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Several Questions on Singularities: Theories and Applications

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Question: Why do you study SINGULARITIES?

Answer 1: Because it's there.

Answer 2: Singularities appear everywhere. We *can not avoid* singularities, for studying regular objects. So studying singularities is *indispensable* in mathematics and other area.

Answer 3: Any information on an object *concentrates on its singularities.* Thus studying singularities is one of fundamental methods in mathematics and other area. We must face with singularities *positively.*

Question: Are there any applications of singularity theory?

Answer: YES. I have collected below some of naïve questions that I have faced during the usual study of applications of singularity theory.

§1. Singularities of Bäcklund Transformations: Classical Theory and Problems.

§2. Frontal Surfaces: Genericity of Mappings to Singular Spaces.

§3. Plane-to-Plane Mappings: Global Configurations.

§4. Singularities in Projective Differential Geometry: Singular Surface Theory.

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1 Singularities of Bäcklund Transformations: Classical Theory and Problems

Bäcklund transformations are transformations of partial differential equations as well as their solutions. They are first introduced around surface theory. See [3]. There are many references on them, related to soliton theory [6]. Recently, Bäcklund transformations have been re-cast in the context of integrable systems in differential geometry [9][2].

In this note we recall the classical definition of Bäcklund transformations following [3], and pose problems related to singularity theory.

A smooth function $z = f(x, y)$, as is well-known, can be described by the surface $\{(x, y, z) \mid z = f(x, y)\}$, the graph of f, in the (x, y, z)-space, endowed with the projection $(x, y, z) \mapsto (x, y)$. If we forget the projection, namely, if we do not distinguish the variables (x, y) and the value z, then the study on functions of two variables is reduced to the study of surfaces in the three space.

A tangent plane to a surface in (x, y, z)-space can be represented by additional two parameters p and q. When the surface is the graph of a function $f(x, y)$, we take $p = f_x, q = f_y$, the partial derivatives. Thus a graphical surface $M = \{z = f(x, y)\}$ can be lifted naturally to a surface

$$\tilde{M} = \{(x, y, f(x, y), f_x(x, y), f_y(x, y)\}$$

in the five dimensional space $\{(x, y, z, p, q)\}$.

Consider the canonical one-form $\alpha = dz - pdx - qdy$. Then α is a contact one-form on this $\mathbf{R}^5$. The canonical *contact structure* on $\mathbf{R}^5$ is defined by the Pfaff equation $\alpha = 0$, namely by the distribution $\{v \in T\mathbf{R}^5 \mid \langle \alpha, v \rangle = 0\} \subset T\mathbf{R}^5$.

Then the lifting $\tilde{M}$ is a *Legendre surface*, namely $\alpha|_{\tilde{M}} = 0$ [1].

To treat non-graphical surfaces, it is natural to introduce the manifold of contact elements of $\mathbf{R}^3$. A contact element of $\mathbf{R}^3$ is, by definition, a linear (hyper)plane of the tangent space to a point in $\mathbf{R}^3$. Since a contact element is defined by a non-zero cotangent vector up to non-zero scalar multiplication, the manifold consisting of all contact elements of $\mathbf{R}^3$ is identified with the fiber-wise projectification $PT^*\mathbf{R}^3$.

Let $\pi : PT^*\mathbf{R}^3 \to \mathbf{R}^3$ be the natural projection, mapping a contact element to its base point. Then each fiber is a projective plane $\mathbf{R}P^2$, which is a compactification of the (p, q)-plane: If we fix the decomposition $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$, we have the natural embedding $\mathbf{R}^5 \hookrightarrow PT^*\mathbf{R}^3$, defined by $(x, y, z, p, q) \mapsto (x, y, z, [p, q, 1])$.

The canonical contact structure on $\mathbf{R}^5$ naturally extends to a contact structure $D \subset TPT^*\mathbf{R}^3$ on the manifold $PT^*\mathbf{R}^3$ of contact elements: A tangent vector $u \in T_c PT^*\mathbf{R}^3$ to PT^*R^3 at a contact element c belongs to D if and only if $\pi_*(u) \subset c(\subset T_{\pi(c)}\mathbf{R}^3)$. Here $\pi_* : TPT^*\mathbf{R}^3 \to T\mathbf{R}^3$ is the linearization of $\pi : PT^*\mathbf{R}^3 \to \mathbf{R}^3$.

Any surface in $\mathbf{R}^3$, then, lifts naturally to a Legendre surface in $PT^*\mathbf{R}^3$ with respect to the contact structure D defined above.

In what follows, we talk on $PT^*\mathbf{R}^3$ for the theoretical naturality, but you may replace it by $\mathbf{R}^5$ without loss of significance of the problem.

Now we consider a transformation of surfaces in $\mathbf{R}^3$. We regard the transformed surfaces lie in another $\mathbf{R}^3$ which is a copy of $\mathbf{R}^3$ with coordinates x', y', z'. Set $M = PT^*\mathbf{R}P^3$ and denote by M' the corresponding copy of M: This M' has the affine coordinate x', y', z', p', q' and the local contact form $\alpha' = dz' - p'dx' - q'dy'$.

Consider the product manifold $M \times M'$ of dimension 10. Thus $M \times M'$ has affine coordinates $x, y, z, p, q, x', y', z', p', q'$.

Denote by $\mathrm{pr} : M \times M' \to M$ and $\mathrm{pr}' : M \times M' \to M'$ the natural projections respectively. Then the contact structures on M and M' provide the distribution $(\mathrm{pr}_*)^{-1}D \cap (\mathrm{pr}'_*)^{-1}D'$ of rank 8, which is locally defined by the Pfaff system

$$\alpha = dz - pdx - qdy = 0, \quad \alpha' = dz' - p'dx' - q'dy' = 0.$$

A Bäcklund transformation is a submanifold B of codimension 4 in $M \times M'$ [3],[4].

Example 1 ([8]). Let N and N' be surfaces in $\mathbf{R}^3$, and $\ell : N \to N'$ a diffeomorphism. Write $P' = \ell(P)$, for $P \in N$. ℓ is called a Bäcklund transformation if the secant $\overline{PP'}$ is tangent to N at P and N' at P', and, the distance $d(P, P') = r$ and the angle angle$(\nu_P, \nu'_P) = \theta$ of normals ν_P, ν'_P is constant $(P \in N)$. If $N : z = z(x, y), N' : z' = z'(x', y')$, and $P = (x, y, z), P' = (x', y', z')$, then ℓ is described by

$$\begin{array}{ll} F_1: & p(x'-x) + q(y'-y) - (z'-z) = 0, \\ F_2: & p'(x-x') + q'(y-y') - (z-z') = 0, \\ F_3: & (x'-x)^2 + (y'-y)^2 = r^2, \\ F_4: & \dfrac{pp'+qq'+1}{\sqrt{p^2+q^2+1}\sqrt{p'^2+q'^2+1}} = \cos\theta, \end{array}$$

in the $(x, y, z, p, q; x', y', z', p', q')$-space.

Remark that a Bäcklund transformation $B \subset M \times M'$ is endowed with a Pfaff system $\alpha = 0, \alpha' = 0$ restricted to it. In the language of tangent vectors, the system defines

$$E = TB \cap (\mathrm{pr}_*)^{-1} D \cap (\mathrm{pr}'_*)^{-1} D' \subset TB,$$

which is a distribution over B with singularities in general.

We impose, in what follows, on a Bäcklund transformation B the condition that

the projections $\mathrm{pr}|_B$ *and* $\mathrm{pr}'|_B$ *are submersions.*

Then we see

Proposition: An integral manifolds of E are at most of dimension 2.

Here is an ad hoc proof of the proposition: Let $S \subset B$ be an integral manifold of E. Since $\mathrm{pr}|_B : B^6 \longrightarrow M^5$ is a submersion, the dimension of the kernel of the differential mapping $(\mathrm{pr}|_B)_*$ is equal to one. Moreover the rank of $(\mathrm{pr}|_S)_*$ must be at most two, since the image satisfies $\alpha = 0$. Therefore $\dim S$ is at most three. Furthermore if $\dim S = 3$, then the image of $(\mathrm{pr}|_S)_*$ is of dimension two, and the inverse image of the image of $(\mathrm{pr}|_S)_*$ coincides with the tangent space to S. This leads to that the dimension of the kernel of $(\mathrm{pr}'|_S)_*$ is at least two, and to a contradiction. $\square$

Now let $I \subset B$ be an integral submanifold of dimension 2 of E:

$$\alpha|_I = 0, \quad \alpha'|_I = 0.$$

Then naturally posed questions are these:

Question: What are generic singularities of $\mathrm{pr}|_I : I \longrightarrow M$ and $\mathrm{pr}'|_I : I \longrightarrow M'$? What are generic singularities of $\pi \circ \mathrm{pr}|_I : I \longrightarrow \mathbf{R}^3$ and $\pi' \circ \mathrm{pr}'|_I : I \longrightarrow \mathbf{R}^3$?

Remark that $\mathrm{pr}|_I$ is *an integral mapping*, namely $(\mathrm{pr}|_I)^*\alpha = 0$, and therefore the image $\mathrm{pr}(I) \subset M = PT^*\mathbf{R}^3$ is a Legendre variety, in other words, the regular part of $\mathrm{pr}(I)$ is an integral manifold (Legendre submanifold) of the contact structure $\alpha = 0$.

Question: Are there generating families for this singularity problem, like in ordinary Legendre singularity theory?

Ideally we wish to find a function of type $F(x, y, z; x', y', z')$, for a given $I \subset B$, which is a generating family (with parameter x, y, z) of $\mathrm{pr}(I)$ with respect to π, and at the same time, is a generating family (with parameter x', y', z') of $\mathrm{pr}'(I)$ with respect to π'. Since $\mathrm{pr}(I)$ and $\mathrm{pr}'(I)$ may have singularities, the generating family may define other extra components than $\mathrm{pr}(I)$ and $\mathrm{pr}'(I)$.

Consider the case that the system of 4 equations defining a Bäcklund transformation B contains $x = x', y = y'$. Then we regard B as a submanifold in the $(x, y, z, z', p, q, p', q')$-space with equations

$$\alpha = dz - pdx - qdy = 0, \quad \alpha' = dz' - p'dx - q'dy = 0,$$

of codimension two, locally defined by two equations, say:

$$f(x, y, z, z', p, q, p', q') = 0, \quad g(x, y, z, z', p, q, p', q') = 0.$$

Question: Are there any local characterizations of the class of differential systems on $(\mathbf{R}^6, 0)$ realized as Bäcklund transformations of above type.

If we eliminate z', p', q' using

$$dz' = p'dx + q'dy, \quad f = 0, \quad g = 0,$$

then we get a 2nd order differential equation of $z = z(x, y)$. If we eliminate z, p, q using

$$dz = pdx + qdy, \quad f = 0, \quad g = 0,$$

then we have a 2nd order differential equation of $z' = z'(x, y)$. Thus a Bäcklund transformation induces a transformation of 2nd order differential equations and solutions. (The graphs of solutions are $\pi \circ \mathrm{pr}(I)$ and $\pi' \circ \mathrm{pr}'(I)$, in our notations.)

Example(Sine-Gordon equation): Let

$$\begin{aligned} f &= p' - p - 2\sin\frac{z'+z}{2} \\ g &= q' + q - 2\sin\frac{z'-z}{2}. \end{aligned}$$

Then we have

$$p'_y = p_y + \left(\cos\frac{z'+z}{2}\right)(q'+q) = p_y + \sin z' - \sin z,$$

and

$$q'_x = -q_x + \left(\cos\frac{z'-z}{2}\right)(p'-p) = -q_x + \sin z' + \sin z.$$

Thus we have

$$p'_y - \sin z' = p_y - \sin z, \quad q'_x - \sin z' = -q_x + \sin z,$$

and two differential equations:

$$z_{xy} = \sin z, \quad z'_{xy} = \sin z',$$

the same sine-Gordon equation. The transformation of solution, then, is closely related the transformation of surfaces with negative curvature.

I believe it is necessary to give the rigorous foundation to the elimination process:

Question: Are there any theory of elimination for partial differential equations, like in algebraic and analytic geometry.

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2 Frontal Surfaces: Genericity of Mappings to Singular Spaces.

A surface in $\mathbf{R}^3 or \mathbf{C}^3$ is called *frontal* if it has "smooth" Nash lifting in $PT^*\mathbf{R}^3$. Exactly, if we give the surface by a parametrization $f : M \to \mathbf{R}^3$ from a C^∞ surface M, then f is called *frontal* if it has a unique frontal lifting $\tilde{f} : M \to PT^*\mathbf{R}^3$. If the surface is an analytic surface in $\mathbf{C}^3$, then, the surface is called *frontal* if the projection from the Nash lifting of the surface to the surface itself is finite to one.

Similarly we define the notion of *frontal hypersurfaces* in $\mathbf{R}^n$ or $\mathbf{C}^n$ and more generally in C^∞ or complex manifolds.

Since the behaivior of tangent spaces to a frontal surfaces is very restrictive, we expect we can apply the stratification theory to studying families of frontal surfaces.

I have applied the stratification theory to verifying the topological triviality of families of tangent developables [5]

Question: Is there any simple criteria for topological triviality of families of frontal (hyper)surfaces?

Remark that frontal surfaces have only non-isolated singularities "generically". However there are examples of frontal surfaces having isolated singularities: $z^2 = x^4 + y^4$.

Also, the following questin should be naturally posed:

Question: Are there any algebraic (ring theoretical) characterization of frontal (hyper)surfaces?

The study on frontal surfaces is closely related to the study on integral mappings.

Givental' conjecture [1]: Generic singularities of integral mappings $\mathbf{R}^2 \to \mathbf{R}^5$ are contact equivalent to the Nash lifting of *folded umbrella*

$$(u, v) \mapsto (x, y, p, q, z) = (u, v^2/2, v^3/3, uv, uv^3/3).$$

The corank one case of Givental' conjecture is proved by Givental' [1][2]. The higher dimensional generalization of corank one case is solved by me [3].

Question: How do we describe the generic conditions for integral mappings of corank > 1.

Here, let us recall the notion of integral jet spaces [4]. In the ordinary jet space $J^r(\mathbf{R}^2, \mathbf{R}^5)$, consider

$$I^r := \{j^r h(x) \mid x \in \mathbf{R}^2, h : \mathbf{R}^2, x \to \mathbf{R}^5 \text{ integral}\}.$$

If $f : \mathbf{R}^2 \to \mathbf{R}^5$ is integral, then the jet extension $j^r f$ is regarded as a mapping to I^r: $j^r f : \mathbf{R}^2, 0 \to I^r$, that we call *the integral jet extension*: $(j^r f)(x) := j^r f(x)$, the r-jet of f at x.

Then a difficulty arises from the fact that the isotropic jet space I^r has quadratic singularities

$$\mathrm{Sing}(I^r) = \{j^r h(x) \mid h\text{: integral of corank} \geq 2\}.$$

Then the natural and important question is this:

Question: Do any transversality theorems exist, for mappings to singular spaces?

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3 Plane-to-Plane Mappings: Global Configurations.

Let $f : \mathbf{R}^2 \to PT^*\mathbf{R}^3$ be a proper generic integral mapping. Consider the projection $\Pi : PT^*\mathbf{R}^3 \to \mathbf{R}^2$, $(x, y, z, p, q) \mapsto (x, y)$ and the composition $\Pi \circ f : \mathbf{R}^2 \to \mathbf{R}^2$, which is called a Lagrange mapping. The critical value set of $\Pi \circ f$ is called *the caustic.*

Question: (The Question on the Topology of Caustics.) Are there any differences on the topology of generic Lagrange mappings and the topology of generic mappings $\mathbf{R}^2 \to \mathbf{R}^2$.

If we pose the condition that f is a Legendre immersion, then the question is classical:

Question: (The Classical Question on the Topology of Caustics.) Are there any differences on the topology of generic Lagrange mappings of *Legendre immersions* and the topology of generic mappings $\mathbf{R}^2 \to \mathbf{R}^2$.

The topology of generic mappings $\mathbf{R}^2 \to \mathbf{R}^2$ itself is also interesting problem. See [1][2] for the characterization of the discriminant set. Even it seems to be not so clearly understood.

The problem should be treated again elsewhere.

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4 Singularities in Projective Differential Geometry: Singular Surface Theory.

Let $f, f' : (\mathbf{R}^2, 0) \to \mathbf{R}P^3$ be map-germs to the projective three space. f and f' are called *projectively equivalent* if there exist a projective transformation $\tau : \mathbf{R}P^3 \to \mathbf{R}P^3$ and a diffeomorphism-germ $\sigma : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that $\tau \circ f = f' \circ \sigma$.

Classical theory treats the projective classification of immersions: There exist relations of classical surface theory to the study on integrable systems, Bäcklund transformations and so on [1].

Question: Are there any generalization of classical theory of projective differential geometry to singular surfaces?

I believe that the projective differential geometry of singularities of ruled surfaces, developable surfaces, and frontal surfaces is a fruitful and promising area for studying; as the manifestation of the "contact nature" of projective geometry.

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