opening of  $f$ , we have that  $\mathcal{R}_f = F^*\mathcal{E}_{m+r}$ .

Let  $\pi : (\mathbf{R}^{m+r}, 0) \rightarrow (\mathbf{R}^m, 0)$  be the projection. Then  $\pi^* : \mathcal{E}_m \rightarrow \mathcal{E}_{m+r}$  is the inclusion. Regard  $F^*\mathcal{E}_{m+r}$  as an  $\mathcal{E}_{m+r}$ -module via  $F^*$ . By the preparation theorem,  $1, h_1, \dots, h_r$  generate  $F^*\mathcal{E}_{m+r} = \mathcal{R}_f$  as  $\mathcal{E}_m$ -module via  $F^* \circ \pi^* = f^*$  if  $1, h_1, \dots, h_r$  generate  $F^*\mathcal{E}_{m+r} / (f^*\mathfrak{m}_m)F^*\mathcal{E}_{m+r}$  over  $\mathbf{R}$ .

We will show that

$$\mathfrak{m}_n^\infty \cap F^*\mathcal{E}_{m+r} \subseteq (f^*\mathfrak{m}_m)F^*\mathcal{E}_{m+r}.$$

Define  $h = \sum_{i=1}^m f_i^2 : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}, 0)$ . Since  $f$  is finite,  $h^{-1}(0) = \{0\}$ , and moreover, the norms of  $1/h$  and its partial derivatives up to order say  $\ell$  are bounded above by  $1/\|x\|^\alpha$  for some  $\alpha = \alpha(\ell) > 0$ . Then  $1/h$  is a multiplier for the ideal  $\mathfrak{m}_n^\infty$  in the sense of Malgrange (Ch. IV §1, p.54, Proposition 1.4



of [20]). Hence, for any  $k \in \mathfrak{m}_n^\infty$ ,  $k/h$  is a  $C^\infty$  function on  $(\mathbf{R}^n, 0)$  and it is an element of  $\mathfrak{m}_n^\infty$ , if we set  $(k/h)(0) = 0$ . Moreover let  $k \in \mathfrak{m}_n^\infty \cap F^*\mathcal{E}_{m+r}$ . Then  $k/h$  formally belongs to  $F^*\mathcal{E}_{m+r}$ . In fact, the Taylor series of  $k/h$  at  $0 \in \mathbf{R}^n$  is 0, and, outside of 0,  $k/h$  is a composite function of  $F$ . Therefore, again by the Glaeser's type theorem as above, we have  $k/h \in F^*\mathcal{E}_{m+r}$ . Then

$$k = \left( \sum_{i=1}^m f_i^2 \right) (k/h) = \sum_{i=1}^m f_i (f_i k/h) \in (f^*\mathfrak{m}_m) F^*\mathcal{E}_{m+r}.$$

Thus we have  $\mathfrak{m}_n^\infty \cap F^*\mathcal{E}_{m+r} \subseteq (f^*\mathfrak{m}_m) F^*\mathcal{E}_{m+r}$ .

Now let  $H \in \mathcal{R}_f = F^*\mathcal{E}_{m+r}$ . Then

$$\begin{aligned} H &\equiv a_0 \circ f + a_1 \circ f \cdot h_1 + \cdots + a_r \circ f \cdot h_r \\ &\equiv a_0(0) + a_1(0)h_1 + \cdots + a_r(0)h_r, \end{aligned}$$

for some functions  $a_0, a_1, \dots, a_r$ , modulo  $(f^*\mathfrak{m}_m) F^*\mathcal{E}_{m+r} + \mathfrak{m}_n^\infty \cap F^*\mathcal{E}_{m+r} \subseteq (f^*\mathfrak{m}_m) F^*\mathcal{E}_{m+r}$ . Thus we see  $1, h_1, \dots, h_r$  generate  $F^*\mathcal{E}_{m+r}/(f^*\mathfrak{m}_m) F^*\mathcal{E}_{m+r}$  over  $\mathbf{R}$ , and therefore they generate  $\mathcal{R}_f$  over  $\mathcal{E}_{\mathbf{R}^m, 0}$  via  $f^*$ .  $\square$

## 6 The cases of corank $\geq 2$

If  $\text{corank}(f) \geq 2$ , then the restriction of a versal opening of an unfolding of  $f$  is not necessarily a versal opening of  $f$ . That phenomenon was observed already in [15]. We utilise Proposition 5.2 if necessary to treat the following examples.

**Example 6.1** (cf. Example 2.8.) Let  $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ ,  $h(x, y) = (\frac{1}{2}x^2, \frac{1}{2}y^2) = (z, w)$ . Then  $\mathcal{R}_f$  is minimally generated by  $1, x^3, y^3, x^3y^3$  over  $f^*\mathcal{E}_{\mathbf{R}^2, 0}$ . Therefore if we set  $F : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^5, 0)$  by

$$F(x, y) = \left( \frac{1}{2}x^2, \frac{1}{2}y^2, x^3, y^3, x^3y^3 \right),$$

then  $F$  is the mini-versal opening of  $f$ .

Here we give a concrete method to find the minimal generators as above.

Let  $h \in \mathcal{E}_{\mathbf{R}^2, 0} = \mathcal{E}_2$ . Then by the preparation theorem we have

$$h \equiv (a \circ f)x + (b \circ f)y + (c \circ f)xy, \pmod{f^*\mathcal{E}_2}.$$

The condition that  $h \in \mathcal{R}_f$  is equivalent to that  $dh$  belongs to Jacobi module  $\mathcal{J}_f$ . We calculate

$$dh \equiv (a \circ f)dx + (b \circ f)dy + (c \circ f)(ydx + xdy), \pmod{\mathcal{J}_f},$$

and set

$$(a \circ f)dx + (b \circ f)dy + (c \circ f)(ydx + xdy) = Axdx + Bydy,$$

for some function  $A, B \in \mathcal{E}_2$ . Again by the preparation theorem, we put

$$\begin{aligned} A &= (a_1 \circ f) + (a_2 \circ f)x + (a_3 \circ f)y + (a_4 \circ f)xy, \\ B &= (b_1 \circ f) + (b_2 \circ f)x + (b_3 \circ f)y + (b_4 \circ f)xy. \end{aligned}$$

Then

$$\begin{aligned} Ax &= (a_1 \circ f)x + (a_2 \circ f)x^2 + (a_3 \circ f)xy + (a_4 \circ f)x^2y \\ &= 2(za_2) + a_1x + 2(za_4)y + a_3xy, \\ By &= (b_1 \circ f)y + (b_2 \circ f)xy + (b_3 \circ f)y^2 + (b_4 \circ f)xy^2 \\ &= 2(wb_3) + 2(wb_4) + b_1y + b_2xy. \end{aligned}$$

omitting “ $\circ f$ ”, where  $z = \frac{1}{2}x^2$  and  $w = \frac{1}{2}y^2$ . Then we have

$$\begin{aligned} a + cy &= 2za_2 + a_1x + (2za_4)y + a_3xy, \\ b + cx &= 2wb_3 + 2(wb_4) + b_1y + b_2xy. \end{aligned}$$

and therefore

$$\begin{aligned} (a - 2za_2) + (-a_1)x + (c - 2za_4)y + (-a_3)xy &= 0, \\ (b - 2wb_3) + (c - 2wb_4)x + (-b_1)y + (-b_2)xy &= 0. \end{aligned}$$

Since  $\mathcal{E}_2$  is free over  $f^*\mathcal{E}_2$  in this example, we have

$$a = 2za_2, a_1 = 0, c = 2za_4, a_3 = 0, b = 2wb_3, c = 2wb_4, b_1 = 0, b_2 = 0.$$

Then we have  $(a_4, b_4) = k(w, z)$  for a function  $k \in \mathcal{E}_2$  and

$$h \equiv (2za_2 \circ f)x + (2wb_3 \circ f)y + (2zwk \circ f)xy, \pmod{f^*\mathcal{E}_2}.$$

Thus we find a minimal system of generators  $1, 2zx, 2wy, 2zwx$ , namely  $1, x^3, y^3, x^3y^3$  of  $\mathcal{R}_f$  over  $f^*\mathcal{E}_2$ .

In each case of the following three examples, we have the mini-versal openings using Proposition 5.2.

**Example 6.2** Let  $g : (\mathbf{R}^3, 0) \rightarrow (\mathbf{R}^3, 0)$  be a map-germ defined by

$$g(x, y, u) = (z, w, u) = \left(\frac{1}{2}x^2 + uy, \frac{1}{2}y^2 + ux, u\right),$$

which is an unfolding of  $f$  in Example 6.1. Then  $\mathcal{R}_g$  is minimally generated over  $g^*\mathcal{E}_{\mathbf{R}^3,0}$  by 1 and

$$\begin{aligned}\psi_3 &= x^3 + y^3 + 3xyu, \\ \psi_5^{5,0} &= x^5 + 5x^3yu - 12x^2u^3 + 9yu^4, \\ \psi_5^{0,5} &= y^5 + 5xy^3u - 12y^2u^3 + 9xu^4, \\ \psi_6^{3,3} &= x^3y^3 - 12x^2y^2u^2 - 11x^3u^3 - 11y^3u^3 - 12xyu^4.\end{aligned}$$

Therefore  $i^*\mathcal{R}_g \subsetneq \mathcal{R}_f$ , where  $i : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$ ,  $i(x, y) = (x, y, 0)$ , and we see that  $g$  is not an extendable unfolding of  $f$ .

The versal opening of  $g$  is given by  $G : (\mathbf{R}^3, 0) \rightarrow (\mathbf{R}^7, 0) = (\mathbf{R}^3 \times \mathbf{R}^4, 0)$ ,

$$\begin{aligned}G(x, y, u) &= (g(x, y, u), x^3 + y^3 + 3xyu, x^5 + 5x^3yu - 12x^2u^3 + 9yu^4, \\ & y^5 + 5xy^3u - 12y^2u^3 + 9xu^4, x^3y^3 - 12x^2y^2u^2 - 11x^3u^3 - 11y^3u^3 - 12xyu^4).\end{aligned}$$

Then

$$G(x, y, 0) = \left(\frac{1}{2}x^2, \frac{1}{2}y^2, x^3 + y^3, x^5, y^5, x^3y^3\right)$$

is not a versal opening of  $f = (\frac{1}{2}x^2, \frac{1}{2}y^2)$ . Note that the element  $\psi_3$  gives a Lagrange immersion of type  $D_4^+$ , which is a Lagrange stable lifting of  $g$ . Other elements are obtained by operating lowerable vector fields of  $g$  to  $\psi_3$ . See §7.

**Example 6.3** (Hyperbolic case.) Let  $h : (\mathbf{R}^4, 0) \rightarrow (\mathbf{R}^4, 0)$  be the stable map-germ

$$h(x, y, \lambda, \mu) = (z, w, \lambda, \mu) = \left(\frac{1}{2}x^2 + y\lambda, \frac{1}{2}y^2 + x\mu, \lambda, \mu\right).$$

of  $\mathcal{K}$ -class  $I_{2,2}$  ([22]). Then  $\mathcal{R}_f$  is minimally generated over  $h^*\mathcal{E}_{\mathbf{R}^4,0}$  by 1 and

$$\begin{aligned}\varphi_4 &= x^3\mu + y^3\lambda + 3xy\lambda\mu \\ \varphi_5^{3,2} &= x^3y^2 - 2x^2y\lambda\mu + x\lambda^2\mu^2 \\ \varphi_5^{2,3} &= x^2y^3 - 2xy^2\lambda\mu + y\lambda^2\mu^2 \\ \varphi_5^{5,0} &= x^5 + 5x^3y\lambda + 15y\lambda^3\mu \\ \varphi_5^{0,5} &= y^5 + 5xy^3\mu + 15x\lambda\mu^3 \\ \varphi_6 &= x^3y^3 - 3xy\lambda^2\mu^2.\end{aligned}$$

We have the mini-versal opening  $H : (\mathbf{R}^4, 0) \rightarrow (\mathbf{R}^4 \times \mathbf{R}^6, 0) = (\mathbf{R}^{10}, 0)$  of  $h$  by

$$H = (f, \varphi_4, \varphi_5^{3,2}, \varphi_5^{2,3}, \varphi_5^{5,0}, \varphi_5^{0,5}, \varphi_6).$$

Moreover we see that

$$j^*\mathcal{R}_h \subsetneq \mathcal{R}_g(\subsetneq \mathcal{E}_{\mathbf{R}^3,0}), \quad (j \circ i)^*\mathcal{R}_h \subsetneq i^*\mathcal{R}_g \subsetneq \mathcal{R}_f(\subsetneq \mathcal{E}_{\mathbf{R}^3,0}),$$

where  $j : (\mathbf{R}^3, 0) \rightarrow (\mathbf{R}^4, 0)$ ,  $j(x, y, u) = (x, y, u, u)$ . Thus the unfolding  $h$  of  $f$  is not extendable, which is also not extendable regarded as an unfolding of  $g$  as well.

Now we show the concrete way of calculations for Example 6.3 to make sure ourselves:

Let  $k \in \mathcal{E}_{\mathbf{R}^4,0} = \mathcal{E}_4$ . By the preparation theorem, we set,

$$k = (a_0 \circ h) + (a_1 \circ h)x + (a_2 \circ h)y + (a_3 \circ h)xy.$$

Then

$$dk \equiv ((a_1 \circ h) + (a_3 \circ h)y)dx + ((a_2 \circ h) + (a_3 \circ h)x)dy \pmod{\mathcal{J}_h}.$$

We suppose  $dk$  is equal to the form

$$\begin{aligned} Adz + Bdw + Cd\lambda + Dd\mu \\ = (Ax + B\mu)dx + (A\lambda + By)dy + (Ay + C)d\lambda + (Bx + D)d\mu. \end{aligned}$$

Then we have

$$\begin{aligned} (a_1 \circ h) + (a_3 \circ h)y = Ax + B\mu, \quad (a_2 \circ h) + (a_3 \circ h)x = A\lambda + By, \\ Ay + C = 0, \quad Bx + D = 0. \end{aligned}$$

Then  $C = -Ay$ ,  $D = -Bx$ . Now set

$$\begin{aligned} A &= (A_0 \circ h) + (A_1 \circ h)x + (A_2 \circ h)y + (A_3 \circ h)xy, \\ B &= (B_0 \circ h) + (B_1 \circ h)x + (B_2 \circ h)y + (B_3 \circ h)xy. \end{aligned}$$

Then we have

$$Ax = (2zA_1 - 4wA_3) + (A_0 + 4\lambda\mu A_3)x + (-2\lambda A_1 + 2zA_3)y + A_2xy,$$

omitting “ $\circ h$ ”. Similarly we have

$$\begin{aligned} By &= (2wB_2 - 4z\mu B_3) + (-2\mu B_2 + 2wB_3)x + (B_0 + 4\lambda\mu B_3)y + B_1xy, \\ A\lambda &= (\lambda A_0) + (\lambda A_1)x + (\lambda A_2)y + (\lambda A_3)xy, \\ B\mu &= (\mu B_0) + (\mu B_1)x + (\mu B_2)y + (\mu B_3)xy. \end{aligned}$$

Then we set, to find analytic or formal generators,

$$\begin{aligned} a_1 &= 2zA_1 - 4w\lambda A_3 + \mu B_0, & a_3 &= -2\lambda A_1 + 2zA_3 + \mu B_2, \\ 0 &= A_0 + 4\lambda\mu A_3 + \mu B_1, & 0 &= A_2 + \mu B_3, \end{aligned}$$

and

$$\begin{aligned} a_2 &= 2wB_2 - 4z\mu B_3 + \lambda A_0, & a_3 &= -2\mu B_2 + 2wB_3 + \lambda A_1, \\ 0 &= B_0 + 4\lambda\mu B_3 + \lambda A_2, & 0 &= B_1 + \lambda A_3. \end{aligned}$$

Then we are led to the relation:

$$3\lambda A_1 - 3\mu B_2 - 2zA_3 + 2wB_3 = 0 \cdots \cdots \cdots (*).$$

If  $A_1, A_2, B_2, B_3$  satisfy the relation (\*), then  $A_0, A_2, B_0, B_1$  are determined from them and so  $a_1 \circ h, a_2 \circ h, a_3 \circ h$ :

$$A_0 = -3\lambda\mu A_3, \quad A_2 = -\mu B_3, \quad B_0 = -3\lambda\mu B_3, \quad B_1 = -\lambda A_3,$$

and

$$\begin{aligned} a_1 &= 2zA_1 - 4w\lambda A_3 - 3\lambda\mu^2 B_3, \\ a_2 &= -3\lambda^2\mu A_3 + 2wB_2 - 4z\mu B_3, \\ a_3 &= -2\lambda A_1 + 2zA_3 + \mu B_2 (= \lambda A_1 - 2\mu B_2 + 2wB_3). \end{aligned}$$

Since  $(3\lambda, -3\mu, -2x, 2w)$  are regular sequence in  $\mathcal{E}_4$ , the first Koszul cohomology for them vanishes (see for instance, [23]). Then, by setting  $(A_1, B_2, A_3, B_3)$  as

$$\begin{aligned} &(\mu, \lambda, 0, 0), (2z, 0, 3\lambda, 0), (2w, 0, 0, -3\lambda), \\ &(0, 2z, -3\mu, 0), (0, 2w, 0, 3\mu), (0, 0, w, z), \end{aligned}$$

respectively, we have elements  $\varphi_4, \varphi_5^{3,2}, \varphi_5^{2,3}, \varphi_5^{5,0}, \varphi_5^{0,5}, \varphi_6$ . such that

$$1, \varphi_4, \varphi_5^{3,2}, \varphi_5^{2,3}, \varphi_5^{5,0}, \varphi_5^{0,5}, \varphi_6$$

generate  $\mathcal{R}_h^\omega$  over  $h^*\mathcal{O}_4$ . Then, by Proposition 5.2, they generate  $\mathcal{R}_h$  over  $h^*\mathcal{E}_4$ .

Similarly we have the following.

**Example 6.4** (Elliptic case.) Let  $k : (\mathbf{R}^4, 0) \rightarrow (\mathbf{R}^4, 0)$  be the stable map-germ given by

$$k(x, y, \lambda, \mu) = \left( \frac{1}{2}(x^2 - y^2) + \lambda x + \mu y, \quad xy + \mu x - \lambda y, \quad \lambda, \quad \mu \right),$$

of  $\mathcal{K}$ -class  $II_{2,2}$ . Then  $\mathcal{R}_k$  is minimally generated over  $k^*\mathcal{E}_4$  by 1 and  $\rho_4, \rho_5^{3,2}, \rho_5^{2,3}, \rho_5^{5,0}, \rho_5^{0,5}, \rho_6$ , where

$$\begin{aligned}\rho &= a_1x + a_2y + \frac{1}{2}a_3(x^2 + y^2), \\ a_1 &= 2zA_1 + 2wA_2 + \left(-\frac{3}{2}\lambda^3 - \frac{3}{2}\lambda\mu^2 - 3z\lambda - 3w\mu\right)A_3 \\ &\quad + \left(-\frac{3}{2}\lambda^2\mu - \frac{3}{2}\mu^3 - 3z\mu + 3w\lambda\right)B_3, \\ a_2 &= -2wA_1 + 2zA_2 + \left(\frac{3}{2}\lambda^2\mu - \frac{3}{2}\mu^3 + z\mu - w\lambda\right)A_3, \\ &\quad + \left(\frac{3}{2}\lambda^3 + \frac{3}{2}\lambda\mu^2 - z\lambda - 3w\mu\right)B_3, \\ a_3 &= -\lambda A_1 - \mu A_2 + \left(z - \frac{1}{2}\lambda^2 + \frac{1}{2}\mu^2\right)A_3 + (w - \lambda\mu)B_3,\end{aligned}$$

and  $\rho = \rho_4, \rho_5^{3,2}, \rho_5^{2,3}, \rho_5^{5,0}, \rho_5^{0,5}, \rho_6$  respectively for

$$\begin{aligned}(A_1, A_2, A_3, B_3) &= (\lambda, \mu, 0, 0), \\ &= (0, z - \frac{3}{2}\lambda^2 + \frac{3}{2}\mu^2, 0, 3\lambda), \\ &= (0, w - 3\lambda\mu, -3\lambda, 0), \\ &= (z - \frac{3}{2}\lambda^2 + \frac{3}{2}\mu^2, 0, 0, -3\mu), \\ &= (w - 3\lambda\mu, 0, 3\mu, 0), \\ &= (0, 0, z - \frac{3}{2}\lambda^2 + \frac{3}{2}\mu^2, w - 3\lambda\mu).\end{aligned}$$

Note that, in the process of calculations, we see that  $A_1, A_2, A_3, B_3$  obey the relation

$$(3\mu)A_1 + (-3\lambda)A_2 + (-w + 3\lambda\mu)A_3 + \left(z - \frac{3}{2}\lambda^2 + \frac{3}{2}\mu^2\right)B_3 = 0.$$

and that  $3\mu, -3\lambda, -w + 3\lambda\mu, z - \frac{3}{2}\lambda^2 + \frac{3}{2}\mu^2$  form a regular sequence in  $\mathcal{E}_4$ .

**Theorem 6.5** *Any stable mono-germ  $(\mathbf{R}^4, a) \rightarrow (\mathbf{R}^4, b)$ , and therefore any stable multi-germ  $(\mathbf{R}^4, A) \rightarrow (\mathbf{R}^4, b)$  has a versal opening.*

*Proof:* Let  $f : (\mathbf{R}^4, a) \rightarrow (\mathbf{R}^4, b)$  be a stable map-germ. Then  $f$  is of corank  $\leq 1$  and is diffeomorphic to a Morin map or  $f$  is of corank 2 and is diffeomorphic to the germ  $h$  of Example 6.3 or  $k$  of Example 6.4 (see for instance [7]). In the case of corank one, we have constructed the versal opening in Proposition 4.1. In the case of corank two, we have constructed the versal opening, using a normal form of  $f$  in Examples 6.3 and Example 6.4. If a map-germ  $\ell$  has a versal opening and  $f$  is diffeomorphic to  $\ell$ , then  $f$  has a versal opening (cf. Lemma 2.2 (3)). Therefore  $f$  has a versal opening. Let  $f : (\mathbf{R}^4, A) \rightarrow (\mathbf{R}^4, b)$  be a stable multi-germ. Then, for each  $a_i \in A$ , the restriction  $f_i : (\mathbf{R}^4, a_i) \rightarrow (\mathbf{R}^4, b)$  of  $f$  to  $(\mathbf{R}^4, a_i)$  is a stable germ. Since each  $f_i$  has a versal opening, we see that  $f$  itself has a versal opening by Remark 2.9.  $\square$

## 7 Openings and lowerable vector fields

Let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  a map-germ. A germ of vector field  $\xi$  over  $(\mathbf{R}^n, A)$  is called *lowerable* for  $f$ , or *f-lowerable*, if there exists a germ of vector field  $\eta$  over  $(\mathbf{R}^m, b)$  such that  $f_*\xi = \eta \circ f$  as a germ of vector field along  $f$ . The lowerable vector fields form an  $f^*\mathcal{E}_{\mathbf{R}^m, b}$ -module, which is denoted by  $\mathfrak{X}_f$ .

**Lemma 7.1** (cf. [14]) *Let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  be a map-germ and  $\xi$  a lowerable vector field for  $f$ . Then  $\xi(f^*\mathcal{E}_{\mathbf{R}^m, b}) \subseteq f^*\mathcal{E}_{\mathbf{R}^m, b}$  and  $\xi(\mathcal{R}_f) \subseteq \mathcal{R}_f$ . Therefore  $\mathfrak{X}_f(f^*\mathcal{E}_{\mathbf{R}^m, b}) \subseteq f^*\mathcal{E}_{\mathbf{R}^m, b}$  and  $\mathfrak{X}_f(\mathcal{R}_f) \subseteq \mathcal{R}_f$ .*

*Proof:* Suppose  $f_*\xi = \eta \circ f$  as in the above definition.

Let  $a \in f^*\mathcal{E}_{\mathbf{R}^m, b}$ . Then

$$\begin{aligned} \xi(f^*a)(x) &= \langle d(f^*a)(x), \xi(x) \rangle = \langle f^*(da)(x), \xi(x) \rangle = \langle (da)(f(x)), f_*(\xi(x)) \rangle \\ &= \langle (da)(f(x)), \eta(f(x)) \rangle = (\eta a)(f(x)) = (f^*(\eta a))(x). \end{aligned}$$

Therefore  $\xi(f^*a) = f^*(\eta a)$ .

Let  $b \in \mathcal{R}_f$ . Then  $db = \sum_{i=1}^m p_i df_i$  for some  $p_i \in \mathcal{E}_{\mathbf{R}^n, A}$ . Then we have

$$d(\xi b) = L_\xi \left( \sum_{i=1}^m p_i df_i \right) = \sum_{i=1}^m (\xi p_i) df_i + \sum_{i=1}^m p_i d(\xi f_i).$$

Since each  $\xi f_i \in f^*\mathcal{E}_{\mathbf{R}^m, b} \subseteq \mathcal{R}_f$ , we see  $d(\xi b) \in \mathcal{J}_f$ . Therefore  $\xi b \in \mathcal{R}_f$ .  $\square$

The differential operator  $D : \mathcal{E}_{\mathbf{R}^n, A} \rightarrow \mathcal{E}_{\mathbf{R}^n, A}$  is called *lowerable* for  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$ , if  $D$  is a finite sum of operators of form

$$\xi_1 \xi_2 \cdots \xi_s$$

with coefficients in  $f^*\mathcal{E}_{\mathbf{R}^m, b}$ , where  $\xi_1, \xi_2, \dots, \xi_s$  are lowerable vector field (regarded as first order differential operators) for  $f$ . The lowerable differential operators form a (non-commutative)  $f^*\mathcal{E}_{\mathbf{R}^m, b}$ -algebra, which is denoted by  $\mathfrak{L}_f$ .

By Lemma 7.1, we have:

**Corollary 7.2** *Let  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b)$  be a map-germ and  $D : \mathcal{E}_{\mathbf{R}^n, A} \rightarrow \mathcal{E}_{\mathbf{R}^n, A}$  be a lowerable differential operator for  $f$ . Then  $D(f^*\mathcal{E}_{\mathbf{R}^m, b}) \subseteq f^*\mathcal{E}_{\mathbf{R}^m, b}$  and  $D(\mathcal{R}_f) \subseteq \mathcal{R}_f$ . Therefore  $\mathfrak{L}_f(f^*\mathcal{E}_{\mathbf{R}^m, b}) \subseteq f^*\mathcal{E}_{\mathbf{R}^m, b}$  and  $\mathfrak{L}_f(\mathcal{R}_f) \subseteq \mathcal{R}_f$ .*

We conclude the present paper by examining how lowerable vector fields act on the ramification modules and the structure of mini-versal openings for Example 6.1, Example 6.2 and Example 6.3.

**Example 7.3** The module of lowerable vector fields for  $f$  in Example 6.1 is generated by

$$\xi_1 = x \frac{\partial}{\partial x}, \quad \xi_2 = y \frac{\partial}{\partial y}$$

over  $f^*\mathcal{E}_2$ . Then we have

$$\begin{aligned} \xi_1(x^3) &= 3x^3, & \xi_2(x^3) &= 0, \\ \xi_1(y^3) &= 0, & \xi_2(y^3) &= 3y^3, \\ \xi_1(x^3y^3) &= 3x^3y^3, & \xi_2(x^3y^3) &= 3x^3y^3. \end{aligned}$$

Define the module  $S = \langle 1, x^3, y^3 \rangle_{f^*\mathcal{E}_m} \subsetneq \mathcal{R}_f$  over  $f^*\mathcal{E}_m$ . Then we see

$$\mathfrak{X}_f(S) = S, \quad \mathfrak{L}_f(S) = S.$$

The module of lowerable vector fields for  $g$  in Example 6.2 is generated by

$$\left\{ \begin{aligned} \xi_0 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + u \frac{\partial}{\partial u}, \\ \xi_1 &= \left(-\frac{3}{2}u^3 + \frac{1}{3}zx\right) \frac{\partial}{\partial x} + \left(-\frac{1}{2}u^2x\right) \frac{\partial}{\partial y} + \left(-\frac{1}{3}zu\right) \frac{\partial}{\partial u}, \\ \xi_2 &= \left(\frac{1}{3}wx + \frac{1}{2}u^2y\right) \frac{\partial}{\partial x} + \left(\frac{3}{2}u^3 - \frac{1}{2}uxy\right) \frac{\partial}{\partial y} + \frac{2}{3}wu \frac{\partial}{\partial u}, \\ \xi_3 &= \left(-\frac{3}{2}wu^2 - \frac{1}{2}zuy + \frac{1}{2}wxy\right) \frac{\partial}{\partial x} + \left(\frac{3}{2}u^3 + \frac{1}{2}zxy\right) \frac{\partial}{\partial y} + (-zw) \frac{\partial}{\partial u}, \end{aligned} \right.$$

over  $g^*\mathcal{E}_2$ , where  $z = \frac{1}{2}x^2 + uy$ ,  $w = \frac{1}{2}y^2 + ux$ . Then we have that

$$1, \xi_0\psi_3, \xi_1\psi_3, \xi_2\psi_3, \xi_3\psi_3$$

generate  $\mathcal{R}_g$  over  $g^*\mathcal{E}_3$ . This fact is a consequence of Lagrange stability of induced Lagrangian immersion from  $(g, \psi_3)$ ,  $\tilde{g} : (\mathbf{R}^3, 0) \rightarrow T^*\mathbf{R}^3 = \mathbf{R}^6$  defined by

$$\tilde{g} = (z, w, u, p_1, p_2, p_3) = \left(z = \frac{1}{2}x^2 + uy, w = \frac{1}{2}y^2 + ux, u, 3x, 3y, -3xy\right).$$

See [11]. Therefore, if we consider the  $g^*\mathcal{E}_3$ -module  $T$  generated by  $1, \psi_3$ , then we have that

$$\mathfrak{X}_g(T) = \mathcal{R}_g, \quad \mathfrak{L}_g(T) = \mathcal{R}_g.$$



The lowerable vector fields for  $h$  in Example 6.3 is generated by

$$\left\{ \begin{array}{l} \xi_0 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \lambda \frac{\partial}{\partial \lambda} + \mu \frac{\partial}{\partial \mu}, \\ \xi_1 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + (3\lambda) \frac{\partial}{\partial \lambda} + (-3\mu) \frac{\partial}{\partial \mu}, \\ \xi_2 = (-3\lambda\mu + xy) \frac{\partial}{\partial x} + (-\mu x) \frac{\partial}{\partial y} + (-x^2 - 2\lambda y) \frac{\partial}{\partial \lambda}, \\ \xi_3 = (-\lambda y) \frac{\partial}{\partial x} + (-3\lambda\mu + xy) \frac{\partial}{\partial y} + (-y^2 - 2\mu x) \frac{\partial}{\partial \mu} \end{array} \right.$$

Then we have  $\xi_0\varphi_4 = 4\varphi_4$ ,  $\xi_1\varphi_4 = 0$  and

$$\begin{aligned} \xi_2\varphi_4 &= -\varphi_5^{2,3} - 8\lambda(w^2 + z\mu^2) \equiv -\varphi_5^{2,3} \pmod{h^*\mathcal{E}_4}, \\ \xi_3\varphi_4 &= -\varphi_5^{3,2} - 8\mu(z^2 + w\lambda^2) \equiv -\varphi_5^{3,2} \pmod{h^*\mathcal{E}_4}. \end{aligned}$$

Moreover

$$\begin{aligned} \xi_0\varphi_5^{3,2} &= 5\varphi_5^{3,2}, \xi_0\varphi_5^{2,3} = 5\varphi_5^{2,3}, \xi_0\varphi_5^{5,0} = 5\varphi_5^{5,0}, \xi_0\varphi_5^{0,5} = 5\varphi_5^{0,5}, \\ \xi_1\varphi_5^{3,2} &= \varphi_5^{3,2}, \xi_1\varphi_5^{2,3} = -\varphi_5^{2,3}, \xi_1\varphi_5^{5,0} = 5\varphi_5^{5,0}, \xi_1\varphi_5^{0,5} = -5\varphi_5^{0,5}, \end{aligned}$$

and

$$\begin{aligned}
\xi_2\varphi_5^{3,2} &= 3\varphi_6^{3,3} + 18\lambda\mu\varphi_4 - 36zw\lambda\mu - 3\lambda^3\mu^3 \\
&\equiv 3\varphi_6^{3,3} + 18\lambda\mu\varphi_4 \pmod{h^*\mathcal{E}_4}, \\
\xi_2\varphi_5^{2,3} &= -4\lambda\varphi_5^{0,5} - 32z^2\mu^2 + 68w\lambda^2\mu^2 \\
&\equiv -4\lambda\varphi_5^{0,5} \pmod{h^*\mathcal{E}_4}, \\
\xi_2\varphi_5^{5,0} &= 5\lambda\varphi_5^{3,2} - 16z^2\mu - 4w\lambda^2\mu \\
&\equiv 5\lambda\varphi_5^{3,2} \pmod{h^*\mathcal{E}_4}, \\
\xi_2\varphi_5^{0,5} &= 15\mu^2\varphi_4 - 60zw\mu^2 - 45\lambda^2\mu^4 \\
&\equiv 15\mu^2\varphi_4 \pmod{h^*\mathcal{E}_4}, \\
\xi_3\varphi_5^{3,2} &= -4\mu\varphi_5^{5,0} - 32w^2\lambda^2 + 68z\lambda^3\mu^2 \\
&\equiv -4\mu\varphi_5^{5,0} \pmod{h^*\mathcal{E}_4}, \\
\xi_3\varphi_5^{2,3} &= 3\varphi_6^{3,3} + 18\lambda\mu\varphi_4 - 36zw\lambda\mu - 3\lambda^3\mu^2 \\
&= 3\varphi_6^{3,3} + 18\lambda\mu\varphi_4 \pmod{h^*\mathcal{E}_4}, \\
\xi_3\varphi_5^{5,0} &= 15\lambda^2\varphi_4 - 60zw\lambda^2 - 45\lambda^4\mu^2 \\
&\equiv 15\lambda^2\varphi_4 \pmod{h^*\mathcal{E}_4}, \\
\xi_3\varphi_5^{0,5} &= 5\mu\varphi_5^{2,3} - 16w^2\lambda - 4z\lambda\mu^2 \\
&\equiv 5\mu\varphi_5^{2,3} \pmod{h^*\mathcal{E}_4}.
\end{aligned}$$

We consider the  $h^*\mathcal{E}_4$ -module  $U$  generated by  $1, \varphi_4$  and  $V$  generated by  $1, \varphi_4, \varphi_5^{3,2}, \varphi_5^{2,3}, \varphi_5^{5,0}, \varphi_5^{0,5}$ . Then we have

$$\mathfrak{X}_h(U) = V, \quad \mathfrak{L}_h(U) = \mathcal{R}_h.$$

It would be an interesting open problem, for the geometry of openings, to find a submodule  $U$ , as small as possible, of  $\mathcal{R}_f$  which satisfies  $\mathfrak{X}_f(U) = \mathcal{R}_f$  or  $\mathfrak{L}_f(U) = \mathcal{R}_f$ , for any stable map-germ  $f : (\mathbf{R}^n, A) \rightarrow (\mathbf{R}^m, b), n \geq 5$ . In this paper we have solved the problem just for the case  $n = 4, m = 4$ . To understand the structure of openings in general case, it seems to be necessary to study more higher dimensional cases  $n \geq 5$ .

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