Chapter 1
Frontal singularities and related problems

Goo Ishikawa

Abstract This is a survey of C^∞ or complex analytic frontal singularities. A mapping from a manifold of dimension n to a manifold of dimension m with $n \leq m$ is called a frontal, if its differential is well-controlled by a field of tangential n-planes on the target manifold along the mapping which contain images by the differentials of tangent spaces to the source manifold. We explain the basic theory on frontal hypersurfaces, the case $m = n + 1$, and then their generalisations in real and complex cases. We mention several notions, topics and problems which are related to frontal singularities from rather wide aspects with future expects.

1.1 Introduction

In the study on geometry of varieties in affine spaces, the behaviour of tangent planes provides significant information on varieties under study. A **frontal hypersurface**, or simply a **frontal**, is a hypersurface, possibly with singularities, which has well-defined tangent planes even along its singular loci (see the following paragraphs for the precise definitions). It should be remarked that frontal map-germs are "special" in the sense that any non-immersive map-germ of finite codimension with respect to right-left equivalence is never frontal, provided the source dimension ≥ 2. However frontal map-germs and their frontal deformations appear, for examples, in the study of wave fronts for mathematical physics and for the generalised theory of manifolds in geometry. We should mention that frontals appear also as discriminant sets of smooth or analytic mappings, as the closures of orbits for reflection groups, as tangent varieties ruled by embedded tangent spaces to a submanifold, e.g. tangent developables to curves, and so on, all of which give motivations to study frontals from

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symplectic-contact geometry, differential analysis, algebraic analysis and complex analytic geometry (see for instance ([9, 6, 8, 7, 98]).

As a background of our study, recall the standard notion in differential geometry. Given a regular surface in Euclidean 3-space, a unit normal vector field along the surface is used, due to Gauss, to study the geometric property of the surface ([34, 116]). Actually there are at least two choices of such a field. Anyway we take one of them and define the mapping from the surface under the study to the unit sphere S^2, the **Gauss map**, and the properties of the surface are studied via the behaviour of the Gauss map. Note that the field of tangent spaces to the surface is recovered from the unit normal vector field as the orthogonal complement to the normal vector field. Then it would be natural to imagine, even when the surface has singularities, a sort of Gauss map can be defined, if a unit "normal" vector field, or a field of "tangent planes" to the surface across the singularities is provided, and then it would be very useful to study the geometry of singular surfaces, as Gauss did for regular surfaces. This is the idea of "frontals".

Moreover, as another background of our study, recall the standard notion in complex analytic geometry. Given a complex analytic hypersurface V in the complex n-space $\mathbb{C}^n$ with singular locus $\Sigma \subset V$, consider tangent planes T_zV to V at regular points z of V. The **Nash blow-up** $\mathrm{Nash}(V)$ of V is defined as the closure of $\{(z, TzV) \mid z \in V \setminus \Sigma\}$ in the space of all complex tangential n-planes over $\mathbb{C}^n$. Then $\mathrm{Nash}(V)|_\Sigma$ consists of the limits of tangent spaces T_zV at $z \in V \setminus \Sigma$ when z tends to a point in Σ. Usually the fibre $\mathrm{Nash}(V)_z$ over $z \in \Sigma$ is not one point set in $\mathrm{Gr}(n, T_z\mathbb{C}^n) \cong P(T_z^*\mathbb{C}^n)$. It would be natural to expect that $\mathrm{Nash}(V)_z$ reflects some properties of the original variety V ([88, 89, 90]). Then it is the case we treat here when $\mathrm{Nash}(V)_z$ is just one point for any $z \in \Sigma$ and Nash blow-up provides a vector bundle of rank n over V, even over the singular locus Σ, which restricts to the tangent bundle to the regular locus $V \setminus \Sigma$. This construction is generalised to to arbitrary variety, not necessarily a hypersurface, to obtain a Legendre varieties in projective cotangent bundle $PT^*\mathbb{C}^n$. This is another origin of the idea of frontals.

Frontals are significant subjects to be studied in singularity theory, analytic geometry and differential geometry and so on.

We call a smooth mapping $f : N \to M$ between smooth manifolds N of dimension n and M of dimension m with $n \leq m$, a **frontal** in a generalised sense, if there exists a smooth field $\{\widetilde{f}(t)\}_{t\in N}$ of tangential n-planes along f with $df(T_tN) \subset \widetilde{f}(t)$ for any, even non-immersive, point $t \in N$ (Definitions 1.2.5 and 1.3.4). The terminology "frontal" comes from "wavefront" which means a front of wave propagations in the case $m = n + 1$. However the above geometric definition of frontals works also for any dimensions n, m with $n \leq m$. Note that, if $n = m$, then any smooth mapping $f : N \to M$ is a frontal in the above sense. Including this trivial case, we are able to treat frontals uniformly via the notion of openings as is explained below (see also §1.4). One might feel discomfort at the terminology "frontal" when $n \leq m - 2$, i.e. when the relative dimension $m - n \geq 2$ or $m = n$, because the terminology "front" is usually used for a hypersurface, to describe of wave propagations or to indicate a boundary, in any mean, of two different regions, say, an in meteorology.

Alternatively, our terminology "frontals" can be called **locally framable maps** in general to make clear their geometrical meaning.

It seems very natural at least to treat generalised frontals of arbitrary relative dimensions. However it should be remarked that there are both *common features* and *different features*, at the same time, on frontal mappings of our generalised sense.

One of common algebraic features on frontals is that the Jacobi ideal, i.e. the ideal generated by maximal minors of the Jacobi matrix is principal (Propositions 1.2.12 and 1.3.23). This leads a unified treatment of frontals using notions of **ramification modules** and **openings** as we are going to introduce (see §1.4.1).

However a crucial difference appears for the study of frontals between the cases $m - n = 1$ and other cases $m - n \geq 2$ or $m - n = 0$. The plane field $\widetilde{f}$ associated to a frontal $f : N \to M$ is regarded as a mapping from N to the Grassmannian bundle $W = \mathrm{Gr}(n, TM)$ over M consisting of tangential n-planes in the tangent bundle of M with the projection $\pi : W \to M$. The Grassmannian bundle $W = \mathrm{Gr}(n, TM)$ has the **canonical**, **tautological**, or **contact distribution** $D \subset TW$ (Definition 1.3.4) and $\widetilde{f}$ is an integral mapping of D, i.e. $\widetilde{f}$ satisfies that $d\widetilde{f}(TN) \subset D$. Any diffeomorphism of M lifts to unique diffeomorphism on W which preserves the contact distribution D. When $m = n + 1$, it is well-known that a diffeomorphism on W preserving the contact structure, i.e. a **contactomorphism**, need not preserve the fibration $\pi : W \to M$; the existence of **Legendre transformations**. Namely the group of contactomorphisms of W is definitely bigger than the group of diffeomorphisms of M, and thus two classifications of singularities of $\widetilde{f}$, the contact equivalence using contactomorphisms of W and the Legendre equivalence using π-preserving contactomorphisms of W are really different ([55]). On the contrary, when $m \geq n+2$, any diffeomorphism preserving D necessarily preserve the fibration $\pi : W \to M$; a kind of Bäcklund's theorem ([122]). In fact the **Cauchy characteristic** $\mathrm{Ch}(D)$ of D is intrinsically defined, for any $w \in W$, as $\mathrm{Ch}(D)_w := \{v \in D_w \mid i_v \mathcal{I}_w \subseteq \mathcal{I}_w\}$, where $\mathcal{I}$ is the differential ideal generated by 1-form-germs annihilating all sections of D and i_v means the interior product of differential forms by v (see [19]). Then $\mathrm{Ch}(D)$ is equal to $\mathrm{Ker}(d\pi : TW \to TM)$ when $m \geq n+2$, while $\mathrm{Ch}(D) = \{0\}$ when $m = n+1$. Therefore, in the generalised case $m \geq n + 2$, contact equivalence and Legendre equivalence coincide. (When $m = n$, they coincide of course.) The difference of the situations between the case $m = n + 1$ and $m \geq n + 2$ or $m = n$ raises difficult and interesting problems in singularity theory of frontals and related area. Some of them will be mentioned in this paper.

Here we would like to present some "history" of investigations on frontals so far, as far as the author knows, with apologies for possible failures to mention.

First of all, V.I. Arnold, V.M. Zakalyukin and other people originate the study of fronts in the framework of Legendre singularity theory [127, 128, 3, 18, 129, 9, 5, 6, 8, 39, 7]. Note that in [6], the terminology "frontal map" is used for a Legendre map, that is a Legendre projection of a Legendre submanifold. The more general terminology "frontal" was defined and used in V.M. Zakalyukin and A.N. Kurbatskiĭ related to control theory [131], and in [54] as far as the author knows.

As for classification problem of frontal singularities, G. Ishikawa studies frontals from the viewpoint of Legendre projections of singular Legendre varieties [49, 55]. Note that singular Lagrange varieties and their Lagrange projections are studied by V. I. Arnold, S. Janeczko, A. B. Givental V. M. Zakalyukin and so on [4, 79, 80, 37, 38, 130, 81, 52], which are closely related to the study on singularities of frontals. In particular works by Givental [37, 38, 39] provide substantial ideas also to the theory of frontals (see §1.5.1, §1.6.1). As for the study on singularities of frontals, J.J. Nuño-Ballesteros, C. Muñoz-Cabello, and R. Oset Sinha study frontal deformations of planar curves, frontal surfaces and frontalisations [104, 98, 100, 101]. K. Saji gives the recognition criterion of cuspidal S_k-frontal and D_4-front [109, 110]. S. Izumiya, K. Saji and M. Takahashi provide the criteria of cuspidal beaks and of cuspidal lips [78]. B. Bruce, T. Fukui, M. Hasegawa and G. Ishikawa study singularities of parallel surfaces [18, 26, 62]. T. Nishimura [102] proves the square of Jacobian ideal is contained in the ramification module and study frontals in the relation with the theory of envelopes [103]. S. Janeczko and T. Nishimura study anti-orthotomics of frontals [82].

As for differential geometric study of frontals, M. Kossowski studies frontal surfaces of codimension 2 in $\mathbb{R}^4$ of corank 1 in [86] and, M. Kossowski and M. Scherfner study total curvatures of frontals in $\mathbb{R}^3$ [87], under the name of "surface with limiting tangent bundle". R. Thom, I. Porteous, C.T.C. Wall, S. Izumiya M. C. Romero Fuster, A. S. Ruas, F. Tari and other people indicate the scope to differential geometry from the viewpoint of singularity theory [114, 105, 115, 117, 106, 75, 77]. M. Kokubu, W. Rossman, K. Saji, M. Umehara, K. Yamada, T. Fukui and other people provide the recognition method of cuspidal edge and swallowtail for the study on differential geometry of fronts and frontals [85, 112, 25, 116]. G. Ishikawa and Y. Machida study singularities of wavefronts appeared in special surfaces [67, 16]. S. Fujimori, K. Saji, M. Umehara, K. Yamada give the recognition criterion of the folded umbrella, or, the cuspidal cross cap, and study differential geometry on surfaces with such singularities [23]. K. Saji, M. Umehara and K. Yamada give Gauss Bonnet formula for frontals and coherent tangent bundles [112, 113]. The case with boundary is studied by W. Domitrz, M. Zwierzyński and K. Hashibori [21, 43]. T, Fukunaga, M. Takahashi and T.A. Medina-Tejeda establish the fundamental theorem of frontal surfaces and related results [31, 95].

Y. Machida, G. Ishikawa and M. Takahashi study tangential frontal surfaces in various geometries, e.g. G_2-geometry, D_4-geometry, null frontals and so on [68, 69, 70, 71, 72]. T, Fukunaga, M. Takahashi, S. Honda and T. Fukui study on differential geometry of frontal curves [27, 28, 29, 45, 30, 24, 32, 46, 33]. G. Ishikawa and T. Yamashita study frontals ruled by geodesics to frontal curves on manifolds with affine connections and on the recognition of open swallowtails [73, 74]. A. Honda, M. Koiso and K. Saji study the fold singularities in Lorentz-Minkowski space [44].

In this survey article, regarding all of the above aspects, first we study frontal hypersurfaces in §1.2 and then introduce their generalisations in §1.3, mainly from the framework of singularity theory of mappings and the modified Thom-Mather theory [93]. Some of above mentioned previous works will be explained in detail. Both the class of frontal hypersurfaces and general frontals are treated uniformly by the notion of **openings**. The related key object named **ramification module** is introduced and the relations with openings and with frontals are studied in §1.4. Moreover in §1.5, we mention several topics related to frontals which might induce fruitful applications in various areas. In particular we touch on some relations of frontals with symplectic geometry, frontal jets and cofrontals, which are not mentioned in previous sections. In the last section §1.6, we present several problems and questions related to frontals. As for descriptions in the exposition, we try to add several basic materials and some detailed calculations which may be not found in previous references and/or which would be helpful to understand and to solve, if possible, future problems presented in this survey article.

Notions and notations. In this article, we use the following notations throughout the paper unless otherwise stated: For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we call a mapping or a manifold *smooth* if it is C^∞ if $\mathbb{K} = \mathbb{R}$ and complex analytic if $\mathbb{K} = \mathbb{C}$. The $\mathbb{K}$-algebra of $\mathbb{K}$-valued smooth function-germs on $(\mathbb{K}^n, a)$ is denoted by $\mathcal{O}_{\mathbb{K}^n,a}$ or $\mathcal{O}_a$. More generally the $\mathbb{K}$-algebra of multi-germs of functions on $(\mathbb{K}^n, A)$ is denoted by $\mathcal{O}_{(\mathbb{K}^n,A)}$ for a finite subset A of $\mathbb{K}^n$. We often abbreviate $\mathcal{O}_{(\mathbb{K}^n,A)}$ by $\mathcal{O}_A$ if there is no risk of confusion. If $A = \{a_1, \dots, a_s\}$, then we denote by $\mathfrak{m}_i$ the maximal ideal consisting of $h \in \mathcal{O}_A$ with $h(a_i) = 0$. We set $\mathfrak{m}_A = \cap_{i=1}^s \mathfrak{m}_i$, Jacobson radical ([94]). Also we consider a map-germ $f : (\mathbb{K}^n, a) \to (\mathbb{K}^m, b)$ and a multi-germ of mapping $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ with $f(A) = \{b\}$. Another germ $g : (\mathbb{K}^n, A') \to (\mathbb{K}^m, b')$ is called **right-left equivalent**, **$\mathcal{A}$-equivalent** or **diffeomorphic** to f if the diagram

$$\begin{array}{ccc}
(\mathbb{K}^n, A) & \xrightarrow{f} & (\mathbb{K}^m, b) \\
\sigma \downarrow & & \downarrow \tau \\
(\mathbb{K}^n, A') & \xrightarrow[g]{} & (\mathbb{K}^m, b')
\end{array}$$

commutes for some smooth $\mathbb{K}$-isomorphisms σ and τ.

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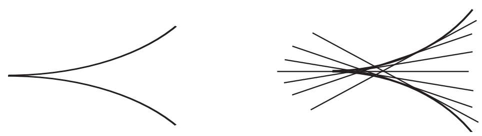
1.2 Frontal hypersurfaces

1.2.1 Examples of frontal singularities

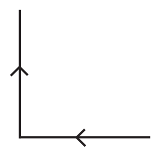
We start this section by introducing typical examples of frontal singularities.

Example 1.2.1 Consider the map-germ $f : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ defined by $t \mapsto (t^2, t^3)$, which gives a **planar cusp**. The tangent vector along f is given by $t \mapsto \dfrac{df}{dt} = (2t, 3t^2)$. Off the origin, the tangent line through $f(t) = (t^2, t^3)$ with direction $\frac{1}{2t}(2t, 3t^2) = (1, \frac{3}{2}t)$ is given by the parametric equation $s \mapsto (x_1, x_2) = (t^2, t^3) + s(1, \frac{3}{2}t) = (t^2 + s, t^3 + \frac{3}{2}st)$. By taking limit $t \to 0$, we have the same equation defining the tangent line $s \mapsto (s, 0)$ even at $t = 0$. Thus the planar cusp turns to be a *frontal* curve.



planar cusp and its tangent lines

As a counter-example, consider the C^∞ curve $g : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ defined by $g(t) = (0, \exp(\frac{1}{t}))$ when $t < 0$, $g(0) = (0, 0)$ and $g(t) = (\exp(-\frac{1}{t}), 0)$ when $t > 0$. Then g is *not* a frontal. In fact the tangent line has direction of $(-1, 0)$ when $t > 0$ and $(0, 1)$ when $t < 0$ and therefore there never exist any C^∞, even C^0 tangent line field along g.



An example of non-frontal curves.

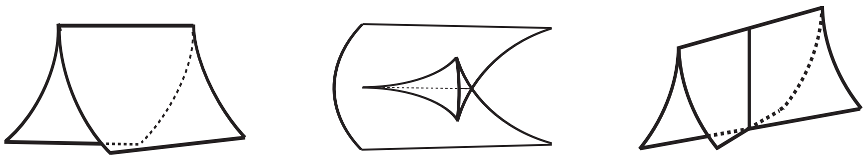
Remark 1.2.2 It is easily shown that any real analytic curve or complex holomorphic curve $(\mathbb{K}, A) \to \mathbb{K}^m$ is a frontal (see [59] §12).

Example 1.2.3 We introduce three typical examples of frontals.

Consider the map-germ $(\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ defined by $(t_1, t_2) \mapsto (t_1, t_2^2, t_2^3)$, which is called a **cuspidal edge**. Then the tangent plane at (t_1, t_2) with $t_2 \neq 0$ is spanned by vectors $(1, 0, 0)$ and $(0, 1, \frac{3}{2}t_2)$, and the tangent plane field is extended across the singular locus $t_1 = 0$.

The map-germ $(\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ defined by $(t_1, t_2) \mapsto (t_1, t_2^3 + t_1 t_2, \frac{3}{4}t_2^4 + \frac{1}{2}t_1 t_2^2)$, which is called a **swallowtail**. Then the tangent plane at (t_1, t_2) with $3t_2^2 + t_1 \neq 0$ is spanned by $(1, t_2, t_2^2)$ and $(0, 1, t_2)$, and the tangent plane field is extended across the singular locus $3t_2^2 + t_1 = 0$.

The map-germ $(\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ defined by $(t_1, t_2) \mapsto (t_1, t_2^2, t_1 t_2^3)$, which is called a **folded umbrella** or a **cuspidal cross-cap**. Then the tangent plane at (t_1, t_2) with $t_2 \neq 0$ is spanned by $(1, 0, 0)$ and $(0, 1, \frac{2}{3}t_2)$, and the tangent plane field is extended across the singular locus $t_2 = 0$.



Cuspidal edge Swallowtail Folded umbrella

One observe that the induced map-germ $(\mathbb{K}^2, 0) \to \mathrm{Gr}(2, T\mathbb{K}^3)$, which is called the **Legendre lift**, is described, via Grassmannian coordinates, $(t_1, t_2) \mapsto (t_1, t_2^2, t_2^3; 0, \frac{3}{2}t_2)$ for the cuspidal edge, $(t_1, t_2) \mapsto (t_1, t_2^3 + t_1t_2, \frac{3}{4}t_2^4 + \frac{1}{2}t_1t_2^2; -\frac{1}{2}t_2^2, t_2)$ for the swallowtail and $(t_1, t_2) \mapsto (t_1, t_2^2, t_1t_2^3; t_2^3, \frac{3}{2}t_1t_2)$ for the folded umbrella. Then the Legendre lift is an immersion for the cuspidal edge and for the swallowtail, but is not an immersion for the folded umbrella.

Example 1.2.4 The map-germ $(\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ defined by $(t_1, t_2) \mapsto (t_1, t_2^2, t_1t_2)$ is called a **cross cap** or a **Whitney umbrella**. It is not a frontal. In fact it is immersive outside of 0 but its Legendre lift $(\mathbb{K}^2 \setminus 0, 0) \to \mathrm{Gr}(2, T\mathbb{K}^3)$ is not extended to $(\mathbb{K}^2, 0)$. Alternative proof is obtained by observing that Jacobi ideal of f turns to be the maximal ideal of $\mathcal{O}_{\mathbb{K}^2,0}$ which is not principal (see Propositions 1.2.12). Moreover it is shown that non-immersive $\mathcal{A}$-finite map-germ $(\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ can not be a frontal (see Proposition 1.5.4).

For details see the following sections.

1.2.2 Frontal hypersurface singularities

Now let us define the class of frontal hypersurface singularities.

Definition 1.2.5 Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^{n+1}, b)$ be a multi-germ of smooth (C^∞ or holomorphic) mapping. The germ f is called a **frontal hypersurface** germ or simply a **frontal** if there exists a smooth (C^∞ or holomorphic) family of n-planes $\widetilde{f}(t) \subseteq T_{f(t)}\mathbb{K}^{n+1}$ along f, $t \in (\mathbb{K}^n, A)$, i.e. there exists a smooth map $\widetilde{f} : (\mathbb{K}^n, A) \to P(T^*\mathbb{K}^{n+1})$ which satisfies the "*integrality condition*"

$$df(T_t\mathbb{K}^n) \subset \widetilde{f}(t) \ (\subset T_{f(t)}\mathbb{K}^{n+1}),$$

for any $t \in \mathbb{K}^n$ nearby A such that $\pi \circ \widetilde{f} = f$, namely the following mapping diagram commutes:

$$\begin{tikzcd} & P(T^*\mathbb{K}^{n+1}) \arrow[d, "\pi"] \\ (\mathbb{K}^n, A) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (\mathbb{K}^{n+1}, b). \end{tikzcd}$$

The lift $\widetilde{f}$ is called a **Legendre lift**, or an **integral lift** of f.

Here $P(T^*\mathbb{K}^{n+1}) = (T^*\mathbb{K}^{n+1} \setminus \{0\})/\mathbb{K}^\times$ is the projective cotangent bundle over $\mathbb{K}^{n+1}$, whose fibre is the projective cotangent space $P((T_x\mathbb{K}^{n+1})^*) = ((T_x\mathbb{K}^{n+1})^* \setminus \{0\})/\mathbb{K}^\times$ of the cotangent space $(T_x\mathbb{K}^{n+1})^* \cong \mathbb{K}^{n+1*}$.

Moreover, if the Legendre lift $\widetilde{f}$ of f can be taken to be an immersion, then f is called a **front**.

The planar cusp, the cuspidal edge and the swallowtail are front. The folded umbrella (cuspidal cross cap) is not a front but a frontal.

Let us consider the Jacobi matrix Jf of f, which is $(n+1)\times n$-matrix consisting of partial derivatives of f for a representative of the map-germ $f : (\mathbb{K}^n, A) \to (\mathbb{K}^{n+1}, b)$. Then the differential map $df : T_t\mathbb{K}^n \to T_{f(t)}\mathbb{K}^{n+1}$, for $t \in \mathbb{K}^n$ nearby a, is defined by $df(v) = (Jf(t))(v), (v \in T_t\mathbb{K}^n)$. Then t is a *regular point* (resp. a *singular point*) of f if the rank of $J(f)$ at t is equal to n (resp. less than n). If t is a regular point of f, then any point t' which is sufficiently near t is a regular point of f as well and df is injective there. Then, in a neighbourhood of a regular point t of f, the Legendre lift $\widetilde{f}$ uniquely exists as $\widetilde{f}(t) = df(T_t\mathbb{K}^n)$. Thus the condition that f is a frontal means that the Legendre lift $\widetilde{f}$ which is uniquely determined on the domain of regular points of f can be extended smoothly across the locus $\mathrm{Sing}(f)$ of singular points.

Definition 1.2.6 We call a frontal f a **proper frontal** or a **fair frontal** if $\mathrm{Sing}(f)$ is nowhere dense.

If $f : (\mathbb{K}^n, A) \to (\mathbb{K}^{n+1}, b)$ is a proper (or a fair) frontal, then Legendre lift $\widetilde{f}$ is uniquely determined, because $\widetilde{f}$ must be continuous.

Remark 1.2.7 In [59], a frontal with nowhere dense singular locus was called *proper*. However in the global study the terminology “proper" can be rather confusing since its usage is different from the ordinary meaning of properness (inverse images of any compact is compact) in the context of differential topology. Therefore here we have suggested the terminology “fair" in addition to “proper".

Note that the projective space $P(T_x^*\mathbb{K}^{n+1})$ of the cotangent space $T_x^*\mathbb{K}^{n+1}$ for any $x \in \mathbb{K}^{n+1}$, which is of dimension n, is regarded as the space of linear hyperplanes of the tangent space $T_x\mathbb{K}^{n+1}$, and $P(T^*\mathbb{K}^{n+1})$ is a manifold of dimension $2n + 1$. Each linear hyperplane of $T_x\mathbb{K}^{n+1}$ is called a **contact element** of $\mathbb{K}^{n+1}$ at x and thus $P(T^*\mathbb{K}^{n+1})$ is regarded as the *space of contact elements* of $\mathbb{K}^{n+1}$. Recall that $P(T^*\mathbb{K}^{n+1})$ has the canonical **contact structure** $D \subset TP(T^*\mathbb{K}^{n+1})$, which is defined, for $(x, [\alpha]) \in P(T^*\mathbb{K}^{n+1})$, by $D_{(x,[\alpha])} := (d\pi)^{-1}(\mathrm{Ker}\,(\alpha))$, where $\pi : P(T^*\mathbb{K}^{n+1}) \to \mathbb{K}^{n+1}$ is the bundle map, $\pi(x, [\alpha]) = x$, and $\alpha : T_x\mathbb{K}^{n+1} \to \mathbb{K}$ is a non-zero linear map, so $[\alpha] \in P(T_x^*\mathbb{K}^{n+1})$. Note that, for $v \in T_{(x,[\alpha])}P(T^*\mathbb{K}^{n+1})$, $v \in D_{(x,[\alpha])}$ if and only if $\alpha(d\pi(v)) = 0$. If $\alpha = \sum_{i=0}^{n+1} a_i dx_i$ and if we regard $a_1, \dots, a_n, a_{n+1}$ as the homogeneous coordinates of fibres, then $D_{(x,[\alpha])} \subset T_{(x,[\alpha])}P(T^*\mathbb{K}^{n+1})$ is defined by the equation $\sum_{i=0}^{n+1} a_i dx_i = 0$, for some representative of $[\alpha]$.

Remark 1.2.8 Since the definition that a multi-germ $f : (\mathbb{K}^n, A) \to (\mathbb{K}^{n+1}, b)$ is a frontal is given by the existence of Legendre lift $\widetilde{f}$ over $(\mathbb{K}^n, A)$, it can be proved that

f is a frontal if and only if each component $f_i : (\mathbb{K}^n, a_i) \to (\mathbb{K}^{n+1}, b)$ of f is a frontal for any $i = 1, \dots, r$, where $A = \{a_1, \dots, a_r\} \subset \mathbb{K}^n$.

Remark 1.2.9 (Grassmannian coordinates). It will be better to recall here how to take coordinates on the bundle $\mathrm{Gr}(n, TM) = PT^*M$ of tangential n-planes in the tangent bundle, or the projective cotangent bundles, over an $(n+1)$-dimensional manifold M. Let $U \subset M$ be a coordinate neighbourhood with coordinates $(x_1, \dots, x_n, x_{n+1})$. Let $\Pi : \mathrm{Gr}(n, TM) \to M$ be the canonical projection. Let $W_i \subset \Pi^{-1}(U)$, $i = 1, 2, \dots, n+1$, be the set of tangential n-planes V over U such that the differential of the projection $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) : U \to \mathbb{K}^n$ induces an isomorphism of V and $\mathbb{K}^n$. For the sake of simplicity, let $i = n+1$, and $V \in W_{n+1}$. Then $(x_1, \dots, x_n)_*|_V : V \to \mathbb{K}^n$ is an isomorphism and we can take unique basis $v_1, \dots, v_n$ of V such that $v_j = \frac{\partial}{\partial x_j} + p_j \frac{\partial}{\partial x_{n+1}}$, $j = 1, \dots, n$. Then $(x_1, \dots, x_n, x_{n+1}; p_1, \dots, p_n)$ gives a system of local coordinates on $\Pi^{-1}(U)$. See also Example 1.2.21.

Remark 1.2.10 (Unit normal). In the real case $\mathbb{K} = \mathbb{R}$, a map-germ $f : (\mathbb{R}^n, A) \to (\mathbb{R}^{n+1}, b)$ is a *frontal* if and only if there exists a smooth *unit* vector field $\nu : (\mathbb{R}^n, A) \to T\mathbb{R}^{n+1}$ along f such that $\nu(t) \cdot f_*(T_t\mathbb{R}^n) = 0$ for any $t \in (\mathbb{R}^n, A)$ for some, or equivalently, for any *Riemannian metric* on $(\mathbb{R}^{n+1}, b)$. Here $\cdot$ means the inner product of the metric. To see this equivalence in the real case, it is sufficient to take $\nu(t)$ as a unit frame of the orthogonal complement to $\widetilde{f}(t)$, i.e. a unit normal, for the metric on $T_{f(t)}\mathbb{R}^{n+1}$, $t \in (\mathbb{R}^n, A)$.

We rewrite the condition of frontal maps in different languages.

Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^{n+1}, b)$ be a smooth map-germ.

Let us take a system of coordinates $t_1, \dots, t_n$ of $\mathbb{K}^n$ and $y_1, \dots, y_{n+1}$ of $\mathbb{K}^{n+1}$. We set $f_i = y_i \circ f$. We write, in coordinates,

$$f(t_1, \dots, t_n) = (f_1(t_1, \dots, t_n), f_2(t_1, \dots, t_n), \dots, f_{n+1}(t_1, \dots, t_n)).$$

Consider the Jacobi matrix

$$Jf := \begin{pmatrix} \dfrac{\partial f_1}{\partial t_1} & \dfrac{\partial f_1}{\partial t_2} & \cdots & \dfrac{\partial f_1}{\partial t_n} \\ \dfrac{\partial f_2}{\partial t_1} & \dfrac{\partial f_2}{\partial t_2} & \cdots & \dfrac{\partial f_2}{\partial t_n} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial f_{n+1}}{\partial t_1} & \dfrac{\partial f_1}{\partial t_2} & \cdots & \dfrac{\partial f_{n+1}}{\partial t_n} \end{pmatrix},$$

which is the matrix representation of the differential map $df : T\mathbb{K}^n \to T\mathbb{K}^{n+1}$ with respect to the frame $\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}$ of $T\mathbb{K}^n$ and $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{n+1}}$ of $T\mathbb{K}^{n+1}$ respectively. Then easily we have

Proposition 1.2.11 The following conditions are equivalent to each other.

(i) f is a frontal in the sense of Definition 1.2.5.

(ii) $\mathrm{Ker}\{(df_t)^* : f^*T^*(\mathbb{K}^{n+1}, b) \to T^*(\mathbb{K}^n, A)\}$ has a smooth section $\nu : (\mathbb{K}^n, A) \to f^*T^*(\mathbb{K}^{n+1}, b) \setminus$ (zero-section).

(iii) There exists a smooth map-germ $\nu = (\nu_1, \nu_2, \ldots, \nu_{n+1}) : (\mathbb{K}^n, A) \to \mathbb{K}^{n+1}\setminus\{0\}$ such that $\nu_1\, df_1 + \nu_2\, df_2 + \cdots + \nu_{n+1}\, df_{n+1} = 0$. □

In the condition (ii), $df : T_t\mathbb{K}^n \to T_{f(t)}\mathbb{K}^{n+1}$ is the differential of f at t, $f^*T^*(\mathbb{K}^{n+1}, b) := \{(t, \alpha) \mid t \in (\mathbb{K}^n, A), \alpha \in T^*_{f(t)}\mathbb{K}^{n+1}\}$ is the pull-back bundle of $T^*(\mathbb{K}^{n+1}, b)$ by f over $(\mathbb{K}^n, A)$ and the dual $(df_t)^*$ of df_t is defined by $(df_t)^*(f(t), \alpha) := (t, \alpha \circ df)$. Here $\alpha : T_{f(t)}\mathbb{K}^{n+1} \to \mathbb{K}$ a co-vector (dual vector) at $f(t) \in \mathbb{K}^{n+1}$.

We observe an algebraic property of frontal-germs.

Let $f : (\mathbb{K}^n, a) \to (\mathbb{K}^{n+1}, b)$ be a mono-map-germ. Consider the **Jacobi ideal** J_f of f in $\mathcal{O}_{\mathbb{K}^n,0}$, which is generated by n-minor determinants

$$\begin{vmatrix} \frac{\partial f_1}{\partial t_1} & \frac{\partial f_1}{\partial t_2} & \cdots & \frac{\partial f_1}{\partial t_n} \\ \frac{\partial f_2}{\partial t_1} & \frac{\partial f_2}{\partial t_2} & \cdots & \frac{\partial f_2}{\partial t_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_{i-1}}{\partial t_1} & \frac{\partial f_{i-1}}{\partial t_2} & \cdots & \frac{\partial f_{i-1}}{\partial t_n} \\ \frac{\partial f_{i+1}}{\partial t_1} & \frac{\partial f_{i+1}}{\partial t_2} & \cdots & \frac{\partial f_{i+1}}{\partial t_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_{n+1}}{\partial t_1} & \frac{\partial f_{n+1}}{\partial t_2} & \cdots & \frac{\partial f_{n+1}}{\partial t_n} \end{vmatrix}, \quad (i = 1, \ldots, n+1),$$

of the Jacobi matrix Jf of f.

Proposition 1.2.12 *Let $f : (\mathbb{K}^n, a) \to (\mathbb{K}^{n+1}, b)$ be a map-germ. If f is a frontal, then Jacobi ideal J_f is principal, i.e. J_f is generated by one element. Conversely, if J_f is principal then f is a frontal, provided the singular locus $S(f)$ is nowhere dense.*

A proof of Proposition 1.2.12 will be given in a more general setting (see Proposition 1.3.23).

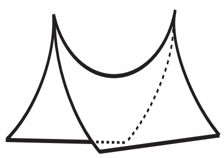
Example 1.2.13 Let $f_{\mathrm{CE}}, f_{\mathrm{SW}}, f_{\mathrm{FU}} : (\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ be defined by $f_{\mathrm{CE}}(t_1, t_2) = (t_1, t_2^2, t_2^3)$, the cuspidal edge, by $f_{\mathrm{SW}}(t_1, t_2) = (t_1, t_2^3 + t_1 t_2, \frac{3}{4}t_2^4 + \frac{1}{2}t_1 t_2^2)$, the swallowtail, and by $f_{\mathrm{FU}}(t_1, t_2) = (t_1, t_2^2, t_1 t_2^3)$, folded umbrella, respectively. Then the Jacobi ideal $J_{f_{\mathrm{CE}}}$ (resp. $J_{f_{\mathrm{SW}}}, J_{f_{\mathrm{FU}}}$) is generated by one element t_2 (resp. $3t_2^2 + t_1,\ t_2$).

Example 1.2.14 The **tangent surface** or **tangent developable** of a space curve is defined as the ruled surface by tangent lines to the curve (see [53, 56, 62]). Some of local models of frontal surfaces are provided by tangent surfaces or tangent developables to space curves in $\mathbb{K}^3$. The tangent surfaces have flat induced metrics and degenerate Gauss mapping. The diffeomorphism classes, **cuspidal edge** (CE), **folded umbrella** (FU), **swallowtail** (SW), **Shcherbak surface** (SB) and **Mond**

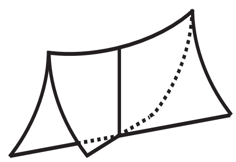
surface (MD) are exactly characterised as those of tangent surfaces in affine space $\mathbb{K}^3$ of curves of type $(1,2,3)$, $(1,2,4)$, $(2,3,4)$, $(1,3,5)$, $(1,3,4)$ respectively. Here a curve $\gamma : (\mathbb{K}, 0) \to \mathbb{K}^3$ is of **type** (a_1, a_2, a_3), for some sequence a_1, a_2, a_3 of integers with $1 \leq a_1 < a_2 < a_3$ if γ is represented as $\gamma(t) = (t^{a_1}, t^{a_2}+$higher order terms, $t^{a_3}+$ higher order terms), up to an affine transformation of $\mathbb{K}^3$ and a diffeomorphism of $(\mathbb{K}, 0)$.

A map-germ $(\mathbb{K}^2, a) \to (\mathbb{K}^3, b)$ is called a **folded pleat** (FP) if it is diffeomorphic to the tangent surface of a curve of type $(2,3,5)$ in $\mathbb{K}^3$. The diffeomorphism classes of folded pleats fall into two classes, the generic folded pleat and the non-generic folded pleat.

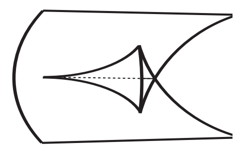
Tangent surfaces to curves in the following pictures are frontals $\mathbb{R}^2 \to \mathbb{R}^3$ which appear in [68].



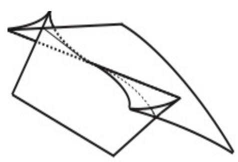
Cuspidal Edge



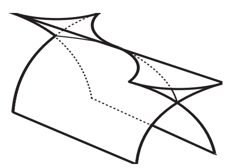
Folded Umbrella (Cuspidal Cross Cap)



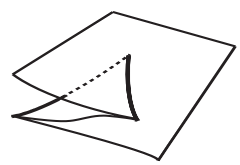
SWallowtail



ShcherBak surface



MonD surface



Folded Pleat

The notion of "global frontality" is defined locally:

Definition 1.2.15 Let N, M be real or complex smooth manifolds with $\dim(N) = n, \dim(M) = n+1$. A smooth ($C^\infty$ or holomorphic) map $f : N \to M$ is called a **frontal map** or simply a **frontal** if the germ of f at any point $t \in N$ is a frontal germ in the sense of Definition 1.2.5 for some (and therefore for any) smooth local coordinates around t of N and around $f(t)$ of M. (See Lemma 1.3.5.)

1.2.3 Lagrangian and Legendrian Geometry

Now, for a moment, recall the notions of symplectic vector spaces and of contact structures. For details see [9] for instance.

Definition 1.2.16 (Symplectic vector space).

(1) Let V be a $\mathbb{K}$-vector space, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A skew-symmetric bilinear form $\Omega : V \times V \to \mathbb{K}$ is called a **symplectic form** if Ω is non-degenerate, i.e. the $\mathbb{K}$-linear map $V \to V^*$ to the dual vector space V^* defined by $u \in V \mapsto (v \in V \mapsto \Omega(u, v) \in \mathbb{K})$

is a $\mathbb{K}$-linear isomorphism. This is equivalent to that Ω is represented as a (skew-symmetric) regular matrix for a basis of V. Then $\dim_{\mathbb{K}} V$ is necessarily even.

(2) A vector space (V, Ω) with a symplectic form Ω is called a **symplectic vector space**.

(3) A subspace $I \subset V$ of is called **isotropic** if $\Omega|_{I\times I}$ is the zero form.

(4) An isotropic subspace $L \subset V$ is called a **Lagrange subspace** if $\dim_{\mathbb{K}} L = n$.

Lemma 1.2.17 Let (V, Ω) be a symplectic vector space of dimension $2n$. Then the dimension of any isotropic subspace $I \subset V$ is less than or equal to n.

Proof Since the inclusion $I \to V$ is injective, the dual map $V^* \to I^*$ is surjective. That $\Omega|_{V\times V}$ is non-degenerate implies that the induced linear map $V \to V^*$ is an isomorphism. Then the composition $V \to V^* \to I^*$ is surjective and its kernel is equal to the skew orthogonal space $I^s = \{u \in V \mid \Omega(u, v) = 0 \text{ for any } v \in I\}$ to I. If $\dim_{\mathbb{R}} I = r$, then $\dim_{\mathbb{R}} I^s = 2n - r$. If I is isotropic, then $I \subset I^s$. Therefore we have $r \leq 2n - r$, which means $r \leq n$. □

Definition 1.2.18 (contact structure, contactomorphism). Let W be a manifold. A smooth (C^∞ or holomorphic) subbundle $D \subset TW$ of codimension 1 of the tangent bundle of W is called a **contact structure** on W if, for any point $w_0 \in W$, there exists a smooth differential 1-form α on an open neighborhood U, called a **contact form**, such that $D|_U$ is equals to $\{(w, v) \in TW|_U \mid \langle\alpha(w), v\rangle = 0\}$ and the exterior differential $d\alpha$ restricted to D_w, $d\alpha|_{D_w\times D_w}$ is a symplectic form on D_w, for any $w \in U$. Then D is of even rank, say $2n$, and therefore W is of odd $2n+1$ dimensional.

A manifold endowed with a contact structure is called a **contact manifold**. Any contact manifold is of odd dimension.

A diffeomorphism $\Phi : (W, D) \to (W', D')$ between contact manifolds (W, D) and (W', D') is called a **contactomorphism** if $d\Phi(D) = D'$.

Definition 1.2.19 (integral mapping, Legendre submanifold, Legendre fibration, Legendre equivalence). Let W be a contact manifold with a contact structure D. A mapping $g : N \to W$ from a manifold N is called an **integral mapping** if $dg(TN) \subset D$, for the differential map $dg : TN \to TW$ of g. If $\dim(N) = n$, then g is called a **Legendre immersion**. A submanifold L of (W, D) is called a **Legendre submanifold** if the inclusion $L \hookrightarrow W$ is a Legendre immersion.

A fibration $\pi : (W, D) \to Z$ from a contact manifold of dimension $2n + 1$ to a manifold Z of dimension $n + 1$ is called a **Legendre fibration** if every fibre $\pi^{-1}(z), (z \in Z)$ is a Legendre submanifold of W. Two mappings $g, h : N \to W$ are called **contact equivalent** or **contactomorphic** if there exist a diffeomorphism $\sigma : N \to N$ and a contactomorphism $\Phi : W \to W$ such that $\Phi \circ g = h\sigma\sigma$. They are called **Legendre equivalent** if moreover Φ is taken to be π-fibre preserving.

Lemma 1.2.20 For any integral mapping $g : N \to (W, D)$ to a contact manifold of dimension $2n + 1$, we have $\dim_{\mathbb{K}}(dg(T_t N)) \leq n$ for any $t \in N$. In particular, if g is an integral immersion, then $\dim(N) \leq n$.

Proof For a contact form α defining D in a neighborhood of $f(t)$, we have $g^*\alpha = 0$, so $g^*(d\alpha) = 0$. This means, for any $t \in N$, $dg(T_tN)$ is an isotropic subspace in the symplectic vector space $D_{g(t)}$ of dimension $2n$. Therefore by Lemma 1.2.17, we have that $\dim_{\mathbb{K}}(dg(T_tN)) \leq n$. □

Example 1.2.21 For $W = P(T^*\mathbb{K}^{n+1})$, the subbundle $D \subset TW$ defined in above is a contact structure. To see this, suppose $a_{n+1} \neq 0$ without loss of generality. If we set $p_i = -\frac{a_i}{a_{n+1}}$, then $x_1, \ldots, x_n, x_{n+1}, p_1, \ldots, p_n$ form a system of local coordinates of $P(T^*\mathbb{K}^{n+1})$. Then D is locally defined by the 1-form $\alpha := dx_{n+1} - \sum_{i=1}^n p_i dx_i$. Then $d\alpha = \sum_{i=1}^n dx_i \wedge dp_i$. Let v be a tangent vector to W over the neighborhood of coordinates and let $v = A\frac{\partial}{\partial x_{n+1}} + \sum_{i=1}^n B_i \frac{\partial}{\partial x_i} + \sum_{j=1}^n C_i \frac{\partial}{\partial p_i}$. Then the condition $\langle \alpha, v \langle = 0$ is equivalent to that $A = \sum_{i=1}^n B_i p_i$. Thus we have that $\frac{\partial}{\partial x_1} + p_1\frac{\partial}{\partial x_{n+1}}, \ldots, \frac{\partial}{\partial x_n} + p_n\frac{\partial}{\partial x_{n+1}}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}$ form a basis of D, and the skew-symmetric form $d\theta|_D$ is represented, for this basis, by the regular matrix

$$\begin{pmatrix} O & I \\ -I & O \end{pmatrix},$$

where I is the unit $n \times n$ matrix.

Remark 1.2.22 By the Darboux theorem, it is known that any contact manifold of dimension $2n + 1$ is locally contactomorphic to the above example $P(T^*\mathbb{K}^{n+1})$. Moreover any Legendre fibration $W \to M$ is locally isomorphic to the projection $\pi : P(T^*\mathbb{K}^{n+1}) \to \mathbb{K}^{n+1}$ ([5]). Note also that, though we do not go into details, the Darboux theorem has generalisations in various directions (see for instance [7, 132, 20]).

1.2.4 Deformations of integral mappings

Singularities of frontals are regarded, via their integral lifts, as those of integral mappings for Legendre equivalence on Legendre fibration of a contact manifold.

Let W be a contact manifold of dimension $2n+1$ with a contact structure $D \subset TW$ and N a manifold of dimension n.

Let $\ell : (N, A) \to W$ be a Legendre (or integral) map-germ, i.e. $\ell^*\alpha = 0$, for some (and any) local contact form α of D defined in a neighbourhood of $f(A)$.

A map-germ $L : (N \times \mathbb{K}, (a, 0)) \to W$ is called a **Legendre deformation** of ℓ if, setting $\ell_u(t) = L(t, u)$ for a representative of L, we have $\ell_0 = \ell$ and ℓ_u is Legendre, $\ell_u^*\alpha = 0$, for any sufficiently small $u \in \mathbb{R}$. A vector field $v : (N, a) \to TW$ along ℓ, ($\pi \circ v = \ell$, $\pi : TW \to W$ is the projection), is called an **infinitesimal Legendre deformation** if there is a Legendre deformation $\ell = (\ell_u)$ of ℓ such that $v = \frac{d\ell_u}{du}|_{u=0}$.

Let us describe infinitesimal Legendre deformations using coordinates.

Let $x_1, \ldots, x_n, x_{n+1}, p_1, \ldots, p_n$ be a system of local coordinates of W such that the contact structure of W is given locally by

$$dx_{n+1} - \sum_{i=1}^{n} p_i dx_i = 0.$$

If we write simply

$$\ell_u = (x_{1u}, \ldots, x_{nu}, x_{n+1\,u}, p_{1u}, \ldots, p_{nu}),$$

then, by the condition that ℓ_u is a Legendre deformation, we have

$$dx_{n+1\,u} - \sum_{i=1}^{n} p_{iu} dx_{iu} = 0.$$

Note that here the exterior differential d is on coordinates $t_1, \ldots, t_n$ of N. By differentiating with respect to u we have

$$d\frac{\partial x_{n+1\,u}}{\partial u} - \textstyle\sum_{i=1}^{n} \frac{\partial p_{iu}}{\partial u} dx_{iu} - \sum_{i=1}^{n} p_{iu} d\frac{\partial x_{iu}}{\partial u} = 0, \cdots\cdots (*).$$

Let $x_1, \ldots, x_n, x_{n+1}, p_1, \ldots, p_n; \xi_1, \ldots, \xi_n, \xi_{n+1}, \varphi_1, \ldots, \varphi_n$ be the induced system of local coordinates of the tangent bundle TW. Then the condition on the coordinates of $v = \frac{d\ell_u}{du}|_{u=0}$ is given, setting $u = 0$ from (*), by

$$d\xi_{n+1} - \sum_{i=1}^{n} \varphi_i dx_i - \sum_{i=1}^{n} p_i d\xi_i = 0.$$

or by

$$d\left(\xi_{n+1} - \sum_{i=1}^{n} p_i \xi_i\right) - \sum_{i=1}^{n} \varphi_i dx_i + \sum_{i=1}^{n} \xi_i dp_i = 0.$$

Note that the function $\xi_{n+1} - \sum_{i=1}^{n} p_i \xi_i$ is equal to the paring $\langle \ell^* \alpha,\ v \rangle$.

Here we digress to recall a general theory on differential analysis on manifolds.

Definition 1.2.23 (**Lie derivative**). Let N, W be a $\mathbb{K}$-manifold, $\ell : N \to W$ a differentiable $\mathbb{K}$-mapping and $v : N \to TW$ be a vector field over ℓ. For a differential p-form α on W, we define a differential p-form $L_v \alpha$ on N by

$$L_v \alpha := \left.\frac{d}{du}\right|_{u=0} \ell_u^* \alpha,$$

for a deformation $\ell_u, (u \in (\mathbb{K}, 0))$ of ℓ with $\left.\frac{d}{du}\right|_{u=0} \ell_u = v$. Then $L_v \alpha$ does not depend on the choice of a deformation but depends only on v (see [52, 55]).

Proposition 1.2.24 [126, 125, 55] *For any differential form α on any manifold W, there is a unique differential form $\widetilde{\alpha}$ on the tangent bundle TW of W which satisfies the followings:*
(0) *Let $1 : TW \to TW$ be the identity regarded as a vector field along $\pi : TW \to W$.*

Then $\widetilde{\alpha} = L_1\alpha$.
(1) *We have* $X^*\widetilde{\alpha} = L_X\alpha$ *for any vector field* $X : W \to TW$. *Here* $L_X\alpha$ *means the Lie derivative of* α *by* X.
(2) *For any smooth map* $f : N \to W$ *and for any vector field* $v : N \to TW$ *along* f, *we have* $v^*\widetilde{\alpha} = L_v\alpha$ *and* $\widetilde{f^*\alpha} = (f_*)^*\widetilde{\alpha}$. *Here* $f_* : TN \to TM$ *is the bundle homomorphism defined by the differential of* f.
(3) *We have* $d\widetilde{\alpha} = \widetilde{d\alpha}$, *for the exterior differential* d. □

We call $\widetilde{\alpha}$ the **natural lifting** or the **complete lift** of α.

Corollary 1.2.25 (1) *For a* 0*-form i.e. a function* h *on* W, *we have* $\widetilde{h}(v) = v(h)$ *for any* $v \in TW$. *Here* $v(h)$ *means the directional derivative of* h *by the tangent vector* v.
(2) $\widetilde{\alpha + \beta} = \widetilde{\alpha} + \widetilde{\beta}$ *for any p-forms* α, β *on* W *for any* $p = 0, 1, 2, 3, \ldots$.
(3) $\widetilde{\alpha \wedge \beta} = \widetilde{\alpha} \wedge \pi^*\beta + \pi^*\alpha \wedge \widetilde{\beta}$, *for any forms* α, β *on* W. *Here* $\wedge$ *means the wedge product of forms and* π^* *the pull-back by the natural projection* $\pi : TW \to W$. □

Example 1.2.26 Let $x = (x_1, x_2, \ldots, x_m)$ be a system of local coordinates on $U \subset W$. Then a point in $\pi^{-1}(U)$ is uniquely represented by $v_1\frac{\partial}{\partial x_1}|_x + \cdots + v_m\frac{\partial}{\partial x_1}|_x$. Let $(x, v) = (x_1, \ldots, x_m; v_1, v_2, \ldots, v_m)$ be the induced system of local coordinates on $\pi^{-1}(U) \subset TW$. For this system of coordinates, we have $\widetilde{h}(x, v) = \sum_{i=1}^m v_i \frac{\partial h}{\partial x_i}(x)$ on $\pi^{-1}(U)$, for any function h on W.

Let $\ell : (N, A) \to W$ be a map-germ to a contact manifold with the contact structure $D \subset TW$. Let D be defied by $\alpha = 0$ locally around $f(A)$. Then we set

$$VI_\ell := \{v : (N, A) \to TW \mid v^*\widetilde{\alpha} = 0, \pi \circ v = \ell\}.$$

We call VI_ℓ the **space of infinitesimal integral deformations** of ℓ.

In the next section, we apply the argument to the case ℓ is a Legendre lift $\widetilde{f}$ of a frontal map-germ f.

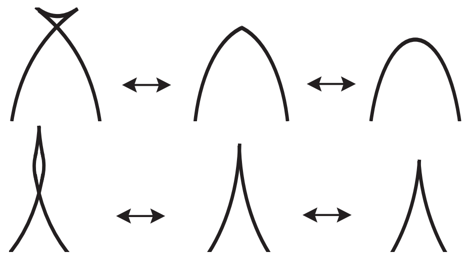
1.2.5 Frontal deformations and frontal stability

Definition 1.2.27 Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^{n+1}, b)$ be a frontal germ. A deformation $F : (\mathbb{K}^n \times \mathbb{K}^r, A \times \{0\}) \to (\mathbb{K}^{n+1}, b)$ of f, $F(t, u) = f_u(t)$, $f_0 = f$, is called a **frontal deformation** if there exists a lift $\widetilde{F} : (\mathbb{K}^n \times \mathbb{K}^r, A \times \{0\}) \to \mathrm{Gr}(n, T\mathbb{K}^{n+1})$,

$$\begin{tikzcd} & \mathrm{Gr}(n, T\mathbb{K}^{n+1}) \arrow[d, "\pi"] \\ (\mathbb{K}^n \times \mathbb{K}, A \times \{0\}) \arrow[ur, "\widetilde{F}"] \arrow[r, "F"'] & \mathbb{K}^{n+1}, \end{tikzcd}$$

satisfying $(f_u)_*(T_t\mathbb{K}^n) \subset \widetilde{f_u}(t)$, for any $(t, u) \in \mathbb{K}^n \times \mathbb{K}^r$ nearby $A \times \{0\}$. Here $\widetilde{F}(t, u) = \widetilde{f_u}(t)$.

Example 1.2.28 The following first two pictures designate examples of frontal deformations, while the third does not.



Frontal deformations.

A non-frontal deformation.

Definition 1.2.29 Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^{n+1}, b)$ be a frontal. Then f is called **frontally stable** if any frontal deformation $F : (\mathbb{K}^n \times \mathbb{K}, (A, 0)) \to (\mathbb{R}^{n+1}, b)$ of f is trivial: i.e. there exist diffeomorphisms

$$\Sigma : (\mathbb{K}^n \times \mathbb{K}, A \times \{0\}) \to (\mathbb{K}^n \times \mathbb{K}, A \times \{0\}), \Sigma(t, u) = (\sigma_u(t), u),$$

$$T : (\mathbb{K}^{n+1} \times \mathbb{K}, (b, 0)) \to (\mathbb{K}^{n+1} \times \mathbb{K}, (b, 0)), T(x, u) = (\tau_u(x), u),$$

such that $(T \circ (F, \mathrm{id}_{\mathbb{R}^r}) \circ \Sigma)(t, u) = (f(t), u)$, or equivalently,

$$\tau_u \circ f_u \circ \sigma_u = f.$$

Here we recall the key notion of contact Hamilton vector fields for the study on frontal stability from [55].

Let (W, D) be a contact manifold. A vector field X over W is called a **contact vector field** if the flow generated by X preserves the contact distribution D. This is equivalent to that the Lie derivative $L_X\alpha = \mu\alpha$ for some (and any) local contact form α giving $D \subset TW$ and for a function μ.

Deleting W if necessary, we assume a contact form α is taken over W. Let $H : W \to \mathbb{K}$ be a smooth function. Then there exists a unique contact vector field $X = X_H$ over W with the condition $\langle \alpha, X \rangle = H$. The contact vector field X_H is called the **contact Hamilton vector field** with **Hamiltonian** H. If $\alpha = dx_{n+1} - \sum_{i=1}^n p_i dx_i$, then X_H is explicitly given by

$$X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial x_i} + p_i \frac{\partial H}{\partial x_{n+1}} \right) \frac{\partial}{\partial p_i} - \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x_i} + \left(H - \sum_{i=1}^n p_i \frac{\partial H}{\partial p_i} \right) \frac{\partial}{\partial x_{n+1}}.$$

Conversely, any contact vector field is locally a contact Hamilton vector field with some Hamilton function.

Associated to a contact form α, we define the **Reeb vector field** R by $\langle \alpha, R\rangle = 1, i_R d\alpha = 0$. Here i means the interior product, i.e. $i_R d\alpha$ is the 1-form which satisfies $i_R d\alpha(v) = d\alpha(R, v)$ for any tangent vector v. Note that, since α is a contact form, R is characterised uniquely. If $\alpha = dx_{n+1} - \sum_{i=1}^{n} p_i dx_i$, then $R = \frac{\partial}{\partial x_{n+1}}$.

Lemma 1.2.30 [55] *Let α be a contact form on W, and $H : W \to \mathbb{K}$ a function. Then we have* (1) *$L_{X_H}\alpha = R(H)\alpha$ and $i_{X_H} d\alpha = R(H)\alpha - dH$.* (2) *Let η be a vector field on W. If $i_\eta d\alpha = 0$, then $\eta = (i_\eta \alpha)R$.* (3) *$X_1 = R$.* □

We have the following formula for the contact Hamilton vector field with the sum (resp. product) of two contact Hamiltonians:

Lemma 1.2.31 [55] *For functions K, H on W, we have $X_{K+H} = X_K + X_H$ and*

$$X_{KH} = K \cdot X_H + H \cdot X_K - (KH) \cdot R = K \cdot X_H + H \cdot X_K - (KH) \cdot X_1.$$

In particular, $X_{aH} = aX_H, (a \in \mathbb{K})$. □

We denote by VH_W the *vector space of contact Hamilton vector fields over W* and by O_W the $\mathbb{K}$-algebra of smooth functions on W. Define a linear map $\Phi : O_W \to VH_W$ by $\Phi(H) = X_H$. Then Φ is an isomorphism of vector spaces. Therefore VH_W is endowed with O_W-module structure induced from Φ, namely, $K * X_H = X_{KH}$. Here, we distinguish this new functional multiplication, using $*$, with the ordinary functional multiplication in V_W.

Let $\pi : W \to M$ be a Legendre fibration (Definition 1.2.19). Then a contact vector field X over W is called a **Legendre vector field** if X is π-lowerable, namely, if there exists a vector field Y over M such that $t\pi(X) = w\pi(Y)$ as vector fields along π. Then we have:

Proposition 1.2.32 [55] *Let $(x_1, \dots, x_n, x_{n+1}, p_1, \dots, p_n)$ be a system of local coordinates of a contact manifold W, so that $\alpha = dx_{n+1} - \sum_{i=1}^{n} p_i dx_i$ is a contact form and π is given by $\pi(x_1, \dots, x_n, x_{n+1}, p_1, \dots, p_n) = (x_1, \dots, x_n, x_{n+1})$. Then a contact Hamilton vector field X_H with Hamiltonian $H = H(x, p)$ is a Legendre vector field if and only if H is an affine function with respect to $\pi : W \to M$, namely, H is of form $H(x, p) = a_0(x_1, \dots, x_n, x_{n+1}) + a_1(x_1, \dots, x_n, x_{n+1})p_1 + \cdots + a_n(x_1, \dots, x_n, x_{n+1})p_n$.*

We denote by $VL_W = VL_{(W,\pi)}$, the totality of Legendre vector fields over W with respect to π.

The following is the infinitesimal characterisation of frontal stability proved in [55, 101].

Theorem 1.2.33 (Mather's type characterisation). *Let $f : N = (\mathbb{K}^n, A) \to M = (\mathbb{K}^{n+1}, b)$ be a frontal map-germ with a Legendre lift of corank ≤ 1. Then the following conditions are equivalent to each other:*

(i) *f is frontally stable.*

(ii) *Any Legendre lift $\widetilde{f} : N \to W = \mathrm{Gr}(n, T\mathbb{R}^{n+1})$ of f is infinitesimally Legendre stable i.e.,*

$$VI_{\widetilde{f}} = t\widetilde{f}(V_N) + w\widetilde{f}(VL_W).$$

(iii) *f is a proper frontal and the unique Legendre lift $\widetilde{f} : N \to W = \mathrm{Gr}(n, T\mathbb{R}^{n+1})$ of f is infinitesimally Legendre stable.*

(iv) *f is a proper frontal and f is infinitesimally frontally stable i.e.,*

$$V_f^{\mathrm{fr}} \subset tf(V_N) + wf(V_M).$$

□

Here V_N, V_M denote the space of vector fields over N, M respectively, V_f the space of vector field along $f : N \to M$, and VL_W the space of Legendre vector fields as is defined in above. Moreover $V_f^{\mathrm{fr}} \subset V_f$ is defined by

$$V_f^{\mathrm{fr}} := \left\{ v \in V_f \;\middle|\; v = \left.\frac{df_u}{du}\right|_{u=0}, \text{ for a frontal deformation } f_u \text{ of } f \right\}$$

Remark 1.2.34 In Theorem 1.2.33 (iv), the converse inclusion $V_f^{\mathrm{fr}} \supset tf(V_N) + wf(V_M)$ holds always.

If f is a frontally stable germ with a Legendre lift of corank ≤ 1, then f is has a proper (= singular locus is nowhere dense) representative, and therefore the Legendre lift $\widetilde{f}$ of f is uniquely determined. Then f is frontally stable if and only if $\widetilde{f}$ is "Legendre stable" in the sense of [55].

Remark that in [55] only the case of single germs and in [101] the case of multi-germs are proved. The detailed proof of Theorem 1.2.33 will not repeated in this survey paper. We only mention following observations (Lemmas 1.2.35, 1.2.37) which play key roles for the proof of Theorem 1.2.33.

Lemma 1.2.35 ([101]) *Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^{n+1}, b)$ be a proper frontal map-germ with the Legendre lift $\widetilde{f}$. Then the linear map $d\pi : VI_{\widetilde{f}} \to V_f$ defined $d\pi(\widetilde{v}) = d\pi \circ \widetilde{v}$ is injective. If $\widetilde{f}$ is of corank ≤ 1, then $d\pi(VI_{\widetilde{f}}) = V_f^{\mathrm{fr}}$. In particular $d\pi$ givens a linear isomorphism between $VI_{\widetilde{f}}$ and V_f^{fr}.*

Here we give an alternative proof of Lemma 1.2.35.

Proof of Lemma 1.2.35. Let $\widetilde{v} \in VI_{\widetilde{f}}$ and take its representative $\widetilde{v} : U \to TW$, where $W = P(T^*\mathbb{K}^{n+1})$ on a sufficiently small open neighbourhood U of a such that $\Pi \circ \widetilde{v} = f : U \to \mathbb{K}^{n+1}$ has dense regular locus in U. Here $\Pi : TW \to \mathbb{K}^{n+1}$ is the composition of the natural projections $\pi : TW \to W$ and $\pi' : W = P(T^*\mathbb{K}^{n+1}) \to \mathbb{K}^{n+1}$. Take any regular point $t_0 \in U$. Then, after taking appropriate local coordinates $t_1, \dots, t_n$ centred at t_0 and $x_1, \dots, x_n, x_{n+1}$ at $f(t_0)$ respectively, we may suppose $f(t) = (t_1, \dots, t_n, 0)$ and then $\widetilde{f}(t) = (t_1, \dots, t_n, 0; [0, \dots, 0, 1])$. Taking induced coordinates $x_1, \dots, x_n, x_{n+1}; p_1, \dots, p_n$ of W such that $\alpha = dx_{n+1} - \sum_{i=1}^n p_i dx_i = 0$ gives the contact structure of W, we have $\widetilde{f}(t) = (t_1, \dots, t_n, 0; 0, \dots, 0)$. Let

$\widetilde{v}(t) = (t_1, \ldots, t_n, 0; 0, \ldots, 0; \xi_1, \ldots, \xi_n, \xi_{n+1}; \varphi_1, \ldots, \varphi_n)$ by using the induced local coordinates on TW. Note that $d\pi(\widetilde{v})(t) = (t_1, \ldots, t_n, 0; \xi_1, \ldots, \xi_n, \xi_{n+1})$. Then the condition $\widetilde{v}^*\widetilde{\alpha} = 0$ reads that $d\xi_{n+1} - \sum_{i=1}^n \varphi_i dt_i = 0$. Therefore each φ_i is determined as $\frac{\partial \xi_{n+1}}{\partial t_i}$. This shows $\widetilde{v}$ is determined by v in a neighbourhood of the regular point t_0. Since the singular locus $\mathrm{Sing}(f)$ is nowhere dense, it can be proved $\widetilde{v}$ is uniquely determined by $v = d\pi(\widetilde{v})$. This means $d\pi$ is injective. Suppose $\widetilde{f}$ is of corank ≤ 1. Then $VI_{\widetilde{f}}$ is equal to the set of $w \in V_{\widetilde{f}}$ such that $w = \frac{d\widetilde{f}_u}{du}|_{u=0}$ for some integral deformation of $\widetilde{f}_u$ of $\widetilde{f}$. Then $\pi \circ \widetilde{f}_u$ gives a frontal deformation of f. Conversely any frontal deformation f_u of f uniquely lifts to an integral deformation $\widetilde{f}_u$ of $\widetilde{f}$. Thus we have the equality $d\pi(VI_{\widetilde{f}}) = V_f^{\mathrm{fr}}$. □

Lemma 1.2.36 [55] *Suppose that corank of $\widetilde{f} \leq 1$. Then, for any $v \in VI_{\widetilde{f}}$, there exist a frontal deformation $F : (\mathbb{K}^n \times \mathbb{K}, A \times 0) \to \mathbb{K}^{n+1}$ of f and an integral deformation $\widetilde{F} : (\mathbb{K}^n \times \mathbb{K}, A \times 0) \to W = P(T^*\mathbb{K}^{n+1})$ of $\widetilde{f}$ covering F such that $\frac{d}{du} f_u\big|_{u=0} = v$.* □

It can be proved that, by Lemma 1.2.35, f is infinitesimally frontally stable if and only if $\widetilde{f}$ is infinitesimally integrally stable. Moreover it can be proved that f is frontally stable if and only if $\widetilde{f}$ is Legendre stable, and then, by Lemma 1.2.36, f (resp. $\widetilde{f}$) must be infinitesimally frontally stable (resp. infinitesimally frontally stable).

Suppose f is infinitesimally frontally stable. This is equivalent to that $\widetilde{f}$ is infinitesimally integrally stable. To show f is frontally stable, we need to find, for any integral deformation $(\widetilde{f}_u)$ of $\widetilde{f}$, a deformation (σ_u) of id_N an integral deformation τ_u of id_W covering a deformation $(\bar{\tau}_u)$ of id_M via $\pi : W \to M$ satisfying $\tau_t^{-1} \circ \widetilde{f}_u \circ \sigma_t = \widetilde{f}$, and thus $\bar{\tau}_u^{-1} \circ f_t \circ \sigma_u = f$. For this, it is sufficient to solve $d\widetilde{f}_u/du = \eta_u \circ \widetilde{f}_u - T\widetilde{f}_u \circ \xi_u (= w\widetilde{f}_u(\eta_u) - t\widetilde{f}_u(\xi_u)) : N \times \mathbb{K} \to TW$ with $\xi_u \in V_N$ and $\eta_u \in VL_W$.

For an unfolding $\mathcal{F} = (\widetilde{f}_u, u) : N \times J \to W \times J$, $u \in J = (\mathbb{K}, 0)$, we set

$$VI_{\mathcal{F}/J} = \{v : N \times J \to TW \mid v_u \in VI_{\widetilde{f}_u}, u \in J\}.$$

If $(\widetilde{f}_u)$ is an integral deformation of $\widetilde{f}$, then we have $(d\widetilde{f}_u/du)_{u \in J} \in VI_{\mathcal{F}/J}$. We define an $\mathcal{O}_{W\times J}$-module structure on $VI_{\mathcal{F}/J}$ by

$$a_u * v_u = (\widetilde{f}_u^* a_u) \cdot v_u + \langle \alpha, i_{v_u} \rangle (X_{a_u} - a_u \cdot R) \circ \widetilde{f}_u,$$

for $v_u \in VI_{\mathcal{F}/J}, a_u \in \mathcal{O}_{W\times J}$, using Hamilton vector field X_{a_u} of a_u and Reeb vector field R. Let $\widetilde{f}$ be finite and of corank at most one. Then it can be proved that the quotient $VI_{\mathcal{F}/J}$ is a finite $\mathcal{O}_{W\times J}$-module. We define $t\mathcal{F}/J : V_N \to VI_{\mathcal{F}/J}$ by $\xi \mapsto (t\widetilde{f}_u(\xi))_{u\in J}$ and $w\mathcal{F}/J : V_W \to VI_{\mathcal{F}/J}$ by $\eta \mapsto (w\widetilde{f}_u(\eta))_{u\in J}$. We set

$$S_{\mathcal{F}/J} = VI_{\mathcal{F}/J}/((w\mathcal{F}/J)(VH_W) + (t\mathcal{F}/J)(V_N)),$$

which is an $\mathcal{O}_{W\times J}$-module, and set

$$S_{\widetilde{f}} = VI_{\widetilde{f}}/(w\widetilde{f}(VL_W) + t\widetilde{f}(V_N)),$$

which is an O_W-module. Then we have:

Lemma 1.2.37 ([55]) *The quotient $S_{\mathcal{F}/J}/m_J S_{\mathcal{F}/J}$ is isomorphic to $S_{\widetilde{f}}$ as an O_W-modules.* □

Now if $\widetilde{f}$ is infinitesimally integrally stable, then $S_{\widetilde{f}} = 0$ and then, by Lemma 1.2.37, we have $m_J S_{\mathcal{F}/J} = S_{\mathcal{F}/J}$, and then, by Nakayama's Lemma, we have $S_{\mathcal{F}/J} = 0$, i.e. $VI_{\mathcal{F}/J} = ((w\mathcal{F}/J)(VH_W) + (t\mathcal{F}/J)(V_N))$. This means that any integral deformation $\widetilde{f_u}$ of $\widetilde{f}$ is recovered by deformations of infinitesimal Legendre equivalences $(\xi_u, \widetilde{\eta}_u)$ such that

$$\frac{d}{du}\widetilde{f_u} = (w\widetilde{f_u})(\widetilde{\eta_u}) - (t\widetilde{f_u})(\xi_u).$$

By integrating on u, we have that $\widetilde{f}$ is Legendre stable and f is frontally stable.

Remark 1.2.38 In [101], a characterisation of frontal stability in terms of $\mathcal{K}$-equivalence is given. Moreover also a characterisation of frontal stability of multi-germs of frontals is given. We can say, in some sense, they complete the frontal analogue of Thom-Mather's theory eventually.

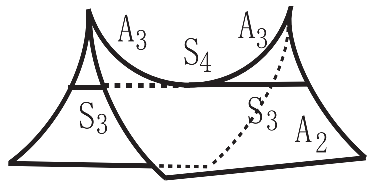
Now we present examples of frontally stable germs. Here is the list of frontally stable frontals of low dimensions. These owe mainly the results by Arnold [2], Zakalyukin [127], Givental [38], Bogaevskii, Ishikawa [15] and Ishikawa [55].

$n = 1$. Any stable frontal $f : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ is right-left ($\mathcal{A}$-) equivalent to (regular) $t_1 \mapsto (t_1, 0)$, or (cusp) (t_1^2, t_1^3).

$n = 2$. Any stable frontal $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ of integral corank ≤ 1 is right-left equivalent to (regular) $(t_1, t_2) \mapsto (t_1,\ 0,\ 0)$, (cuspidal edge) $(t_1, t_2) \mapsto (t_1,\ t_2^2,\ t_2^3)$, (swallowtail) $(t_1, t_2) \mapsto (t_1,\ t_2^3 + t_1 t_2,\ \frac{3}{4}t_2^4 + \frac{1}{2}t_1 t_2^2)$, or (folded umbrella, or, cuspidal cross-cap) $(t_1, t_2) \mapsto (t_1,\ t_2^2,\ t_1 t_2^3)$.

$n = 3$. Any stable frontal $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^4, 0)$ of integral corank ≤ 1 is right-left equivalent to regular $A_1 : (t_1, t_2, t_3) \mapsto (t_1,\ 0,\ 0,\ 0)$, (three dimensional cuspidal edge) $A_2 : (t_1, t_2, t_3) \mapsto (t_1,\ t_2, t_3^2,\ t_3^3)$, (three dimensional swallowtail) $A_3 : (t_1, t_2, t_3) \mapsto (t_1,\ t_2, t_3^3 + t_1 t_3,\ \frac{3}{4}t_3^4 + \frac{1}{2}t_1 t_3^2)$, (butterfly) $A_4 : (t_1, t_2, t_3) \mapsto (t_1,\ t_2, t_3^4 + t_1 t_3^2 + t_2 t_3,\ \frac{4}{5}t_3^5 + \frac{1}{2}t_1 t_2^2)$,
(hyperbolic umbilical point, wallet D_4^+) and (elliptic umbilical point, pyramid D_4^-) : $(t_1, t_2, t_3) \mapsto (t_1,\ t_2^2 \pm t_3^2,\ t_2 t_3 + t_1 t_2,\ t_2^2 t_3 \pm \frac{1}{3}t_3^3 + \frac{1}{2}t_1 t_2^2)$,
(folded umbrella $S_3 = A_{2,1}$) : $(t_1, t_2, t_3) \mapsto (t_1,\ t_2,\ t_3^2,\ t_1 t_3^3)$, or
$S_4 = A_{3,1} : (t_1, t_2, t_3) \mapsto (t_1,\ t_2,\ t_3^3 + t_1 t_3,\ \frac{3}{5}t_3^5 + \frac{3}{4}t_2 t_3^4 + \frac{1}{3}t_1 t_3^3 + \frac{1}{2}t_1 t_2 t_3^2)$.

We do not give the three dimensional pictures in $\mathbb{R}^4$. Instead, here we present the bifurcation diagram of S_4-singularities.



Bifurcation around S_4-singularities.

1.3 General frontals

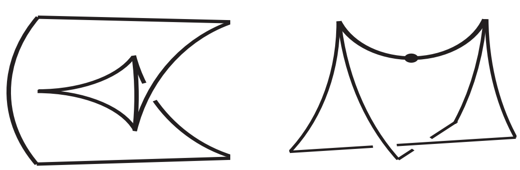
1.3.1 Examples of general frontals

Before introducing a generalization of the notion of frontals in the next section, we mention several typical examples of frontal surface singularities in $\mathbb{K}^4$ in a generalised sense.

Example 1.3.1 (Embedded cuspidal edges). A map-germ $f : (\mathbb{K}^2, a) \to (\mathbb{K}^4, b)$ is called an **embedded cuspidal edge** if f is right-left equivalent to the germ defined by $(t_1, t_2) \mapsto (t_1,\ t_2^2,\ t_2^3,\ 0)$. More generally a map-germ $f : (\mathbb{K}^2, a) \to (\mathbb{K}^m, b)$, $m \geq 4$ is called an embedded cuspidal edge if f is right-left equivalent to the germ defined by $(t_1, t_2) \mapsto (t_1,\ t_2^2,\ t_2^3,\ 0, \dots, 0)$. This is a cuspidal edge in $\mathbb{K}^3$ composed with the embedding $\mathbb{K}^3 \to \mathbb{K}^3 \times \{0\} \subset \mathbb{K}^m$.

Example 1.3.2 (Open swallowtails (OSW) and open folded umbrellas (OFU)). A map-germ $f : (\mathbb{K}^2, a) \to (\mathbb{K}^4, b)$ is called an **open swallowtail** if f is right-left equivalent to the germ defined by $(t_1, t_2) \mapsto (t_1,\ t_2^3 + t_1 t_2,\ \frac{3}{4}t_2^4 + \frac{1}{2}t_1 t_2^2,\ \frac{3}{5}t_2^5 + \frac{1}{3}t_1 t_2^3)$. It will be seen that f is a frontal in the sense of §1.3.2 and has an injective representative as depicted "virtually" in $\mathbb{R}^4$ (see also [80]).

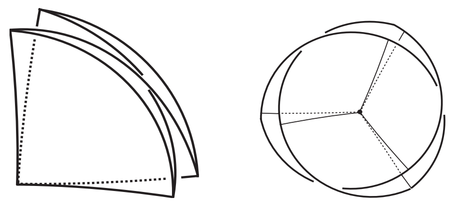
A map-germ $f : (\mathbb{K}^2, a) \to (\mathbb{K}^4, b)$ is called an **open folded umbrella** or an **open cuspidal cross-cap** if f is right-left equivalent to the germ defined by $(t_1, t_2) \mapsto (t_1,\ t_2^2,\ t_1 t_2^3,\ t_2^5)$, which is also a frontal and has an injective representative.



Open swallowtail and open folded umbrella.

Example 1.3.3 (Products of cusps (PCU) and complex cusps (CCU)). A map-germ $f : (\mathbb{K}^2, a) \to (\mathbb{K}^4, b)$ is called a **product of cusps** if f is right-left equivalent to the germ defined by $(t_1, t_2) \mapsto (t_1^2, t_2^2, t_1^3, t_2^3)$. It is a frontal (front) which has an injective representative.

Consider, in complex case, the map-germ $(\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ defined by $z \to (z^2, z^3)$. We regard it as a map-germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^4, 0)$ by taking $z = t_1 + \sqrt{-1}t_2$, which is given by $(t_1, t_2) \mapsto (t_1^2 - t_2^2,\ 2t_1t_2,\ t_1^3 - 3t_1t_2^2,\ 3t_1^2t_2 - t_2 - 3)$. Then a map-germ $f : (\mathbb{K}^2, a) \to (\mathbb{K}^4, b)$ is called a **complex cusp**, in each case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, if f is right-left equivalent to the germ $(\mathbb{K}^2, 0) \to (\mathbb{K}^4, 0)$ which is defined by $(t_1, t_2) \mapsto (t_1^2 - t_2^2,\ 2t_1t_2,\ t_1^3 - 3t_1t_2^2,\ 3t_1^2t_2 - t_2^3)$. It is also a frontal (front) which has an injective representative.



Product of cusps and complex cusp.

The above examples Examples 1.3.2 and 1.3.3 are obtained by the process of "openings" (see §1.4).

1.3.2 General frontal singularities

We generalise the notion of frontals defined in §1.2.

Definition 1.3.4 Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be a map-germ. Suppose $n \le m$. Then f is called a **frontal map-germ** or, briefly a **frontal** in short, if there exists a smooth family of n-planes $\widetilde{f}(t) \subseteq T_{f(t)}\mathbb{K}^m$ along f, $t \in (\mathbb{K}^n, A)$, i.e. there exists a smooth lift $\widetilde{f} : (\mathbb{K}^n, A) \to \mathrm{Gr}(n, T\mathbb{K}^m)$ satisfying the "*integrality condition*"

$$df(T_t\mathbb{K}^n) \subset \ \widetilde{f}(t)\ (\subset T_{f(t)}\mathbb{K}^m),$$

for any $t \in \mathbb{K}^n$ nearby A, such that $\pi \circ \widetilde{f} = f$, namely the following mapping diagram commutes:

$$\begin{tikzcd} & \mathrm{Gr}(n, T\mathbb{K}^m) \arrow[d, "\pi"] \\ (\mathbb{K}^n, A) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (\mathbb{K}^m, b). \end{tikzcd}$$

Here $\mathrm{Gr}(n, T\mathbb{K}^m)$ is the **Grassmannian bundle** consisting of n-dimensional linear $\mathbb{K}$-subspaces $V \subset T_x\mathbb{K}^m$ $(x \in \mathbb{K}^m)$ with the canonical projection $\pi(x, V) = x$, and $df : T_t\mathbb{K}^n \to T_{f(t)}\mathbb{K}^m$ is the differential map of f at $t \in (\mathbb{K}^n, A)$. The lift $\widetilde{f}$ is called a **Legendre lift** or an **integral lift** of the frontal f. Actually $\widetilde{f}$ is an integral mapping to the canonical or contact distribution on $\mathrm{Gr}(n, T\mathbb{K}^m)$ (cf. [56]). The canonical distribution $D \subset T\mathrm{Gr}(n, T\mathbb{K}^m)$ is defined, for $(x, V) \in \mathrm{Gr}(n, T\mathbb{K}^m)$, by

$$D_{(x,V)} := \{w \in T_{(x,V)}\mathrm{Gr}(n, T\mathbb{K}^m) \mid d\pi(w) \in V\},$$

where $\pi : \mathrm{Gr}(n, T\mathbb{K}^m) \to \mathbb{K}^m$ is the projection defined by $\pi(x, V) = x$, therefore $d\pi : T_{(x,V)}\mathrm{Gr}(n, T\mathbb{K}^m) \to T_x\mathbb{K}^m$ maps $w \in T_{(x,V)}\mathrm{Gr}(n, T\mathbb{K}^m)$ to $d\pi(w) \in T_x\mathbb{K}^m$. Then $\widetilde{f}$ satisfies that $d\widetilde{f}(T_t\mathbb{K}^n) \subset D_{(x,V)}$. This means that $df(T_t\mathbb{K}^n) \subset \widetilde{f}(t)$ for any $t \in \mathbb{K}^n$ nearby A.

A frontal $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ is called a **front** if there exists an immersive Legendre lift of f.

Lemma 1.3.5 *If f is a frontal (resp. a front) and g is right-left equivalent to f, then g is a frontal (resp. a front).*

Proof The diffeomorphism-germ τ induces the isomorphism $\tau_* : T\mathbb{K}^m \to T\mathbb{K}^m$ covering τ and the diffeomorphism $\widetilde{\tau} : \mathrm{Gr}(n, T\mathbb{K}^m) \to \mathrm{Gr}(n, T\mathbb{K}^m)$ covering τ, where $\widetilde{\tau}$ is defined by $\widetilde{\tau}(V) = \tau_*(V)$ for $V \in \mathrm{Gr}(n, T\mathbb{K}^m)$. Then, if there is a Legendre lift $\widetilde{f}$ of f, then $\widetilde{g} := \widetilde{\tau} \circ \widetilde{f} \circ \sigma^{-1}$ is a Legendre lift of g. Moreover if $\widetilde{f}$ is an immersion, then $\widetilde{g}$ is an immersion. □

Remark 1.3.6 As is remarked in Remark 1.2.8, $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ is a frontal if only if each component $f_i : (\mathbb{K}^n, a_i) \to (\mathbb{K}^m, b)$ is a frontal for any $i = 1, \dots, r$, where $A = \{a_1, \dots, a_r\} \subset \mathbb{K}^n$. We would like to keep to treat multi-germs for the generality of the exposition as far as possible.

The notion of "global frontality" is defined locally in the general cases as in Definition 1.2.15 :

Definition 1.3.7 Let N (resp. M) be a manifold of dimension n (resp. m). Suppose $n \leq m$. A map $f : N \to M$ is called a **frontal map** or simply a **frontal** (resp. a **front**) if, for any $a \in N$, the germ of f at a is frontal (resp. front) under local coordinates around a and $f(a)$.

We rewrite the condition of frontal maps to different languages.

Let $f : (\mathbb{K}^n, a) \to (\mathbb{K}^m, b)$ be a smooth map. We write, in coordinates,

$$f(t_1, \dots, t_n) = (f_1(t_1, \dots, t_n), f_2(t_1, \dots, t_n), \dots, f_m(t_1, \dots, t_n)).$$

Proposition 1.3.8 The following conditions are equivalent to each other.

(i) f is a frontal in the sense of Definition 1.2.5.

(ii) $\mathrm{Ker}\{(df_t)^* : f^*T^*(\mathbb{K}^m, b) \to T^*(\mathbb{K}^n, a)\}$ has a smooth section $\nu : (\mathbb{K}^n, a) \to f^*T^*(\mathbb{K}^m, b) \setminus$ (zero-section).

(iii) There exists a smooth map-germ $\nu = (\nu_1, \nu_2, \dots, \nu_m) : (\mathbb{K}^n, a) \to \mathbb{K}^m \setminus \{0\}$ such that $\nu_1\, df_1 + \nu_2\, df_2 + \cdots + \nu_m\, df_m = 0$.

In the condition (ii), $df : T_t\mathbb{K}^n \to T_{f(t)}\mathbb{K}^m$ is the differential of f at t, $f^*T^*(\mathbb{K}^m, b) := \{(t, \alpha) \mid t \in (\mathbb{K}^n, a), \alpha \in T^*_{f(t)}\mathbb{K}^m$ is the pull-back bundle of $T^*(\mathbb{K}^m, b)$ by f and the dual $(df_t)^*$ of df is defined by $(df)^*(x, \alpha) := (x, \alpha \circ df)$.

Proof of Proposition 1.3.8. The condition (iii) is equivalent to that

$$\begin{cases} v_1 \dfrac{\partial f_1}{\partial t_1} + v_2 \dfrac{\partial f_2}{\partial t_1} + \cdots + v_m \dfrac{\partial f_m}{\partial t_1} = 0, \\ v_1 \dfrac{\partial f_1}{\partial t_2} + v_2 \dfrac{\partial f_2}{\partial t_2} + \cdots + v_m \dfrac{\partial f_m}{\partial t_2} = 0, \\ \cdots\cdots\cdots \\ v_1 \dfrac{\partial f_1}{\partial t_n} + v_2 \dfrac{\partial f_2}{\partial t_n} + \cdots + v_m \dfrac{\partial f_m}{\partial t_n} = 0, \end{cases}$$

or, to that

$$(v_1, v_2, \ldots, v_m) \begin{pmatrix} \frac{\partial f_1}{\partial t_1} & \frac{\partial f_1}{\partial t_2} & \cdots & \frac{\partial f_1}{\partial t_n} \\ \frac{\partial f_2}{\partial t_1} & \frac{\partial f_2}{\partial t_2} & \cdots & \frac{\partial f_2}{\partial t_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial t_1} & \frac{\partial f_m}{\partial t_2} & \cdots & \frac{\partial f_m}{\partial t_n} \end{pmatrix} = (0, \ldots, 0),$$

for some $v = (v_1, \ldots, v_m) : (\mathbb{K}^n, a) \to \mathbb{K}^m \setminus \{0\}$. Thus we see (ii) and (iii) are equivalent.

The condition (iii) is a rewrite of the condition (i), $\widetilde{f}$ being $(f, [v])$. $\square$

Now we recall the notion of contact structures in a generalised sense.

Definition 1.3.9 (Generalised contact distribution [121, 122, 123, 124]). Let M be a manifold of dimension m. Suppose $n \leq m$. Let $W = \mathrm{Gr}(n, TM)$ be the Grassmann bundle with the projection $\pi : W = \mathrm{Gr}(n, TM) \to M$. The fibre of π is given by the Grassmannian $\mathrm{Gr}(n, \mathbb{K}^m)$ consisting of $\mathbb{K}$-linear subspaces of dimension n in $\mathbb{K}^m$. Note that $\dim_{\mathbb{K}} \mathrm{Gr}(n, \mathbb{K}^m) = n(m-n)$ and $\dim(W) = n(m-n) + m$. The **contact distribution** $D \subset TW$ is defined by, for any $(x, V) \in W, V \subset T_x M,\ \dim(V) = n$,

$$D_{(x,V)} := (d\pi)^{-1}(V) = \{v \in T_{(x,V)}W \mid (d\pi)(v) \in V\} \subset T_{(x,V)}W.$$

Note that D is a subbundle of TW of corank $m - n$ and of rank $n(m - n + 1)$.

To simplify the story, consider the case $M = \mathbb{K}^m$. Decompose $\mathbb{K}^m = \mathbb{K}^n \times \mathbb{K}^{m-n}$. Let Ω be the open subset of $\mathrm{Gr}(n, T\mathbb{K}^m)$ consisting of tangential n-planes V such that the projection of V to the former component $\mathbb{K}^n$ in $\mathbb{K}^m = \mathbb{K}^n \times \mathbb{K}^{m-n}$ is a $\mathbb{K}$-isomorphism. Then $V \in \Omega$ is regarded as the graph of a $\mathbb{K}$-linear map $T_x\mathbb{K}^n \to T_y\mathbb{K}^{m-n}$ for some $(x, y) \in \mathbb{K}^n \times \mathbb{K}^{m-n} = \mathbb{K}^m$, and Ω is identified with the 1-jet space $J^1(\mathbb{K}^n, \mathbb{K}^{m-n}) = \mathbb{K}^n \times \mathbb{K}^{m-n} \times \mathbb{K}^{n(m-n)}$. Taking the system of coordinates on $\Omega \subset \mathrm{Gr}(n, T\mathbb{K}^m)$, $x_1, \ldots, x_n, y_1, \ldots, y_{m-n},\ p_{ij}$ $(1 \leq i \leq m-n, 1 \leq j \leq n)$, we set

$$\alpha_i := dy_i - \textstyle\sum_{j=1}^n p_{ij} dx_j, \quad (1 \leq i \leq m - n).$$

Then the contact distribution is given by the Pfaff system

$$D: \ \alpha_1 = 0,\ \alpha_2 = 0,\ \ldots,\ \alpha_{m-n} = 0.$$

Lemma 1.3.10 *Let $f : N \to M$ be a frontal map. Then for any $a \in N$, there exists an open neighbourhood U of $a \in N$ and smooth vector fields $X_1, \dots X_n : U \to TM$ along $f|_U$ such that they are linearly independent point-wise on U and $df(T_tN) \subset \langle X_1(t), \dots, X_n(t)\rangle_{\mathbb{K}}$ for any $t \in U$.*

Proof Let $a \in N$ and $\widetilde{f} : U \to \mathrm{Gr}(n, TM)$ be an integral map covering f (Legendre lift) on a neighbourhood U of a. We write $V_t := \widetilde{f}(t) \subset T_{f(t)}M$. After a linear change of coordinates of M, we may suppose V_a is mapped isomorphically to $\mathbb{K}^n$ by the projection to the first n-components. Then so is V_t for any $t \in U$, after shrinking U if necessary. Then there exists a frame $(X_1, \dots X_n)$ of V_t of the form

$$X_1 = \frac{\partial}{\partial x_1} + \sum_{j=2}^{m} a_{j1}(t)\frac{\partial}{\partial x_j},\ X_2 = \frac{\partial}{\partial x_2} + \sum_{j=3}^{m} a_{j1}(t)\frac{\partial}{\partial x_j},\ \dots,\ X_n = \frac{\partial}{\partial x_n} + \sum_{j=n+1}^{m} a_{jn}(t)\frac{\partial}{\partial x_j}.$$

where $a_{ji}(t)$ are smooth function on U. □

Example 1.3.11 (1) Any *immersion* is a frontal. In that case, the Legendre lift is given by $\widetilde{f}(t) := T_t f(T_t\mathbb{K}^n)$, the image of the differential map $T_t f : T_t\mathbb{K}^n \to T_{f(t)}\mathbb{K}^m$.
(2) When $n = m$, namely when the dimension of source equals to that of target, any map-germ $(\mathbb{K}^n, a) \to (\mathbb{K}^n, b)$ is a frontal. In fact the Legendre lift is given by $\widetilde{f}(t) := T_{f(t)}\mathbb{K}^n$.
(3) Any *constant map-germ* is a frontal. In fact we can take any lift $\widetilde{f}$ of f.
(4) Any *wavefront* $(\mathbb{K}^n, a) \to (\mathbb{K}^{n+1}, b)$, that is a Legendre projection of a Legendre submanifold in $\mathrm{Gr}(n, T\mathbb{K}^{n+1}) = PT^*\mathbb{K}^{n+1}$, is a frontal. The inclusion of the Legendre submanifold is regarded as the Legendre lift $\widetilde{f}$.

Example 1.3.12 (Singularities of tangent surfaces to curves). Let $\gamma : (\mathbb{K}, a) \to \mathbb{K}^m$ be a smooth curve-germ in the affine space $\mathbb{K}^m$. The curve-germ γ is called of **type** $\mathbb{L} = (\ell_1, \ell_2, \ell_3, \dots,), (1 \le \ell_1 < \ell_2 < \ell_3 < \cdots)$, if

$$\gamma(t) = (t^{\ell_1},\ t^{\ell_2} + \cdots,\ t^{\ell_3} + \cdots,\ \dots)$$

for a system of affine coordinates of $\mathbb{K}^m$ centred at $\gamma(a)$ and a smooth coordinate t of $(\mathbb{K}.a)$ centred at a. Then the velocity vector of γ is given by

$$\gamma'(t) = (\ell_1 t^{\ell_1 - 1},\ \ell_2 t^{\ell_2 - 1} + \cdots,\ \ell_3 t^{\ell_3 - 1} + \cdots,\ \cdots),$$

and the tangent line to γ at t is given by $\gamma(t) + s\frac{1}{\ell_1 t^{\ell_1 - 1}}\gamma'(t)$ using an affine parameter s. Note that, if $\ell_1 \ge 2$, then $\gamma'(t)$ vanishes at $t = 0$, i.e. at a. Therefore we divide by $\ell_1 t^{\ell_1 - 1}$ to get a limiting direction vector $\frac{1}{\ell_1 t^{\ell_1 - 1}}\gamma'(t)$.

The **tangent surface** $\mathrm{Tan}(\gamma) : (\mathbb{K}, a) \times \mathbb{K} \to \mathbb{K}^m$ of γ is defined as the ruled surface generated by tangent lines along the curve, i.e.

$$\mathrm{Tan}(\gamma)(t, s) := \gamma(t) + s\frac{1}{\ell_1 t^{\ell_1 - 1}}\gamma'(t).$$

The parametrisation of tangent surface depends on the choice of coordinate t, direction vector and affine coordinates of $\mathbb{K}^m$. The tangent surface turns to be a frontal.

In fact a Legendre lift $\widetilde{\mathrm{Tan}(\gamma)}$ is given by the planes spanned by

$$X_1(t) = \frac{1}{\ell_1 t^{\ell_1 - 1}}\gamma'(t), \quad X_2(t) = \frac{\ell_1}{\ell_2(\ell_2 - \ell_1)t^{\ell_2-\ell_1-1}}X_1'(t),$$

at $\mathrm{Tan}(\gamma)(t, s)$.

Then it is known that the singularity of $\mathrm{Tan}(\gamma)$ is *uniquely determined* by the type $\mathbb{L}$ and called cuspidal edge (*CE*) if $\mathbb{L} = (1, 2, 3, \dots)$, folded umbrella (*FU*) or cuspidal cross cap (*CCC*) if $(1, 2, 4)$, swallowtail (*SW*) if $(2, 3, 4)$, Mond (*MD*) or cuspidal beaks (*CB*) if $(1, 3, 4)$, Shcherbak (*SB*) if $(1, 3, 5)$, cuspidal swallowtail (*CS*) if $(3, 4, 5)$, open folded umbrella (*OFU*) if $(1, 2, 4, 5, \dots)$, open swallowtail (*OSW*) if $(2, 3, 4, 5, \dots)$, open Mond (*OMD*) or open cuspidal beaks (*OCB*) if $(1, 3, 4, 5, \dots)$ (see [56]).

Note that the classification in the paper [56] is performed over the real, i.e. in C^∞ case. However the same proofs work over the complex analytic case.

Remark 1.3.13 In the real case, there is an equivalent definition to the above definition of frontals: $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ is a *frontal* if and only if there exists a system of smooth *orthonormal* vector fields $\nu_1, \dots, \nu_{m-n} : (\mathbb{R}^n, a) \to T\mathbb{R}^m$ along f such that

$$\nu_i(t) \cdot f_*(T_t\mathbb{R}^n) = 0, \quad (1 \le i \le m - n),$$

for any $t \in (\mathbb{R}^n, a)$ (for some, or equivalently, for any Riemannian metric on $(\mathbb{R}^m, b)$). In other words, f is *frontal* if and only if, there exists a system of smooth vector fields $V_1, \dots, V_n : (\mathbb{R}^n, a) \to T\mathbb{R}^m$ along f, which are linearly independent point-wise, and satisfy that, any $t \in (\mathbb{R}^n, a)$,

$$f_*(T_t\mathbb{R}^n) \subset \langle V_1(t), \dots, V_n(t)\rangle_{\mathbb{R}} \ (\subset T_{f(t)}\mathbb{R}^m).$$

1.3.3 Stability of general frontals

Ler $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be a frontal.

Definition 1.3.14 A deformation $F : (\mathbb{K}^n \times \mathbb{K}^r, A \times \{0\}) \to (\mathbb{K}^m, b)$ of f, $F(t, u) = f_u(t)$, $f_0 = f$, is called a **frontal deformation**, similarly to the case $m = n + 1$, if there exists a lift $\widetilde{F} : (\mathbb{K}^n \times \mathbb{K}^r, A \times \{0\}) \to \mathrm{Gr}(n, T\mathbb{K}^m)$,

$$\begin{tikzcd} & \mathrm{Gr}(n, T\mathbb{K}^m) \arrow[d, "\pi"] \\ (\mathbb{K}^n \times \mathbb{K}^r, A \times \{0\}) \arrow[ur, "\widetilde{F}"] \arrow[r, "F"'] & \mathbb{K}^m, \end{tikzcd}$$

satisfying $(f_u)_*(T_t\mathbb{K}^n) \subset \widetilde{f_u}(t)$, for any $(t, u) \in \mathbb{K}^n \times \mathbb{K}^r$ nearby $A \times \{0\}$. Here $\widetilde{F}(t, u) = \widetilde{f_u}(t)$.

A germ of frontal $f:(\mathbb{K}^n,A)\to(\mathbb{K}^m,b)$ is called **frontally stable** if any frontal deformation $F:(\mathbb{K}^n\times\mathbb{K}^r,A\times\{0\})\to(\mathbb{R}^m,b)$ of f is trivialised: i.e. there exist diffeomorphisms

$$\Sigma:(\mathbb{K}^n\times\mathbb{K}^r,A\times\{0\})\to(\mathbb{K}^n\times\mathbb{K}^r,A\times\{0\}),\Sigma(t,u)=(\sigma_u(t),u),$$

$$T:(\mathbb{K}^m\times\mathbb{K}^r,(b,0))\to(\mathbb{K}^m\times\mathbb{K}^r,(b,0)),T(x,u)=(\tau_u(x),u),$$

such that $(T\circ(F,\mathrm{id}_{\mathbb{R}^r})\circ\Sigma)(t,u)=(f(t),u)$, or equivalently, $\tau_u\circ f_u\circ\sigma_u=f$.

The characterisation of frontal stability of frontal curves is given directly as follows.

Proposition 1.3.15 *Let $f:(\mathbb{K},a)\to(\mathbb{K}^m,b), m\geq 2$ be a frontal curve-germ. Then f is frontally stable if and only if f is right-left equivalent to $t\mapsto(t,0)$* (immersion) *or $t\mapsto(t^2,t^3,0,\dots,0)$* (embedded cusp).

Proof Suppose that f is frontally stable. It is clear that f is immersion, then f is frontally stable. Suppose f is not an immersion. Up to right-left equivalence, we may suppose $f:(\mathbb{K},0)\to(\mathbb{K}^m,0)$ and $2\leq\mathrm{ord}_0 f_1<\mathrm{ord}_0 f_i (i=2,\dots,m)$. Since f is a frontal curve, there exists $p_i:(\mathbb{K},0)\to(\mathbb{K},0)$ such that $f_i'(t)=p(t)f_1'(t)$. Consider any deformation $F_1(t,u)$ of $f_1(t)$ and $P_i(t,u)$ of $p_i(t)$ with $F_1(0,u)=0, P_i(0,u)=0, u\in(\mathbb{K},0)$. Then we have a frontal deformation $F(t,u)$ of $f(t)$ which is defined by

$$F_i(t,u):=\int_0^t P(s,u)\frac{\partial F_1}{\partial s}(s,u),(i=2,\dots,m).$$

In particular take F_1 and P_i such that, for any (t,u) sufficiently near $(0,0)$ and for fixed $u\neq 0$, there exists a critical point t_0 of $F_1(t,u)$ and moreover any critical point t_0 of $F_1(t,u)$, $\frac{dF_1}{dt}(t_0)=0$ is non-degenerate $\frac{d^2F_1}{dt^2}(t_0)\neq 0$. Moreover take P_i such that

$$(F_1(t,u),P_2(t,u),\dots,P_m(t,u))$$

is an immersion. The existence of such a deformation is guaranteed by Thom's transversality theorem for $\mathbb{K}=\mathbb{R}$ and by Bertini's type theorem for $\mathbb{K}=\mathbb{C}$, for instance. Then, for any critical point t_0 of $F_1(t,u)$, there exist a coordinate t centred at t_0 and a system of coordinates $(x_1,\dots,x_m)$ centred at $F(0,u)$ such that $F_1(t,u)=t^2$ and, for some $i, 2\leq i\leq m$, $F_i(t,\lambda)$ has a Taylor expansion at $t=0$ like $F_i(t,u)=a_0t^2+a_1t^3+$ higher order terms, with $a_0,a_1\in\mathbb{K}, a_1\neq 0$. Then it can be proved that $F(\cdot,u)$ is left equivalent to $t\mapsto(t^2,t^3,0,\dots,0)$. In the case $\mathbb{K}=\mathbb{R}$, we use Malgrange-Mather preparation theorem if necessary ([17, 92]). Thus we have that the original f must be right-left equivalent to $t\mapsto(t^2,t^3,0,\dots,0)$ by the frontal stability condition, provided f is not immersive. The proof that the embedded cusp germ is actually frontally stable will be given by the above argument. We omit here the details, which is left to the readers. (See also [55]). □

We give an formulation on infinitesimal frontal stability of frontal-germs in general case.

Here we recall some materials on natural liftings of differential forms to tangent bundle for the basic on Legendre stability or frontal stability.

Let $W = \mathrm{Gr}(n, T\mathbb{K}^m)$ be the Grassmann bundle with the projection $\pi : W = \mathrm{Gr}(n, T\mathbb{K}^m) \to \mathbb{K}^m$. Then we define the **natural lifting** $\widetilde{D} \subset T(TW)$ of the contact distribution $D \subset TW$ locally by

$$\widetilde{D} : \ \widetilde{\alpha}_1 = 0, \ \widetilde{\alpha}_2 = 0, \ \dots, \ \widetilde{\alpha}_{m-n} = 0.$$

using the natural lifting of the Pfaff system defining D,

$$D : \ \alpha_1 = 0, \ \alpha_2 = 0, \ \dots, \ \alpha_{m-n} = 0.$$

See Proposition 1.2.24.

For the system of local coordinates

$$x_1, \dots, x_n, y_1, \dots, y_{m-n}, \ p_{ij}; \ \dot{x}_1, \dots, \dot{x}_n, \dot{y}_1, \dots, \dot{y}_{m-n}, \ \dot{p}_{ij}$$

on the tangent bundle TW of $W = \mathrm{Gr}(n, T\mathbb{K}^m)$, we have

$$\widetilde{\alpha}_i = d\left(y_i - \sum_{j=1}^{n} p_{ij}\dot{x}_j \right) + \sum_{j=1}^{n} \dot{x}_j dp_{ij} - \sum_{j=1}^{n} \dot{p}_{ij} dx_j.$$

We explain infinitesimal integral and frontal deformations in general case.

Definition 1.3.16 A vector field $\widetilde{v} : (\mathbb{K}^n, A) \to TW = T\mathrm{Gr}(n, T\mathbb{K}^m)$ along a map-germ $\ell = \widetilde{f} : (\mathbb{K}^n, A) \to W = \mathrm{Gr}(n, T\mathbb{K}^m)$ is called **integral** if $\widetilde{v}$ is $\widetilde{D}$-integral, i.e., $\widetilde{v}_*(T_t\mathbb{K}^n) \subset \widetilde{D}_{\widetilde{v}(t)}$.

Let $\ell : (N, A) \to W$ be a map-germ to W with the (generalised) contact structure $D \subset TW$. Let D be defined by $\alpha_1 = 0, \dots, \alpha_{m-n} = 0$ locally around $f(A)$. Then we set, as similarly as in the case $m = n + 1$,

$$\begin{aligned} VI_\ell &:= \{\widetilde{v} : (N, A) \to TW \mid \widetilde{v} \text{ is a } \widetilde{D}\text{-integral vector field along } \ell\} \\ &= \{\widetilde{v} : (N, A) \to TW \mid \widetilde{v}^*\widetilde{\alpha}_i = 0, \ i \leq i \leq m-n, \ \pi \circ \widetilde{v} = \ell\}, \end{aligned}$$

where $\pi : TW \to W$ is the natural bundle projection and $\widetilde{\alpha}_i$ is the natural lifting of α_i. We call VI_ℓ the *space of infinitesimal integral deformations* of ℓ.

A vector field $v : (\mathbb{K}^n, A) \to T\mathbb{K}^m$ along a frontal map-germ $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ is called an **infinitesimal frontal deformation** of f if v lifts to an infinitesimal integral deformation $\widetilde{v}$ of a Legendre lift $\widetilde{f}$ of f, i.e. $\widetilde{v} \in VI_{\widetilde{f}}$. We denote by V_f^{fr} the space of infinitesimal frontal deformations of f.

Here we present the natural generalisation on Mather's type characterisation of the frontal stability in the general case.

Theorem 1.3.17 (Infinitesimal characterisation of frontal stability II). Let $f : N = (\mathbb{K}^n, A) \to M = (\mathbb{K}^m, b)$, $(m \geq n+2)$ be a frontal map-germ of corank ≤ 1. Then the following conditions are equivalent to each other:

(i) f is frontally stable.

(ii) $VI_{\widetilde{f}} = t\widetilde{f}(V_N) + w\widetilde{f}(VL_{W,\widetilde{f}(b)}$, for any Legendre lift $\widetilde{f} : N \to W = \mathrm{Gr}(n, T\mathbb{K}^m)$ of f.

(iii) f is a proper frontal and $VI_{\widetilde{f}} = t\widetilde{f}(V_N) + w\widetilde{f}(VL_{W,\widetilde{f}(b)})$, for unique Legendre lift $\widetilde{f} : N \to W = \mathrm{Gr}(n, T\mathbb{K}^m)$ of f.

(iv) f is a proper frontal and $V_f^{\mathrm{fr}} \subset tf(V_N) + wf(V_M)$.

Here we introduce several notions and give an outline of the proof for Theorem 1.3.17.

Definition 1.3.18 A germ of vector field X over W is a (**generalised**) **contact vector field** if X generates contactomorphisms of (W, D), i.e. $L_X D \subseteq D$. Let $b \in \mathbb{K}^m$ and $\widetilde{b} \in W$ with $\pi(\widetilde{b}) = b$. We denote by $VC_{W,\widetilde{b}}$ the set of germs of contact vector fields over $(W, \widetilde{b})$. A contact vector filed $X \in VC_{W,\widetilde{b}}$ is called a **Legendre vector field** if X is π-**lowerable**, i.e. if there exists a vector field η over $M = (\mathbb{K}^m, b)$ such that $d\pi((X)(w)) = \eta(\pi(w))$ for any $w \in (W, \widetilde{b})$. The set of Legendre vector fields over $(W, \widetilde{b})$ is denoted by $VL_{W,\widetilde{b}}$.

Given a function-germ $h : (\mathbb{K}^m, b) \to \mathbb{K}$, we define Hamilton vector fields $X = X_{i,h}$, $1 \leq i \leq m-n$, over $(\mathbb{K}^m, b)$ by

$$X_{i,h} := h\frac{\partial}{\partial y_i} + \sum_{j=1}^{n}\left(\frac{\partial h}{\partial x_j} + \sum_{k=1}^{m-n}\frac{\partial h}{\partial y_k}p_{ik}\right)\frac{\partial}{\partial p_{ij}}.$$

Then it can be proved that $X_{i,h}$ is a contact vector field and $\langle \alpha_k, X_{i,h}\rangle = \delta_{ki}h$. Compare with the explanation after Definition 1.2.29.

Define the $\mathcal{O}_{\mathbb{K}^m,b}$-module structure of $VC_{W,\widetilde{b}}$ by

$$h * X := (h \circ \pi)X + \sum_{i=1}^{m-n}\langle \alpha_i, X\rangle \left(X_{i,h} - hX_{i,1}\right).$$

Here we present several preliminary results.

Lemma 1.3.19 ([101]) *Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be a proper frontal map-germ with the Legendre lift $\widetilde{f}$. Then the linear map $d\pi : VI_{\widetilde{f}} \to V_f$ defined $d\pi(\widetilde{v}) = d\pi \circ \widetilde{v}$ is injective. If f is of corank ≤ 1, then $d\pi(VI_{\widetilde{f}}) = V_f^{\mathrm{fr}}$. In particular $d\pi$ givens a linear isomorphism between $VI_{\widetilde{f}}$ and V_f^{fr}.*

The proof of Lemma 1.3.19 is similarly established as that of Lemma 1.2.35 supposing f itself is of corank ≤ 1. Also we have

Lemma 1.3.20 *Suppose $f : (\mathbb{R}^n, A) \to (\mathbb{R}^m, b)$ is a frontal of corank ≤ 1. Then, for any $v \in VI_{\widetilde{f}}$, there exist a frontal deformation $F : (\mathbb{K}^n \times \mathbb{K}, A \times 0) \to \mathbb{K}^m$ of f and* *an integral deformation* $\widetilde{F} : (\mathbb{K}^n \times \mathbb{K}, A \times 0) \to W = P(T^*\mathbb{K}^{n+1})$ *of* $\widetilde{f}$ *covering* F *such that* $\frac{d}{du} f_u\big|_{u=0} = v$. □

We have that, by Lemma 1.3.19, f is infinitesimally frontally stable if and only if $\widetilde{f}$ is infinitesimally integrally stable. Moreover it can be proved that f is frontally stable if and only if $\widetilde{f}$ is Legendre stable, and then, by Lemma 1.2.36, f (resp. $\widetilde{f}$) must be infinitesimally frontally stable (resp. infinitesimally frontally stable).

Then the proof of Theorem 1.3.17 is completed by using the following: For an unfolding $\mathcal{F} = (\widetilde{f_u}, u) : N \times J \to W \times J, \ \ u \in J = (\mathbb{K}, 0)$, we set

$$VI_{\mathcal{F}/J} = \{v : N \times J \to TW \ \mid \ v_u \in VI_{\widetilde{f_u}},\, u \in J\}.$$

If $(\widetilde{f_u})$ is an integral deformation of $\widetilde{f}$, then we have $(d\widetilde{f_u}/du)_{u\in J} \in VI_{\mathcal{F}/J}$. We define an $\mathcal{O}_{W\times J}$-module structure on $VI_{\mathcal{F}/J}$ by

$$a_u * v_u = (\widetilde{f_u}^* a_u) \cdot v_u + \sum_{i=1}^{m-n} \langle \alpha_i, v_u \rangle (X_{i,a_u} - a_u \cdot X_{i,1}) \circ \widetilde{f_u},$$

for $v_u \in VI_{\mathcal{F}/J}, a_u \in \mathcal{O}_{W\times J}$, using Hamilton vector fields X_{i,a_u} of a_u. Let $\widetilde{f}$ be finite and of corank at most one. Then it can be proved that the quotient $VI_{\mathcal{F}/J}$ is a finite $\mathcal{O}_{W\times J}$-module. We define $t\mathcal{F}/J : V_N \to VI_{\mathcal{F}/J}$ by $\xi \mapsto (t\widetilde{f_u}(\xi))_{u\in J}$ and $w\mathcal{F}/J : V_W \to VI_{\mathcal{F}/J}$ by $\eta \mapsto (w\widetilde{f_u}(\eta))_{u\in J}$. We set

$$S_{\mathcal{F}/J} = VI_{\mathcal{F}/J}/((w\mathcal{F}/J)(VH_W) + (t\mathcal{F}/J)(V_N)),$$

which is an $\mathcal{O}_{W\times J}$-module, and set

$$S_{\widetilde{f}} = VI_{\widetilde{f}}/(w\widetilde{f}(VL_W) + t\widetilde{f}(V_N)),$$

which is an $\mathcal{O}_W$-module. Then we have:

Lemma 1.3.21 ([55]) *The quotient* $S_{\mathcal{F}/J}/m_J S_{\mathcal{F}/J}$ *is isomorphic to* $S_{\widetilde{f}}$ *as an* $\mathcal{O}_W$-*modules.* □

Proof of Theorem 1.3.17. Let f be a frontal map-germ of corank ≤ 1. First note that, if a frontal f is frontally stable, then f must be a proper frontal, i.e. the singular locus $S(f)$ is nowhere dense. Actually, using notions in the next section, any front of corank ≤ 1 is an opening of a germ $g : (\mathbb{R}^n, A) \to (\mathbb{R}^n, c)$ of corank ≤ 1. Take any deformation of g, say, a stable deformation g_u of g with nowhere dense singular locus. Then (g_u, u) lifts to an opening $(G_u, u) : (\mathbb{R}^n, A) \to \mathbb{R}^m$ with $G_0 = f$. The germ G_u is a proper frontal. and f must be right-left equivalent to the germ of G_u at some multi-point of $\mathbb{R}^n$. Thus f must be a proper frontal.

Then we have the implications (i) ⇒ (iii) ⇒ (ii), (i) ⇒ (iv) and (iii) ⇔ (iv). Therefore it is enough to show that (ii) ⇒ (i).

Now if $\widetilde{f}$ is infinitesimally integrally stable, then $S_{\widetilde{f}} = 0$ and then, by Lemma 1.3.21, we have $m_J S_{\mathcal{F}/J} = S_{\mathcal{F}/J}$, and then, by Nakayama's Lemma, we have $S_{\mathcal{F}/J} = 0$, i.e. $VI_{\mathcal{F}/J} = ((w\mathcal{F}/J)(VH_W) + (t\mathcal{F}/J)(V_N))$. This means that any integral deformation $\widetilde{f_u}$ of $\widetilde{f}$ is recovered by deformations of infinitesimal Legendre equivalences $(\xi_u, \widetilde{\eta}_u)$ such that

$$\frac{d}{du}\widetilde{f_u} = w(\widetilde{f_u})(\widetilde{\eta_u}) - t(\widetilde{f_u})(\xi_u).$$

By integrating on u, we have that $\widetilde{f}$ is Legendre stable and f is frontally stable for 1-parameter deformation. The case of arbitrary number of parameters follows from the triviality of 1-parameter deformations. □

Example 1.3.22 Here are the list of frontally stable frontals in low dimensions.

$n = 1, m \geq 2$. $f : (\mathbb{K}, a) \to (\mathbb{K}^m, b)$ is **frontally stable** if and only if f is right-left equivalent to $t_1 \mapsto (t_1, 0, \dots, 0)$ or $(t_1^2, t_1^3, 0, \dots, 0)$.

$n = 2, m = 4$. If $f : (\mathbb{K}^2, a) \to (\mathbb{K}^4, b)$ is **frontally stable** and of *corank* ≤ 1, then f is right-left equivalent to (regular germ) : $(t_1, t_2) \mapsto (t_1, t_2, 0, 0)$, (cuspidal edge) : $(t_1, t_2) \mapsto (t_1,\ t_2^2,\ 0,\ t_2^3)$, or (open swallowtail) : $(t_1, t_2) \mapsto (t_1,\ t_2^3 + t_1 t_2,\ \frac{3}{4}t_2^4 + \frac{1}{2}t_1 t_2^2,\ \frac{3}{5}t_2^5 + \frac{1}{3}t_1 t_2^3)$.

1.3.4 An algebraic characterisation of frontals

Here we present a simple criterion of frontality in the general case. It suffices to treat the case of mono-germs (cf. Remarks 1.2.8, 1.3.6).

Denote by Γ the set of subsets $I \subseteq \{1, 2, \dots, m\}$ with $\#(I) = n$. For a map-germ $f : (\mathbb{K}^n, a) \to (\mathbb{K}^m, b), n \leq m$ and $I \in \Gamma$, we set $D_I = \det(\partial f_i/\partial t_j)_{i \in I, 1 \leq j \leq n}$. Then **Jacobi ideal** J_f of f is defined as the ideal generated in $\mathcal{O}_a = \mathcal{O}_{\mathbb{K}^n, a}$ by all n-minor determinants D_I $(I \in \Gamma)$ of Jacobi matrix J_f of f. Then we have:

Proposition 1.3.23 (Criterion of frontality [61]). *Let $f : (\mathbb{K}^n, a) \to (\mathbb{K}^m, b)$ be a map-germ. If f is a general frontal, then the Jacobi ideal J_f of f is principal. In fact J_f is generated by D_I for some $I \in \Gamma$. Conversely, if J_f is principal and the singular locus*

$$S(f) = \{t \in (\mathbb{K}^n, a) \mid \operatorname{rank}(T_t f : T_t\mathbb{K}^n \to T_{f(t)}\mathbb{K}^m) < n\}$$

of f is nowhere dense in $(\mathbb{K}^n, a)$, then f is a general frontal.

Because of the importance of Proposition 1.3.23, we will repeat the proof here.

Proof of Proposition 1.3.23. Let f be a frontal and $\widetilde{f}$ a Legendre lift of f. Take $I_0 \in \Gamma$ such that $\widetilde{f}(a)$ projects isomorphically by the projection $\mathbb{K}^m \to \mathbb{K}^n$ to the components belonging to I_0. Let $(p_I)_{I \in \Gamma}$ be the Plücker coordinates of $\widetilde{f}$. Then $p_{I_0}(a) \neq 0$. This implies that for any $I \in \Gamma$, there exists $h_I \in \mathcal{O}_a$ such that $D_I = h_I D_{I_0}$. Set $\lambda = D_{I_0}$. Then the Jacobi ideal J_f is generated by λ.

Conversely suppose J_f is generated by one element $\lambda \in \mathcal{O}_a$. Since J_f is generated by λ, we have that there exists $k_I \in \mathcal{O}_a$ for any $I \in \Gamma$ such that $D_I = k_I \lambda$. Since $\lambda \in J_f$, there exists $\ell_I \in \mathcal{O}_a$ for any $I \in \Gamma$ such that $\lambda = \sum_{I \in \Gamma} \ell_I D_I$. Therefore

$(1 - \sum_{I\in\Gamma} \ell_I k_I)\lambda = 0$. Suppose $(\ell_I k_I)(a) = 0$ for any $I \in \Gamma$. Then $1 - \sum_{I\in\Gamma} \ell_I k_I$ is a unit and therefore $\lambda = 0$. Thus we have $J_f = 0$. This contradicts to the assumption that $S(f)$ is nowhere dense. Hence there exists $I_0 \in \Gamma$ such that $(\ell_{I_0} k_{I_0})(a) \neq 0$. Then $k_{I_0}(a) \neq 0$. Therefore J_f is generated by D_{I_0}. Hence $D_I = h_I D_{I_0}$ for any $I \in \Gamma$ with $h_{I_0}(a) = 1$. Then the Legendre lift $\widetilde{f}$ on $\mathbb{K}^n \setminus S(f)$ extends to $(\mathbb{K}^n, a)$, which is given by Plücker coordinates $(h_I)_{I\in\Gamma}$. □

Similarly we have

Proposition 1.3.24 *Let* $F : (\mathbb{K}^n \times \mathbb{K}^r, (a, 0)) \to (\mathbb{K}^m, b), F(t, u) = (f_u(t), u)$ *be a deformation of a mono-germ of mapping. Then* F *is a frontal deformation of a frontal map-germ if and only if the Jacobi ideal* J *generated by* n*-minor determinants of Jacobi matrix* $J_t F = (\partial(f_u)_i/\partial t_j)$ *of* F *on* $t = (t_1, \dots, t_n)$ *is a principal (= with a single generator) ideal* $J = \Lambda\, \mathcal{O}_{t,u}$ *in* $\mathcal{O}_{t,u}$ *for some* $\Lambda \in \mathcal{O}_{t,u}$.

The function $\Lambda : (\mathbb{K}^n \times \mathbb{K}^r, (a, 0)) \to \mathbb{K}$ is uniquely determined up to $\mathcal{PK}$-*equivalence* ([77]). We call $\Lambda = \Lambda_F$ *Jacobian* of the frontal deformation F.

1.4 Openings

1.4.1 Jacobi modules, ramification modules and openings

Now, as the key constructions to study frontals generally, we would like to introduce the notion of **openings** of multi-germs of mappings. An origin of the notion of openings can be found in author's earlier work [48]. By this notion of openings we can treat arbitrarily codimensional "frontals" uniformly. To do this, first we recall auxiliary notions according to the paper [57]. We start with an example.

Example 1.4.1 There is a sequence of well-known singularities of map-germs: The **Whitney cusp** $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$, $f(t, u) = (t^3 + ut,\ u)$, the **swallowtail** $F : (\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$, $F(t, u) = (f(t, u),\ t^4 + \frac{2}{3}ut^2)$, and the **open swallowtail** $\widetilde{F} : (\mathbb{K}^2, 0) \to (\mathbb{K}^4, 0)$, $\widetilde{F}(t, u) = (F(t, u),\ t^5 + \frac{4}{9}ut^3)$.

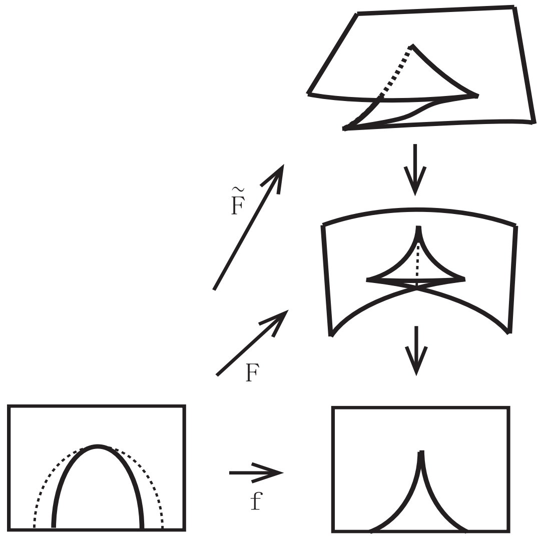
They have the same singular locus and the same kernel field of the differential along the singular locus, while the self-intersections are resolved.

We are then led to ask the natural question: What is the common algebraic structure behind these singularities?

Definition 1.4.2 Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be a multi-germ of a smooth map with $n \leq m$. We define the **Jacobi module** $\mathcal{J}_f$ of f by

$$\mathcal{J}_f = \{ \sum_{j=1}^{m} p_j\, df_j \ \mid\ p_j \in \mathcal{O}_{\mathbb{K}^n, A}\ (1 \leq j \leq m) \} \subset \Omega^1_{\mathbb{K}^n, A}$$

in the space $\Omega^1_{\mathbb{K}^n, A}$ of differential 1-form-germs on $(\mathbb{K}^n, A)$.



As for the notations, we distinguish $\mathcal{J}_f$ from the Jacobi matrix Jf and the Jacobi ideal J_f which appeared in previous sections §1.2 and §1.3.

Note that $\mathcal{J}_f$ is the first order component of the graded differential ideal $\mathcal{J}_f^\bullet$ in $\Omega^\bullet_{\mathbb{K}^n,A}$ generated by $df_1, \dots, df_m$. Then the singular locus, the non-immersive locus, of f is given by

$$S(f) = \{t \in (\mathbb{K}^n, A) \mid \operatorname{rank} \mathcal{J}_f(t) < n\}.$$

Also we consider the **kernel field** $\operatorname{Ker}(f_* : T\mathbb{K}^n \to T\mathbb{K}^m)$ of the differential of f, along Σ_f. Note that $\operatorname{Ker}(f_*)$ is a subset-germ of $(T\mathbb{K}^n, T\mathbb{K}^n|_A)$.

For another map-germ $f' : (\mathbb{K}^n, A) \to (\mathbb{K}^{m'}, b')$, $n \leq m'$, if $\mathcal{J}_{f'} = \mathcal{J}_f$, then $S(f') = S(f)$ and $\operatorname{Ker}(f'_*) = \operatorname{Ker}(f_*)$.

Definition 1.4.3 The **ramification module** $\mathcal{R}_f$ of f is defined by $\mathcal{R}_f = \{h \in \mathcal{O}_{\mathbb{K}^n,A} \mid dh \in \mathcal{J}_f\}$ (cf. ([49, 52, 61]).

It is shown that $\mathcal{R}_f$ is an $\mathcal{O}_{\mathbb{K}^m,b}$-module. Moreover $\mathcal{R}_f$ is a C^∞ or an analytic ring in the sense of if $g_1, \dots, g_k \in \mathcal{R}_f$ and $h \in \mathcal{O}_{\mathbb{K}^k,g(A)}$ then $h(g_1, \dots, g_k) \in \mathcal{R}_f$, where $g = (g_1, \dots, g_k)$.

Remark 1.4.4 In the case $n > m$, the module $\mathcal{R}_f$ was studied by Moussu, Tougeron (see [99]), related to the characterisation of composite differentiable functions and structure of relative de Rham cohomologies of map-germs. In the case $n \leq m$, the module $\mathcal{R}_f$ appears in singularity theory in symplectic and contact geometries. Also we remark that a closed related ring $R_0 f$ was introduced and studied already by David Mond in [97] from a different motivation.

Lemma 1.4.5 *For map-germs $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$, $f' : (\mathbb{K}^n, A) \to (\mathbb{K}^{m'}, b')$, we have that $\mathcal{J}_{f'} = \mathcal{J}_f$ if and only if $\mathcal{R}_{f'} = \mathcal{R}_f$.*

Proof It is clear that $\mathcal{J}_{f'} = \mathcal{J}_f$ implies $\mathcal{R}_{f'} = \mathcal{R}_f$. Conversely suppose $\mathcal{R}_{f'} = \mathcal{R}_f$. Then any component f'_j of f' belongs to $\mathcal{R}_{f'} = \mathcal{R}_f$, hence $df'_j \in \mathcal{J}_f$. Therefore $\mathcal{J}_{f'} \subseteq \mathcal{J}_f$. By the symmetry we have $\mathcal{J}_{f'} = \mathcal{J}_f$. □

We mention several properties on the introduced notions.

Lemma 1.4.6 *Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be a map-germ. Then we have*
(1) $f^*\mathcal{O}_{\mathbb{K}^m,b} \subseteq \mathcal{R}_f \subseteq \mathcal{O}_{\mathbb{K}^n,A}$.
(2) *$\mathcal{R}_f$ is an $\mathcal{O}_{\mathbb{K}^m,b}$-submodule via $f^* : \mathcal{O}_{\mathbb{K}^m,b} \to \mathcal{O}_{\mathbb{K}^n,A}$ of $\mathcal{O}_{\mathbb{K}^n,A}$.*
(3) *If $\tau : (\mathbb{K}^m, b) \to (\mathbb{K}^m, b')$ is a diffeomorphism-germ, then $\mathcal{R}_{\tau\circ f} = \mathcal{R}_f$. If $\sigma : (\mathbb{K}^n, A') \to (\mathbb{K}^n, A)$ is a diffeomorphism-germ, then $\mathcal{R}_{f\circ\sigma} = \sigma^*(\mathcal{R}_f)$.*
(4) *$\mathcal{R}_f$ is a C^∞-subring of $\mathcal{O}_{\mathbb{K}^n,A}$ in the case $\mathbb{K} = \mathbb{R}$.*

For the notion of C^∞-rings, see [48] for instance.

Proof of Lemma 1.4.6.
The assertions (1) and (2) follow from that, if $h \in \mathcal{R}_f$ and $dh = \sum_{j=1}^m p_j df_j$, then

$$d\{(k \circ f)h\} = \sum_{j=1}^{m} \left\{(k \circ f)p_j + h\left(\partial k/\partial y_j\right)\right\} df_j.$$

The assertion (3) follows from that $\mathcal{J}_{\tau\circ f} = \mathcal{J}_f$ and $\mathcal{J}_{f\circ\sigma} = \sigma^*(\mathcal{J}_f)$. The assertion (4) follows from that, if $h_1, \dots, h_r \in \mathcal{R}_f$ and if $\tau : \mathbb{R}^r \to \mathbb{R}$ is a C^∞ function, then

$$d\{\tau(h_1, \dots, h_r)\} = \sum_{i=1}^{r} \frac{\partial \tau}{\partial y_i}(h_1, \dots, h_r)\, dh_i \in \mathcal{J}_f.$$

□

Lemma 1.4.7 *Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be a map-germ with $A = \{a_1, \dots, a_r\}$. We denote by $f_i : (\mathbb{K}^n, a_i) \to (\mathbb{K}^m, b)$ the restriction of f to $(\mathbb{R}^n, a_i)$. Then $\mathcal{R}_f \cong \prod_{i=1}^r \mathcal{R}_{f_i}$ as $\mathcal{O}_b$-module.*

Proof We have the isomorphism $\varphi : \mathcal{R}_f \to \prod_{i=1}^r \mathcal{R}_{f_i}$ defined by $\varphi(h) = (h|_{(\mathbb{R}^n,a_i)})_{i=1}^r$. □

Definition 1.4.8 Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be a map-germ. Given $h_1, \dots, h_r \in \mathcal{R}_f$, the map-germ $F : (\mathbb{K}^n, A) \to \mathbb{K}^m \times \mathbb{K}^r = \mathbb{K}^{m+r}$ defined by $F = (f_1, \dots, f_m, h_1, \dots, h_r)$ is called an **opening** of f. Then f is called a **closing** of F.

Openings of map-germs appear as typical singularities in several problems of geometry and its applications. They appear also in the classification problem of "tangent varieties" [56][58]. Open swallowtails, open folded umbrellas, etc. appear as tangent varieties. We have applied opening constructions to solve the "stable" classification problem of tangent varieties to generic submanifolds in [58]. Moreover openings are related to singularities of isotropic mappings in symplectic spaces. See §1.5.1.

Example 1.4.9 Consider Whitney cusp map $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ of the form $f(t,u) = (t^3 + ut,\ u)$. For the swallowtail $F(t,u) = (t^3 + ut,\ u,\ t^4 + \frac{2}{3}ut^2)$, we have that

$$d(t^4 + \frac{2}{3}ut^2) = \frac{4}{3}t\, d(t^3 + ut) - \frac{4}{9}t^2\, du \ \in \ \left\langle d(t^3 + ut),\ du \right\rangle_{\mathcal{O}_2}.$$

Therefore $t^4 + \frac{2}{3}ut^2 \in \mathcal{R}_f$ and the swallowtail is an opening of Whitney cusp map.

For the open swallowtail $\widetilde{F}(t,u) = (t^3 + ut,\ u,\ t^4 + \frac{2}{3}ut^2,\ t^5 + \frac{4}{9}ut^3)$, we have that

$$d(t^5 + \frac{4}{9}ut^3) = \frac{5}{3}t^2\, d(t^3 + ut) - \frac{10}{9}t^3\, du \ \in \ \left\langle d(t^3 + ut),\ du \right\rangle_{\mathcal{O}_2}.$$

Therefore the open swallowtail is an opening of the swallowtail and of Whitney cusp map as well.

Note that an opening of an opening of f is an opening of f. We have

Lemma 1.4.10 *For any opening F of f, we have $\mathcal{R}_F = \mathcal{R}_f$, $\mathcal{J}_F = \mathcal{J}_f$, $S(F) = S(f)$ and* $\mathrm{Ker}(F_*) = \mathrm{Ker}(f_*)$. □

Let $g : (\mathbb{K}^n, A) \to (\mathbb{K}^n, c)$ be a map-germ and $f = (g, h_1, \dots, h_r) : (\mathbb{K}^n, A) \to \mathbb{K}^n \times \mathbb{K}^r = \mathbb{K}^m$, $m = n + r$, be an opening of g. Let $F = (f_u) : (\mathbb{K}^n \times \mathbb{K}^k, A \times 0) \to \mathbb{K}^m$, $F(t,u) = (G(t,u), H_1(t,u), \dots, H_r(t,u))$ be a deformation of f. Then we have

Lemma 1.4.11 *The deformation F is a frontal deformation of f if and only if $H_1, \dots, H_r \in \mathcal{R}_{(G,u)}$.* □

1.4.2 Versal openings

Definition 1.4.12 An opening $F = (f, h_1, \dots, h_r)$ of f is called a **versal opening** (resp. a **mini-versal opening**) of $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$, if $1, h_1, \dots, h_r$ form a system (resp. a minimal system) of generators of $\mathcal{R}_f$ as an $\mathcal{O}_{\mathbb{K}^m,b}$-module via $f^* : \mathcal{O}_{\mathbb{K}^m,b} \to \mathcal{O}_{\mathbb{K}^n,A}$.

Note that a versal opening of an opening of f is a versal opening of f. An opening of a versal opening of f is a versal opening of f.

A mini-versal opening $F : (\mathbb{K}^n, A) \to \mathbb{K}^{m+r}$ of f is unique up to left-equivalence and a versal opening $G : (\mathbb{K}^n, A) \to \mathbb{K}^{m+s}$ of f is left-equivalent to a mini-versal opening composed with an immersion $(\mathbb{K}^n, A) \to \mathbb{K}^{m+r} \hookrightarrow \mathbb{R}^{m+s}$ (Corollary 1.4.18).

A map-germ $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ is called **finite** if $\mathcal{O}_{\mathbb{K}^n,A}$ is a finite $\mathcal{O}_{\mathbb{K}^m,b}$-module via f^*. The condition is equivalent to that $\dim_{\mathbb{K}} \mathcal{O}_{\mathbb{K}^n,A}/(f^*\mathfrak{m}_{\mathbb{K}^m,b})\mathcal{O}_{\mathbb{K}^n,A} < \infty$ by Nakayama's lemma (see for example [17]). Moreover f is finite if and only if f is $\mathcal{K}$-finite ([118]).

Proposition 1.4.13 *Suppose $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ is finite. In the C^∞ case, $\mathbb{K} = \mathbb{R}$, we assume f is of corank at most one. Then we have*

(1) *$\mathcal{R}_f$ is a finite $\mathcal{O}_{\mathbb{K}^m,b}$-module. Therefore there exists a versal opening of f.*
(2) *$1, h_1, \dots, h_r \in \mathcal{R}_f$ generate $\mathcal{R}_f$ as $\mathcal{O}_{\mathbb{K}^m,b}$-module if and only if they generate the vector space $\mathcal{R}_f/(f^*\mathfrak{m}_{\mathbb{K}^m,b})\mathcal{R}_f$ over $\mathbb{K}$.*

Proof (1) In the complex case, since $\mathcal{O}_{\mathbb{C}^n,A}$ is a finite $\mathcal{O}_{\mathbb{K}^m,b}$-module, $\mathcal{R}_f$ is an $\mathcal{O}_{\mathbb{K}^m,b}$-submodule of $\mathcal{O}_{\mathbb{C}^n,A}$ and $\mathcal{O}_{\mathbb{K}^m,b}$ is Noetherian, we have that $\mathcal{R}_f$ is a finite $\mathcal{O}_{\mathbb{K}^m,b}$-module.

In the real case of mono-germs, the assertion is proved in Theorem 1.3 of [50] and Corollary 2.4 of [52]. For the case of multi-germs, the assertions are reduced to the case that A consists of a point.

(2) follows from Nakayama's lemma. □

Example 1.4.14 Let us consider the following five map-germs: $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$, $g : (\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$, $h : (\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$, $k : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$, $\ell : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ defined by

$$f(t,s) = (t,\ s^2), \quad g(t,s) = (t,\ ts,\ s^2), \quad h(t,s) = (t^2,\ ts,\ s^2),$$
$$k(t,s) = (t^2,\ s^2), \quad \ell(t,s) = (t^2 - s^2,\ ts).$$

Then we have

$$\mathcal{R}_k \subsetneq \mathcal{R}_h \subsetneq \mathcal{R}_g, \quad \mathcal{R}_\ell \subsetneq \mathcal{R}_h, \quad \mathcal{R}_f \subsetneq \mathcal{R}_g.$$

In fact

$$\mathcal{J}_f = \langle dt, sds\rangle_{\mathcal{O}_2}, \quad \mathcal{J}_g = \langle dt, tds, sds\rangle_{\mathcal{O}_2}, \quad \mathcal{J}_h = \langle tdt, tds + sdt, sds\rangle_{\mathcal{O}_2},$$
$$\mathcal{J}_k = \langle tdt, sdt\rangle_{\mathcal{O}_2}, \quad \mathcal{J}_\ell = \langle tdt - sds, sdt + tds\rangle_{\mathcal{O}_2}.$$

Then it can be proved that $\mathcal{R}_f$ is minimally generated by $1, s^3$ over $f^*\mathcal{O}_2$, $\mathcal{R}_g$ is minimally generated by $1, s^3$ over $g^*\mathcal{O}_3$, and $\mathcal{R}_h$ is minimally generated by $1, t^3, t^2s, ts^2, s^3$ over $h^*\mathcal{O}_3$. Moreover it can be proved that $\mathcal{R}_k$ is minimally generated by $1, t^3, s^3, t^3s^3$ over $k^*\mathcal{O}_2$ and that $\mathcal{R}_\ell$ is minimally generated by $1, t^3 - 3ts^2, 3t^2s - s^3, t^2(t^2 + s^2)^2$ over $\ell^*\mathcal{O}_2$.

By Proposition 1.4.13, we have

Corollary 1.4.15 *Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be finite and of corank at most one. Then there exists a versal opening of f.* □

Moreover we have the following:

Corollary 1.4.16 *Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be finite and of corank at most one. Then an opening $F = (f, h_1, \dots, h_r)$ of f is a mini-versal opening of f, namely, $1, h_1, \dots, h_r \in \mathcal{R}_f$ form a minimal system of generators of $\mathcal{R}_f$ as $\mathcal{O}_b$-module if and only if they form a basis of $\mathbb{K}$-vector space $\mathcal{R}_f/(f^*\mathfrak{m}_b)\mathcal{R}_f$.*

The following is useful for the classification problem of map-germs in a geometric context ([56][58]).

Proposition 1.4.17 *Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$, $n \leq m$ be a smooth map-germ.*

(1) *For any versal opening $F : (\mathbb{K}^n, A) \to (\mathbb{K}^{m+r}, F(A))$ of f and for any opening $G : (\mathbb{K}^n, A) \to (\mathbb{K}^{m+s}, G(A))$, there exists an affine bundle map $\Psi : (\mathbb{K}^{m+r}, F(A)) \to (\mathbb{K}^{m+s}, G(A))$ over $(\mathbb{K}^m, b)$ such that $G = \Psi \circ F$.*

(2) *For any mini-versal openings $F : (\mathbb{K}^n, A) \to (\mathbb{K}^{m+r}, F(A))$ and $F' : (\mathbb{K}^n, A) \to (\mathbb{K}^{m+r}, F'(A))$ of f, there exists an affine bundle isomorphism $\Phi : (\mathbb{K}^{m+r}, F(A)) \to (\mathbb{K}^{m+r}, F'(A))$ over $(\mathbb{K}^m, b)$ such that $F' = \Psi \circ F$. In particular, the diffeomorphism class of mini-versal opening of f is unique.*

(3) *Any versal openings $F'' : (\mathbb{K}^n, A) \to (\mathbb{K}^{m+s}, F''(A))$ of f is diffeomorphic to $(F, 0)$ for a mini-versal opening F of f.* $\square$

Two map-germs $F : (\mathbb{K}^n, A) \to (\mathbb{K}^p, B)$ and $G : (\mathbb{K}^n, A) \to (\mathbb{K}^q, C)$ is called $\mathcal{L}$-equivalent, or, left-equivalent, if there exists a diffeomorphism-germ $\Psi : (\mathbb{K}^p, B) \to (\mathbb{K}^q, C)$ such that $G = \Psi \circ F$.

Then, by Proposition 1.4.17, we have:

Corollary 1.4.18 *Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be a smooth map-germ ($n \leq m$). Then a mini-versal opening of f is unique up to $\mathcal{L}$-equivalence. A versal opening of f is $\mathcal{L}$-equivalent to a mini-versal opening composed with an immersion.* $\square$

As for a geometric property, we show injectivity of versal openings.

Proposition 1.4.19 (Corollary 1.2 of [50], Proposition 2.16 [57]) *Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be a finite map-germ. Suppose $F : (\mathbb{K}^n, A) \to (\mathbb{K}^{m+r}, F(A))$ is a versal opening of f. Then F has an injective representative.*

Proof First we reduce Proposition 1.4.19 to the case of mono-germs.

Let $A = \{a_1, \dots a_s\}$. Suppose each restriction $F|_{(\mathbb{K}^n, a_i)}$ for $i = 1, 2, \dots, s$ is injective. Then we show that F is injective. Suppose F is not injective. Then $F|_A$ must be not injective. In fact, suppose F is injective and $F|_A$ is not injective. Then there exists $i \neq j$ with $F(a_i) = F(a_j)$. We take $v \neq 0 \in \mathbb{K}^{m+r}$ and take a map-germ $F^i : (\mathbb{K}^n, A) \to (\mathbb{K}^m, F^i(A))$ which coincides with F near a_j and $F + v$ near a_i. Since F^i is an opening of f, and F is a versal opening of f, we must have $F^i = g \circ F$ for a function $g \in \mathcal{O}_{\mathbb{K}^{m+r}, F(A)}$. Then $F^i(a_i) = g(F(a_i)) = g(F(a_j)) = F^i(a_j)$, while $F^i(a_i) = F(a_i) + v = F(a_j) + v \neq F(a_j) = F^i(a_j)$, which leads a contradiction. (See Proposition 1.4.17 (1)).

Second we treat mono-germs in holomorphic case, $\mathbb{K} = \mathbb{C}$, following [50]. Suppose that f is a mono-germ $f : (\mathbb{C}^n, a) \to (\mathbb{C}^m, b)$. Let us take sufficiently small neighbourhoods U of a in $\mathbb{C}^n$, V of b in $\mathbb{C}^m$ and a representative $f : U \to V$ of the germ f which is proper and finite-to-one and $f^{-1}(b) = \{a\}$ in U. Define the sheaf $\mathcal{R}_f$ over U, for any open subset $U' \subset U$, by

$$\mathcal{R}_f(U') := \{h \in \mathcal{O}_{U'} \mid \text{ for any } x \in U', dh_x \in \sum_{i=1}^{m} \mathcal{O}_{\mathbb{C}^m, x} df_{ix}\}.$$

Let Ω^1_U be the sheaf of holomorphic differential forms on U and $\mathcal{J}_U$ the sheaf of O_U-submodules generated by $df_1, \dots, df_m$ (cf. Definition 1.4.2). Consider the quotient sheaf $\Omega^1_U/\mathcal{I}_U$. Then both Ω^1_U and $\Omega^1_U/\mathcal{I}_U$ are coherent O_U-modules. By Grauert's finite coherence theorem ([41, 42]), $f_*(\Omega^1_U)$ and $f_*(\Omega^1_U/\mathcal{I}_U)$ are coherent O_V-modules. Using the exterior differential, define $d : f_*(O_U) \to f_*(\Omega^1_U/\mathcal{I}_U)$ by $d(h) := [dh]$, the residue class of dh. Then d turns out to be an O_V-homomorphism. In fact we have

$$d((k \circ f)h) = \sum_{j=1}^{s}(\frac{\partial k}{\partial y_j} \circ f)h\, df_j + (k \circ f)dh \equiv (k \circ f)dh, \text{ mod. } \mathcal{J}_U.$$

Moreover we have that $f_*(\mathcal{R}_f) = \mathrm{Ker}(d)$ and that $f_*(\mathcal{R}_f)$ is a coherent O_V-module.

Set $F = (f, h_1, \dots, h_r)$, a versal opening of f. Then $1, h_1, \dots, h_r$ generate $f_*(\mathcal{R}_f)_b$ over O_b. Since $f_*(\mathcal{R}_f)$ is a coherent O_V-module, we have that $1, h_1, \dots, h_r$ generate $f_*(\mathcal{R}_f)_V$ over O_V, if we shrink U, V if necessary. Then we have, for $y \in V$, that

$$f_*(\mathcal{R}_f)_y = \{h \in O_{\mathbb{C}^n, f^{-1}(y)} \mid dh \in \sum_{i=1}^{m} O_{\mathbb{C}^n, f^{-1}(y)} df_i\}$$

contains the stalk of push-forward $f_*\mathbb{C}_U$ at y, where $\mathbb{C}_U$ is the constant sheaf over U. Therefore $h_1, \dots, h_r$ separate points of $f^{-1}(y)$. Hence the representative $F : U \to \mathbb{C}^{m+r}$ is injective. This proves Proposition 1.4.19 in the complex case.

Lastly we treat mono-germs in real C^∞ case, $\mathbb{K} = \mathbb{R}$, following [57] with additional technical modifications. Suppose that f is a mono-germ $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$. Assume the versal opening map-germ $F : (\mathbb{R}^n, a) \to (\mathbb{R}^{m+r}, F(a))$ of f has no injective representative. Take any representative $F : \mathbb{R}^n \to \mathbb{R}^{m+r}$. Then there must be a sequence of points b_i in $\mathbb{R}^{m+r}$ which tends to $\widetilde{b} = F(a)$ when $i \to \infty$, and a_i^1, a_i^2 in $\mathbb{R}^n$ which tend to a respectively when $i \to \infty$, such that $F(a_i^1) = F(a_i^2) = \widetilde{b}_i$. We may suppose $\widetilde{b}_i \neq \widetilde{b}_j$ if $i \neq j$. Take a C^∞ function h on $\mathbb{R}^{m+r}$ such that (1) the support of h is a disjoint union of small balls B_i centred at $\widetilde{b}_i$, $i = 1, 2, \dots$. We define C^∞ functions h_i on $\mathbb{R}^{m+r}$ such that $h_i = h|_{B_i}$ and $h_i = 0$ outside of B_i. Then we impose, by modifying the functions h_i if necessary, that (2) $\sum_{i=1}^{\ell} h_i$ and each of its partial derivative converges uniformly to h when $\ell \to \infty$. Note that such a function h must be flat at $\widetilde{b}$, i.e., all partial derivatives vanish at $\widetilde{b}$. Take the C^∞ function $k = F^*h$ on $\mathbb{R}^n$. Since F is finite, the support of k is a disjoint union of compact neighbourhoods W_i^1 of a_i^1 and W_j^2 of a_i^2, possibly with other additional compact set. Modify the function k to k' such that the support of k' equals that of k, k' coincides with k on W_i^1 and $k' = F \circ h'$ on W_i^2 for some C^∞ function h' on $\mathbb{R}^{m+r}$ with the same properties (1),(2) as above and with $h'(b_i) \neq h(b_i)$ for each i. We set k' identically 0 outside of W_i^1 and W_i^2. Define K_i^1 (resp. K_i^2) by $K_i^1(x) = h_i \circ F(x)$ $(x \in W_i^1)$, 0 (otherwise) (resp. $K_i^2(x) = h_i' \circ F(x)$ $(x \in W_i^2)$, 0 (otherwise)). Then K_i^1, K_i^2 are C^∞ functions and the sequence $\sum_{i=1}^{\ell}(K_i^1 + K_i^2)$ and all partial derivatives converge to k' uniformly when $\ell \to \infty$. Therefore we have that k' is C^∞. Moreover if we set $a_j(x) := \frac{\partial h}{\partial y_j} \circ F(x)$ $(x \in W_i^1)$, $\frac{\partial h'}{\partial y_j} \circ F(x)$ $(x \in W_i^2)$, 0 (otherwise), then it can be shown, similarly as for k', that each a_j is a C^∞ function on $\mathbb{R}^n$ and $dk' = \sum_{j=1}^{m+r} a_j dF_j$. Thus we see that k' belongs to $\mathcal{R}_F = \mathcal{R}_f$. Further we have $k'(a_i^1) \neq k'(a_i^2)$ for any $i = 1, 2, \ldots$. Consider the opening $(f, k') : (\mathbb{R}^n, a) \to (\mathbb{R}^{m+1}, (b, 0))$ of f. Since F is a versal opening of f, we have (f, k') is a composite mapping of F. Since $F(a_i^1) = F(a_i^2)$, we must have $k'(a_i^1) = k'(a_i^2)$. This leads a contradiction and we complete the proof of Proposition 1.4.19. □

We recall the notion of unfolding of map-germs ([92]).

Definition 1.4.20 Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be a multi-germ of map. An unfolding of f is a map-germ $F : (\mathbb{K}^{n+\ell}, A \times \{0\}) \to (\mathbb{K}^{m+\ell}, (b, 0))$ of form $F(x, u) = (F_1(x, u), u)$ and $F_1(x, 0) = f(x)$, for $(x, u) \in (\mathbb{K}^{n+\ell}, A \times \{0\})$.

For another unfolding $G : (\mathbb{K}^{n+\ell}, A \times \{0\}) \to (\mathbb{K}^{m+\ell}, (b, 0))$ of f, F and G are called **isomorphic** as unfoldings if there exist an unfolding $\Sigma : (\mathbb{K}^{n+\ell}, A \times 0) \to (\mathbb{K}^{n+\ell}, A \times 0)$ of the identity map on $(\mathbb{K}^n, A)$ and an unfolding $T : (\mathbb{K}^{m+\ell}, (b, 0)) \to (\mathbb{K}^{m+\ell}, (b, 0))$ of the identity map on $(\mathbb{K}^m, b)$ such that $G \circ \Sigma = T \circ F$.

Proposition 1.4.21 (Unfoldings and openings) *Let $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be a smooth map-germ and $F : (\mathbb{K}^{n+\ell}, A \times \{0\}) \to (\mathbb{K}^{m+\ell}, (b, 0))$ be an unfolding of f. Let $i : (\mathbb{K}^n, A) \to (\mathbb{K}^{n+\ell}, A \times \{0\})$ be the inclusion, $i(x) = (x, 0)$. Then we have:*

(1) $i^*\mathcal{R}_F \subset \mathcal{R}_f$.

(2) *If f is of corank ≤ 1 with $n \leq m$, then $i^*\mathcal{R}_F = \mathcal{R}_f$. If $1, H_1, \ldots, H_r$ generate $\mathcal{R}_F$ via F^*, then $1, i^*H_1, \ldots, i^*H_r$ generate $\mathcal{R}_f$ via f^*.*

Proof For the mono-germ case the assertions are proved in Proposition 1.6 of [50], Lemma 2.4 of [51]. Here we present the proof for the general case: (1) is clear. (2) Let $H \in \mathcal{R}_F$. Then $dH \in \mathcal{J}_F$. Hence $d(i^*H) = i^*(dH) \in i^*\mathcal{J}_F \subset \mathcal{J}_f$. Therefore $i^*H \in \mathcal{R}_f$. Let f be of corank at most one. Suppose $h \in \mathcal{R}_f$. Then $dh = \sum_{j=1}^m a_j df_j$ for some $a_j \in \mathcal{O}_a$. There exist $A_j, B_k \in \mathcal{O}_{(a,0)}$ such that $i^*A_j = a_j$ and the 1-form $\sum_{j=1}^m A_j d(F_1)_j + \sum_{k=1}^\ell B_k d\lambda_k$ is closed (cf. Lemma 2.5 of [52]). Then there exists an $H \in \mathcal{O}_{(a,0)}$ such that $dH = \sum_{j=1}^m A_j d(F_1)_j + \sum_{k=1}^\ell B_k d\lambda_k \in \mathcal{J}_F$ and $d(i^*H) = i^*(dH) = dh$. Then there exists $c \in \mathbb{R}$ such that $h = i^*H + c = i^*(H + c)$, and $H + c \in \mathcal{R}_F$. Therefore $h \in i^*\mathcal{R}_F$. Since i^* is a homomorphism over $j^* : \mathcal{O}_{(b,0)} \to \mathcal{O}_b$, where $j : (\mathbb{R}^m, 0) \to (\mathbb{R}^{m+\ell}, 0)$ is the inclusion $j(y) = (y, 0)$, we have the consequence. □

Definition 1.4.22 An unfolding $F : (\mathbb{K}^{n+\ell}, A \times \{0\}) \to (\mathbb{K}^{m+\ell}, (b, 0))$ of a map-germ $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ is called **extendable** if $i^*\mathcal{R}_F = \mathcal{R}_f$ for the inclusion $i : (\mathbb{K}^n, A) \to (\mathbb{K}^{n+\ell}, A \times \{0\})$.

By Proposition 1.4.21, we have:

Corollary 1.4.23 ([52]) *If corank of f is at most one, then any unfolding of f is extendable.* □

In §1.4.3, we will see that there exist non-extendable unfoldings for map-germs of corank ≥ 2. Therefore the opening constructions do not behave well under unfoldings in general.

We will describe versal openings in the case of corank (kernel rank) one explicitly. It is sufficient to treat the case of mono-germs, namely, germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^m, 0)$ of corank one. Moreover, by Corollary 1.4.23, it is sufficient to treat the case that f is stable, namely, f is a Morin map.

We present the normal forms of Morin maps. Let $k \geq 0, m \geq 0$ and consider variables $t, \lambda = (\lambda_1, \dots, \lambda_{k-1}), \mu = (\mu_{ij})_{1 \leq i \leq m, 1 \leq j \leq k}$ and polynomials

$$F(t, \lambda) = t^{k+1} + \sum_{i=1}^{k-1} \lambda_j t^j, \quad G_i(t, \mu) = \sum_{j=1}^{k} \mu_{ij} t^j, (1 \leq i \leq m).$$

Then the normal forms of Morn maps $f : (\mathbb{K}^{k+km}, 0) \to (\mathbb{K}^{m+k+km}, 0)$ is given by

$$f(t, \lambda, \mu) := (F(t, \lambda), G(t, \mu), \lambda, \mu).$$

For $\ell \geq 0$, we denote by $F_{(\ell)}, G_{i(\ell)}$ the polynomials

$$F_{(\ell)}(t, \lambda) = \int_0^t s^\ell F(s, \lambda) ds, \quad G_{i\,(\ell)}(t, \mu) = \int_0^t s^\ell G_i(s, \mu) ds.$$

Then we have

Proposition 1.4.24 (Theorem 3 of [49]) *The ramification module $\mathcal{R}_f$ of the Morin map f is minimally generated over $f^* \mathcal{O}_{m+k+km}$ by the $1 + k + (k-1)m$ elements*

$$1,\ F_{(1)},\ \dots,\ F_{(k)},\ G_{1\,(1)},\ \dots,\ G_{1\,(k-1)},\ \dots,\ G_{m\,(1)},\ \dots,\ G_{m\,(k-1)}.$$

The map-germ $\mathbb{F} : (\mathbb{R}^{k+mk}, 0) \to (\mathbb{R}^{m+k+km} \times \mathbb{R}^{k+(k-1)m}, 0) = (\mathbb{R}^{2(k+km)}, 0)$ defined by

$$\mathbb{F} = \left(f,\ F_{(1)}, \dots, F_{(k)}, G_{1\,(1)}, \dots, G_{1\,(k-1)}, \dots, G_{m\,(1)}, \dots, G_{m\,(k-1)}\right)$$

is a mini-versal opening of f. □

Remark 1.4.25 It is shown in [36, 49] moreover that $\mathbb{F}$ is an isotropic map for a symplectic structure on $\mathbb{R}^{2(k+km)}$. □

In particular we have:

Lemma 1.4.26 *Let ℓ be a positive integer and $F = (F_1(t, u), u) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ an unfolding of $f : (\mathbb{K}, 0) \to (\mathbb{K}, 0), f(t) = F_1(t, 0) = t^\ell$. Suppose $H_1, \dots, H_r \in \mathcal{R}_F \cap \mathfrak{m}_n$. Then $1, H_1, \dots, H_r$ generate $\mathcal{R}_F$ via F^* if and only $i^* H_1, \dots, i^* H_r$ generate $\mathfrak{m}_1^{\ell+1}/\mathfrak{m}_1^{2\ell}$. In particular $F_{1(1)}, \dots, F_{1(\ell-1)}$ form a system of generators of $\mathcal{R}_F$ via F^* over $\mathcal{O}_n$.*

Proof It is easy to show that $\mathcal{R}_f = \mathbb{R} + \mathfrak{m}_1^\ell$. By Proposition 1.4.13 (2), $1, H_1, \dots, H_r$ generate $\mathcal{R}_F$ as O_n-module via F^* if and only if they generate $\mathcal{R}_F / F^*(\mathfrak{m}_n)\mathcal{R}_F$ over $\mathbb{R}$. Since $\mathcal{R}_F / F^*(\mathfrak{m}_n)\mathcal{R}_F \cong (\mathbb{R} + \mathfrak{m}_1^\ell)/(f^*\mathfrak{m}_1)(\mathbb{R} + \mathfrak{m}_1^\ell) \cong \mathfrak{m}_1^{\ell+1}/\mathfrak{m}_1^{2\ell}$, we have the consequence. □

Now let $f : (\mathbb{R}^n, A) \to (\mathbb{R}^m, b)$ be a finite *real* analytic map-germ. Remember that, in this survey, we denote by $O_{\mathbb{R}^n,A}$ (resp. $O_{\mathbb{R}^m,b}$) the $\mathbb{R}$-algebra of C^∞-function-germs at $(\mathbb{R}^n, A)$ (resp. $(\mathbb{R}^m, b)$. Then we denote by $O^\omega_{\mathbb{R}^n,A}$ (resp. $O^\omega_{\mathbb{R}^m,b}$) the germ of sheaf of *real analytic* functions on $(\mathbb{R}^n, A)$ (resp. $(\mathbb{R}^m, b)$). Besides with $\mathcal{R}_f$, here we consider the sheaf

$$\mathcal{R}^\omega_f := \{h \in O^\omega_{\mathbb{R}^n,A} \mid dh \in \langle df_1, \dots, df_m \rangle_{O^\omega_{\mathbb{R}^n,A}}\}$$

and the direct image $f_*(\mathcal{R}^\omega_f)$ as $O^\omega_{\mathbb{R}^m,b}$-module. Then it can be proved that, in particular, $f_*(\mathcal{R}^\omega_f)$ is a finite $O^\omega_{\mathbb{R}^m,b}$-module. Thus we have that $f_*(\mathcal{R}^\omega_f)$ is generated over $O^\omega_{\mathbb{R}^m,b}$ by some $1, h_1, \dots, h_r \in \mathcal{R}^\omega_f$. Moreover it can be proved that $F = (f, h_1, \dots, h_r) : (\mathbb{R}^n, A) \to (\mathbb{R}^{m+r}, b \times h(A))$ is injective (See [50], Proposition 1.4.19).

Then we show the following

Proposition 1.4.27 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ be a finite analytic mono map-germ. Suppose $1, h_1, \dots, h_r$ generate $\mathcal{R}^\omega_f$ over $O^\omega_{\mathbb{R}^m,0}$ via $f^* : O^\omega_{\mathbb{R}^m,0} \to O^\omega_{\mathbb{R}^n,0}$. Then $1, h_1, \dots, h_r$ generate $\mathcal{R}_f$ over $O_{\mathbb{R}^m,0}$ via $f^* : O_{\mathbb{R}^m,0} \to O_{\mathbb{R}^n,0}$. That is, analytic generators become C^∞ generators.*

Proof First we may suppose $h_i(0) = 0, (1 \leq i \leq r)$. Then we remark that the opening

$$F = (f_1, \dots, f_m, h_1, \dots, h_r) : (\mathbb{R}^n, 0) \to (\mathbb{R}^{m+r}, 0)$$

of f is injective.

Let $\mathcal{F}_p$ stand for the $\mathbb{R}$-algebra of formal functions on $(\mathbb{R}^n, p)$, and set $\widetilde{\mathcal{F}}_n = \prod_{p\in(\mathbb{R}^n,0)} \mathcal{F}_p$. Then $\widetilde{\mathcal{F}}$ is faithfully flat over O_n (Cor. 4.13 of [91]). Define the formal counterpart

$$\widetilde{\mathcal{R}}_f = \{(\widehat{h}_p)_{p\in(\mathbb{R}^n,0)} \in \widetilde{\mathcal{F}}_n \mid dh_p \in \langle d\widehat{f}_{1,p}, \dots, d\widehat{f}_{n,p} \rangle_{\mathcal{F}_p}, p \in (\mathbb{R}^n, 0)\}$$

of $\mathcal{R}_f$. Then $1, h_1, \dots, h_r$ generate $\widetilde{\mathcal{R}}_f$ over $\widetilde{\mathcal{F}}_m$. Then, since F is injective, we have $\mathcal{R}_f \subseteq F^* O_{m+r}$ by Gleaser's type theorem (see [13]). Since F is an opening, we have $\mathcal{R}_f = F^* O_{m+r}$

Let $\pi : (\mathbb{R}^{m+r}, 0) \to (\mathbb{R}^m, 0)$ be the projection. Then $\pi^* : O_m \to O_{m+r}$ is the inclusion. Then regard $F^* O_{m+r}$ as an O_{m+r}-module via F^*. By the preparation theorem, $1, h_1, \dots, h_r$ generate $F^* O_{m+r} = \mathcal{R}_f$ as O_m-module via $F^* \circ \pi^* = f^*$ if $1, h_1, \dots, h_r$ generate $F^* O_{m+r}/(f^*\mathfrak{m}_m)F^* O_{m+r}$ over $\mathbb{R}$. We will show

$$\mathfrak{m}_n^\infty \cap F^* O_{m+r} \subseteq (f^*\mathfrak{m}_m)F^* O_{m+r}.$$

Set $h = \sum_{i=1}^{m} f_i^2 : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. Since f is finite, $h^{-1}(0) = \{0\}$, and moreover, the norms of $1/h$ and its partial derivatives up to say ℓ are estimated from above $1/\|x\|^\alpha$ for some $\alpha = \alpha(\ell) > 0$.

Now let $k \in \mathfrak{m}_n^\infty \cap F^*\mathcal{O}_{m+r}$. Then k/h is regarded as a C^∞ function on $(\mathbb{R}^n, 0)$ and an element of $\mathfrak{m}_n^\infty$ (see [91]). Moreover $k/h \in F^*\mathcal{O}_{m+r}$. Then

$$k = (\sum_{i=1}^{m} f_i^2)(k/h) = \sum_{i=1}^{m} f_i(f_i k/h) \in (f^*\mathfrak{m}_m)F^*\mathcal{O}_{m+r}.$$

Let $h \in \mathcal{R}_f$ again. Then

$$\begin{aligned} h &\equiv a_0 \circ f + a_1 \circ f \cdot h_1 + \cdots + a_r \circ f \cdot h_r \\ &\equiv a_0(0) + a_1(0)h_1 + \cdots + a_r(0)h_r, \end{aligned}$$

modulo $(f^*\mathfrak{m}_m)F^*\mathcal{O}_{m+r} + \mathfrak{m}_n^\infty \cap F^*\mathcal{O}_{m+r} \subseteq (f^*\mathfrak{m}_m)F^*\mathcal{O}_{m+r}$.

Thus we see $1, h_1, \ldots, h_r$ generate $F^*\mathcal{O}_{m+r}/(f^*\mathfrak{m}_m)F^*\mathcal{O}_{m+r}$ over $\mathbb{R}$, and they generate $\mathcal{R}_f$ over $\mathcal{O}_{\mathbb{R}^m,0}$ via f^*. □

Remark 1.4.28 By a theorem by Bierstone and Milman([13]), we have the following

Lemma 1.4.29 *Suppose $\mathcal{R}_f$ is formally generated by $\mathcal{R}_f^\omega$, namely, if, for any $k \in \mathcal{R}_f$ and for $x \in (\mathbb{R}^n, A)$, $\widehat{k}_x \in \langle 1, \widehat{h}_1, \ldots, \widehat{h}_r \rangle_{\widehat{f}^*\mathcal{F}_{\mathbb{R}^m, f(x)}}$. Then $\mathcal{R}_f = \langle 1, h_1, \ldots, h_r \rangle_{f^*\mathcal{O}_{\mathbb{R}^m, b}}$ and $F = (f, h_1, \ldots, h_r)$ is a versal opening of f.* □

We can utilise both Lemma 1.4.29 and a direct method for the construction concrete generators of $\mathcal{R}_f$ and therefore for the construction of versal openings.

1.4.3 The cases of corank ≥ 2

The existence of versal openings of a smooth map-germ of corank ≥ 2 is still open. One of the difficulties lies on the fact that if $\mathrm{corank}(f) \geq 2$, then the restriction of a versal opening of an unfolding of f is not necessarily a versal opening of f. That phenomenon was observed already in [37, 52]. We utilise Proposition 1.4.27 if necessary to treat the following examples.

Lemma 1.4.30 (cf. Example 1.4.14.) *Let $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$, $f(t, s) = (\frac{1}{2}t^2, \frac{1}{2}s^2) = (z, w)$. Then $\mathcal{R}_f$ is minimally generated by $1, t^3, s^3, t^3s^3$ over $f^*\mathcal{O}_{\mathbb{K}^2,0}$. Therefore if we set $F : (\mathbb{K}^2, 0) \to (\mathbb{K}^5, 0)$ by*

$$F(t, s) = (\frac{1}{2}t^2,\ \frac{1}{2}s^2,\ t^3,\ s^3,\ t^3s^3),$$

then F is the mini-versal opening of f.

Here we give a concrete method to find the minimal generators as above.

Proof Let $h \in \mathcal{O}_{\mathbb{K}^2,0} = \mathcal{O}_2$. Then by the preparation theorem we have

$$h \equiv (a \circ f)t + (b \circ f)s + (c \circ f)ts, \text{mod.} f^*\mathcal{O}_2.$$

The condition that $h \in \mathcal{R}_f$ is equivalent to that dh belongs to Jacobi module $\mathcal{J}_f$. Now assume that $h \in \mathcal{R}_f$. Then we calculate

$$dh \equiv (a \circ f)dt + (b \circ f)ds + (c \circ f)(sdt + tds), \text{mod.}\mathcal{J}_f,$$

and set

$$(a \circ f)dt + (b \circ f)ds + (c \circ f)(sdt + tds) = Atdt + Bsds,$$

for some function $A, B \in \mathcal{O}_2$. Again by the preparation theorem, we put

$$\begin{aligned} A &= (a_1 \circ f) + (a_2 \circ f)t + (a_3 \circ f)s + (a_4 \circ f)ts, \\ B &= (b_1 \circ f) + (b_2 \circ f)t + (b_3 \circ f)s + (b_4 \circ f)ts. \end{aligned}$$

Then

$$\begin{aligned} At &= (a_1 \circ f)t + (a_2 \circ f)t^2 + (a_3 \circ f)ts + (a_4 \circ f)t^2 s \\ &= 2(za_2) + a_1 t + 2(za_4)s + a_3\, ts, \\ Bs &= (b_1 \circ f)s + (b_2 \circ f)ts + (b_3 \circ f)s^2 + (b_4 \circ f)ts^2 \\ &= 2(wb_3) + 2(wb_4)t + b_1 s + b_2 ts. \end{aligned}$$

omitting "$\circ f$", where $z = \frac{1}{2}t^2$ and $w = \frac{1}{2}s^2$. Then we have

$$\begin{aligned} a + cs &= 2za_2 + a_1 t + (2za_4)s + a_3 ts, \\ b + ct &= 2wb_3 + 2(wb_4) + b_1 s + b_2 ts. \end{aligned}$$

and therefore

$$\begin{aligned} (a - 2za_2) + (-a_1)t + (c - 2za_4)s + (-a_3)ts &= 0, \\ (b - 2wb_3) + (c - 2wb_4)t + (-b_1)s + (-b_2)ts &= 0. \end{aligned}$$

Since $\mathcal{O}_2$ is free over $f^*\mathcal{O}_2$ in this example, we have

$$a = 2za_2, a_1 = 0, c = 2za_4, a_3 = 0, b = 2wb_3, c = 2wb_4, b_1 = 0, b_2 = 0.$$

Therefore $wb_4 = za_4$ and hence b_4 is divisible by z, and we can write $b_4 = zk(z, w)$ for some function $k \in \mathcal{O}_2$. Then we have that $a_4 = wk(z, w)$ and that

$$h \equiv (2za_2 \circ f)t + (2wb_3 \circ f)s + (2zwk \circ f)ts, \text{mod.} f^*\mathcal{O}_2.$$

Thus we find a minimal system of generators $1, 2zt, 2ws, 2zwts$, namely $1, t^3, s^3, t^3s^3$ of $\mathcal{R}_f$ over $f^*\mathcal{O}_2$. □

Similarly we find the mini-versal openings in the following three cases. For details see [57].

Lemma 1.4.31 *Let* $g : (\mathbb{K}^3, 0) \to (\mathbb{K}^3, 0)$ *be a map-germ defined by*

$$g(t,s,u) = (z,w,u) = (\frac{1}{2}t^2 + us, \frac{1}{2}s^2 + ut,\ u),$$

which is an unfolding of f in Example 1.4.30. Then $\mathcal{R}_g$ is minimally generated over $g^*\mathcal{O}_{\mathbb{K}^3,0}$ by 1 *and* $\psi_3 = t^3 + s^3 + 3tsu$, $\psi_5^{5,0} = t^5 + 5t^3su - 12t^2u^3 + 9su^4$, $\psi_5^{0,5} = s^5 + 5ts^3u - 12s^2u^3 + 9tu^4$, $\psi_6^{3,3} = t^3s^3 - 12t^2s^2u^2 - 11t^3u^3 - 11s^3u^3 - 12tsu^4$. □

Therefore $i^*\mathcal{R}_g \subsetneq \mathcal{R}_f$, where $i : (\mathbb{K}^2,0) \to (\mathbb{K}^3,0), i(t,s) = (t,s,0)$, and we see that g is not an extendable unfolding of f.

The versal opening of g is given by $G : (\mathbb{K}^3,0) \to (\mathbb{K}^7,0) = (\mathbb{K}^3 \times \mathbb{K}^4,0)$,

$$\begin{aligned} G(t,s,u) = (&g(t,s,u),\ t^3+s^3+3tsu,\ t^5+5t^3su-12t^2u^3+9su^4,\\ &s^5+5ts^3u-12s^2u^3+9tu^4, t^3s^3-12t^2s^2u^2-11t^3u^3-11s^3u^3-12tsu^4).\end{aligned}$$

Then

$$G(x,y,0) = (\frac{1}{2}t^2, \frac{1}{2}s^2, t^3+s^3, t^5, s^5, t^3s^3)$$

is not a versal opening of $f = (\frac{1}{2}t^2, \frac{1}{2}s^2)$. Note that the element ψ_3 gives a Lagrange immersion of type D_4^+, which is a Lagrange stable lifting of g. Other elements are obtained by operating lowerable vector fields of g to ψ_3.

Lemma 1.4.32 (Hyperbolic case). *Let $h : (\mathbb{K}^4,0) \to (\mathbb{K}^4,0)$ be the stable map-germ*

$$h(t,s,\lambda,\mu) = (z,w,\lambda,\mu) = (\frac{1}{2}t^2 + s\lambda,\ \frac{1}{2}s^2 + t\mu,\ \lambda,\ \mu).$$

of $\mathcal{K}$-class $I_{2,2}$ (V of [93], see also [108]). *Then $\mathcal{R}_h$ is minimally generated over $h^*\mathcal{O}_{\mathbb{K}^4,0}$ by* 1 *and* $\varphi_4 = t^3\mu + s^3\lambda + 3ts\lambda\mu$, $\varphi_5^{3,2} = t^3s^2 - 2t^2s\lambda\mu + t\lambda^2\mu^2$, $\varphi_5^{2,3} = t^2s^3 - 2ts^2\lambda\mu + s\lambda^2\mu^2$, $\varphi_5^{5,0} = t^5 + 5t^3s\lambda + 15s\lambda^3\mu$, $\varphi_5^{0,5} = s^5 + 5ts^3\mu + 15t\lambda\mu^3$, $\varphi_6 = t^3s^3 - 3ts\lambda^2\mu^2$.

We have the mini-versal opening $H : (\mathbb{K}^4,0) \to (\mathbb{K}^4 \times \mathbb{K}^6,0) = (\mathbb{K}^{10},0)$ of h by

$$H = (h, \varphi_4, \varphi_5^{3,2}, \varphi_5^{2,3}, \varphi_5^{5,0}, \varphi_5^{0,5}, \varphi_6).$$

□

Then we observe

$$j^*\mathcal{R}_h \subsetneq \mathcal{R}_g (\subsetneq \mathcal{O}_{\mathbb{K}^3,0}), \qquad (j \circ i)^*\mathcal{R}_h \subsetneq i^*\mathcal{R}_g \subsetneq \mathcal{R}_f (\subsetneq \mathcal{O}_{\mathbb{K}^2,0}),$$

where $j : (\mathbb{K}^3,0) \to (\mathbb{K}^4,0), j(x,y,u) = (x,y,u,u)$. Thus the unfolding h of f is not extendable, which is also not extendable regarded as an unfolding of g as well.

Similarly we have the following.

Lemma 1.4.33 (Elliptic case.) *Let $k : (\mathbb{K}^4,0) \to (\mathbb{K}^4,0)$ be the stable map-germ given by*

$$k(t,s,\lambda,\mu) = (\frac{1}{2}(t^2 - s^2) + \lambda t + \mu s,\ ts + \mu t - \lambda s,\ \lambda,\ \mu),$$

of $\mathcal{K}$-class $II_{2,2}$. Then $\mathcal{R}_k$ is minimally generated over $k^*\mathcal{O}_4$ by 1 *and* $\rho_4, \rho_5^{3,2}, \rho_5^{2,3}, \rho_5^{5,0}, \rho_5^{0,5}, \rho_6$, *where*

$$
\begin{aligned}
\rho &= a_1 t + a_2 s + \tfrac{1}{2}a_3(t^2 + s^2),\\
a_1 &= 2zA_1 + 2wA_2 + (-\tfrac{3}{2}\lambda^3 - \tfrac{3}{2}\lambda\mu^2 - 3z\lambda - 3w\mu)A_3 + (-\tfrac{3}{2}\lambda^2\mu - \tfrac{3}{2}\mu^3 - 3z\mu + 3w\lambda)B_3,\\
a_2 &= -2wA_1 + 2zA_2 + (\tfrac{3}{2}\lambda^2\mu - \tfrac{3}{2}\mu^3 + z\mu - w\lambda)A_3, +(\tfrac{3}{2}\lambda^3 + \tfrac{3}{2}\lambda\mu^2 - z\lambda - 3w\mu)B_3,\\
a_3 &= -\lambda A_1 - \mu A_2 + (z - \tfrac{1}{2}\lambda^2 + \tfrac{1}{2}\mu^2)A_3 + (w - \lambda\mu)B_3,
\end{aligned}
$$

and $\rho = \rho_4, \rho_5^{3,2}, \rho_5^{2,3}, \rho_5^{5,0}, \rho_5^{0,5}, \rho_6$ *respectively for* $(A_1, A_2, A_3, B_3) = (\lambda, \mu, 0, 0)$, $(0, z - \frac{3}{2}\lambda^2 + \frac{3}{2}\mu^2, 0, 3\lambda)$, $(0, w - 3\lambda\mu, -3\lambda, 0)$, $(z - \frac{3}{2}\lambda^2 + \frac{3}{2}\mu^2, 0, 0, -3\mu)$, $(w - 3\lambda\mu, 0, 3\mu, 0)$, $(0, 0, z - \frac{3}{2}\lambda^2 + \frac{3}{2}\mu^2, w - 3\lambda\mu))$. □

From the above results, combined with known classification of stable germs $\mathbb{K}^4 \to \mathbb{K}^4$ of corank 2 under right-left equivalence [40], we have the following

Theorem 1.4.34 *Any stable mono-germ* $(\mathbb{K}^4, a) \to (\mathbb{K}^4, b)$*, and therefore any stable multi-germ* $(\mathbb{K}^4, A) \to (\mathbb{K}^4, b)$ *has a versal opening.* □

1.5 Other topics

1.5.1 Relation to symplectic geometry

The notion of openings, studied in the previous section, has a close relation also to the singularity theory on "isotropic" map-germs in a symplectic space (see [37, 38, 55, 64, 65, 66]).

In classical mechanics, the canonical one-form, or Liouville one form θ and the symplectic two-form $\omega = d\theta$ on the cotangent bundle over a configuration space play basic roles [1]. For the canonical coordinates $p_1, \dots, p_n; x_1, \dots, x_n$ on the cotangent bundle $T^*\mathbb{K}^m$ of $\mathbb{K}^m$, θ and ω are given by

$$\theta = \sum_{i=1}^{m} p_i dx_i, \quad \omega = \sum_{i=1}^{m} dp_i \wedge dx_i.$$

Recall that a smooth map-germ $L : (\mathbb{K}^n, A) \to T^*\mathbb{K}^m = \mathbb{K}^{2m}$ is called isotropic if $L^*\omega = 0$. Moreover if $n = m$, L is called a **Lagrange map**. We denote by $\mathcal{I}_{n,m}$ (resp, $\mathcal{L}_n$) the set of all isotropic (resp. Lagrangian) map-germs $(\mathbb{K}^n, A) \to T^*\mathbb{K}^m$. If $L \in \mathcal{I}_{n,m}$, then $L^*\omega = 0$, so $d(L^*\theta) = 0$. Then there exists a function-germ $e : (\mathbb{K}^m, A) \to \mathbb{K}$ such that $de = L^*\theta = \sum_{i=1}^{m}(p_i \circ L)d(x_i \circ L)$. Note that e is determined up to an addition of locally constant functions. We set $f = (x_1 \circ L, \dots, x_m \circ L) : (\mathbb{K}^n, A) \to \mathbb{K}^m$ and recall the ramification module

$$\mathcal{R}_f := \{h \in \mathcal{O}_{(\mathbb{K}^n,0)} \mid dh = \sum_{i=1}^{n} a_i df_i, \text{for some } a_i \in \mathcal{O}_{(\mathbb{K}^n,A)}, 1 = 1, 2, \ldots, n\}.$$

Then $e \in \mathcal{R}_f$. Therefore (f, e) is an opening of f. See §1.4.1. Recall that, if the locus of immersion $\mathrm{Reg}(f)$ of f is dense in $\mathbb{K}^n$ near A, then $(f, e) : (\mathbb{K}^n, A) \to \mathbb{K}^{m+1}$ is a proper (fair) frontal.

Note that, for a smooth map-germ $f : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$, $\mathcal{R}_f$ is equal to the set of generating functions of isotropic liftings $(\mathbb{K}^n, A) \to T^*\mathbb{K}^m$ of f.

Example 1.5.1 (1) Define $L : (\mathbb{K}^2, 0) \to T^*\mathbb{K}^2$ by

$$L(u, v) = (x_1 \circ L, x_2 \circ L, p_1 \circ f, p_2 \circ L) = (u, v^2 + uv, -\frac{1}{2}v^2, v)$$

Then L is a Lagrangian immersion and $e = \frac{3}{4}v^4 + \frac{1}{2}uv^2$ is a generating function of L.

(2) Define $L : (\mathbb{K}^2, 0) \to T^*\mathbb{K}^2$ by

$$L(u, v) = (u, v^2 + uv, -\frac{3}{10}v^5 - \frac{1}{6}uv^3, \frac{3}{4}v^4 - \frac{1}{2}uv^2).$$

Then L is a Lagrangian map-germ and is called the **open swallowtail**, which is introduced by Arnol'd ([4]).

Suppose $n = m$. Then The Grassmann lift (Legendre lift) of (f, e) is given by $(L, e) : (\mathbb{K}^n, A) \to T^*\mathbb{K}^n \times \mathbb{K}$. The mapping (L, e) is called a **Legendrisation** of L. For example the Legendrisation of a open Whitney umbrella is a Legendre lift of the folded umbrella.

Let $T^*\mathbb{K}^m = \mathbb{K}^{2m}$ be the $2m$-dimensional symplectic space with coordinates $x_1, \ldots, x_m; p_1, \ldots, p_m$ such that symplectic form $\omega = \sum_{i=1}^m dp_i \wedge dx_i$. A multi-germ of mapping $L : (\mathbb{K}^n, A) \to T^*\mathbb{K}^m = \mathbb{K}^{2m}$ is called **isotropic** if $L^*\omega = \sum_{i=1}^m (p_i \circ L)d(x_i \circ L) = 0$. Then, since $L^*\omega = 0$, we have that $\sum_{i=1}^m (p_i \circ L)d(x_i \circ L)$ is closed, so it is exact by Poincaré's lemma, and therefore there exists $e \in \mathcal{O}_{\mathbb{K}^n,A}$ such that

$$de = \sum_{i=1}^{m} (p_i \circ L)d(x_i \circ L).$$

Set $f : (\mathbb{K}^n, A) \to \mathbb{K}^m$ by $f(x) = (x_1 \circ L(x), \ldots, x_m \circ L(x))$. Then $e \in \mathcal{R}_f$. Conversely, given $e \in \mathcal{R}_f$, we have $de = \sum_{i=1}^m a_i df_i$ for some functions $a_1, \ldots, a_m \in \mathcal{O}_{\mathbb{K}^n,A}$, and we obtain an isotropic multi-germ $L : (\mathbb{K}^n, A) \to \mathbb{K}^{2m}$ by defining $p_i \circ L = a_i, x_i \circ L = g_i, (1 \leq i \leq m)$. Thus we obtain, associated to L, an opening $(f, e) : (\mathbb{K}^n, A) \to \mathbb{K}^{m+1}$ of f from L. Moreover (f, e) has the Legendre lift $(L, e) : T^*\mathbb{K}^m \times \mathbb{K} \subset PT^*\mathbb{K}^{m+1}$ and (f, e) is a frontal. We call (L, e) a Legendrisation of L. For example, associated to open Whitney umbrella $L : (\mathbb{K}^2, 0) \to T^*\mathbb{K}^2$ ([52]), we obtain the folded umbrella and its Legendre lift called the open folded umbrella appears also as a "frontal-symplectic singularity"[63].

Here we make clear the relation of generalised frontals and openings with a generalisation of symplectic geometry, which is called **poly-symplectic geometry** ([11, 12, 54]).

Let M be a manifold of dimension m. For a positive integer k, consider the Whitney sum

$$T^{*(k)}M := T^*M \oplus \cdots \oplus T^*M \to M,$$

endowed with the system of closed 2-forms $\omega_i = d\theta_i$, $1 \leq i \leq k$, where θ_i is the Liouville 1-form on the i-th factor. Given a system of local coordinates $x_1, \ldots, x_m$ on M, we have the system of local coordinates $x_1, \ldots, x_m$; $p_{11}, \ldots, p_{1m}$; $\ldots$; $p_{k1}, \ldots, p_{km}$, on $T^{*(k)}M$. Then $\theta_i = \sum_{j=1}^m p_{ij}dx_j$ and $\omega_i = \sum_{j=1}^m dp_{ij} \wedge dx_j$.

A smooth mapping $\varphi : N \to T^{*(k)}M$ from an n-dimensional manifold N with $k = m - n$ is called **isotropic** if $\varphi^*\omega_i = 0, 1 \leq i \leq r$. If we take the universal covering $\rho : \widetilde{N} \to N$ of N, then there exist functions $e_i : \widetilde{N} \to \mathbb{K}$ such that $de_i = (\varphi \circ \rho)^*\theta_i, 1 \leq i \leq r$. We define the **graph** of φ by $f = (\pi \circ \varphi \circ \rho, e_1, \ldots e_k) : \widetilde{N} \to M \times \mathbb{K}^k =: W$. If $\Sigma(f)$ is nowhere dense in $\widetilde{M}$, then f is frontal: The Legendre lift is given by

$$\widetilde{f} = (\varphi \circ \rho, e) : \overline{N} \longrightarrow T^{*(r)}M \times \mathbb{K}^k \hookrightarrow \mathrm{Gr}(n, TM).$$

We compare equivalence relations for isotropic mappings, integral mappings and frontal mappings.

Two isotropic mappings $\varphi, \varphi' : N \to T^{*(k)}M$ are called **Lagrange equivalent** if there exist diffeomorphisms $\sigma : N \to N$ and $\tau : T^{*(k)}M \to T^{*(k)}M$ such that $\tau^*\omega_i = \omega_i, 1 \leq i \leq r, \tau$ covers a diffeomorphism $\overline{\tau} : M \to M$ with respect to $\pi : T^{*(k)}M \to M$, and that $\tau \circ \varphi = \varphi' \circ \sigma$.

Two integral mappings $L, L' : N \to T^{*(k)}M \times \mathbb{K}^k$ are called **s-Legendre equivalent** if there exist diffeomorphisms $\sigma : N \to N$ and $\widetilde{\tau} : T^{*(k)}M \times \mathbb{K}^k \to T^{*(k)}M \times \mathbb{K}^k$ such that $\overline{\tau}$ preserves the canonical distribution and the fibration $\Pi : T^{*(k)} \times \mathbb{K}^k \to M \times \mathbb{K}^k$ and that $\widetilde{\tau} \circ L = L' \circ \sigma$.

Two frontal mappings $f, f' : N \to M \times \mathbb{K}^k$ are called **s-equivalent** if there exist diffeomorphisms $\sigma : N \to N$ and $\kappa : M \times \mathbb{K}^k \to M \times \mathbb{K}^k$ of the form $\kappa(y, z) = (\overline{\tau}(y), z + \rho(y))$, and that $\kappa \circ f = f' \circ \sigma$.

Then we are naturally led to the following, the proof of which is left to the readers. See also [76].

Proposition 1.5.2 *Let $\varphi : N \to T^{*(k)}M$ be an isotropic mapping with nowhere dense singular set $\Sigma(\pi \circ \varphi)$. Then the following conditions are equivalent to each other:*
(1) *Isotropic mappings φ and $\varphi' : N \to T^{*(r)}M$ are Lagrange equivalent.*
(2) *Legendre lifts $\widetilde{f}, \widetilde{f}' : \widetilde{N} \to T^{*(r)}M \times \mathbb{K}^k$ are s-Legendre equivalent.*
(3) *Frontal mappings $f, f' : \widetilde{M} \to M \times \mathbb{K}^k$ are s-equivalent.* $\square$

1.5.2 Frontal jets

In this subsection, we present some unpublished joint works with K. Hashibori. The details will be given in a forthcoming paper.

Recall that two map-germs $f, g : (\mathbb{K}^n, 0) \to (\mathbb{K}^m, 0)$ are said to define the same r-jet if they have the same partial derivatives at 0 up to order $\leq r$. The equivalence class of f is called an r-jet of f at 0 and written by $j^r f(0)$.

Let $J^r(n, m)$ denote the space of r-jets of map-germs $(\mathbb{K}^n, 0) \to (\mathbb{K}^m, 0)$.

Definition 1.5.3 A jet $z \in J^r(n, m), n \leq m$ is called a **frontal jet** if there exists a frontal $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^m, 0)$ such that $z = j^r f(0)$.

Let $L^r(n)$ denote the real algebraic group consisting of r-jets of diffeomorphism-germs on $(\mathbb{K}^n, 0)$. Then the group $L^r(n) \times L^r(m)$ acts on $J^r(n, m)$ by

$$(j^r\sigma(0), j^r\tau(0))(j^r f(0)) := j^r(\tau \circ f \circ \sigma^{-1})(0),$$

which induces $\mathcal{A}^r$-equivalence on $J^r(n, m)$.

The property that a germ is frontal, is invariant under the right-left equivalence, however the frontality condition for a germ is “transcendental", i.e. not finitely determined by its jet, provided the germ is not immersive.

Proposition 1.5.4 *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^{n+1}, 0)$ be a frontal of corank 1 and $s \in \mathbb{N}$. Suppose f is non-immersive. Then there exist $r \in \mathbb{N}$ and a smooth map-germ $g : (\mathbb{K}^n, a) \to (\mathbb{K}^{n+1}, b)$ such that $s \leq r$, $j^r g(a) = j^r f(a)$ and that g is not a frontal.*

Proof We may suppose that $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^{n+1}, 0)$, $f = (t_1, \dots, t_{n-1}, h(t), k(t))$, $\frac{\partial h}{\partial t_n}(0) = 0$ and that $\frac{\partial k}{\partial t_n} \in \frac{\partial h}{\partial t_n}\mathcal{O}_n$. Since $\dim_{\mathbb{K}} \mathcal{O}_n / \frac{\partial h}{\partial t_n}\mathcal{O}_n = \infty$, there exists a monomial function $\ell(t)$ of degree $\geq s$ such that $\ell \notin \frac{\partial h}{\partial t_n}\mathcal{O}_n$. Then, by taking a monomial $m(t)$ with $\frac{\partial m}{\partial t_n} = \ell$, we may put $g = (t_1, \dots, t_{n-1}, h, k + m)$, which is not a frontal. $\square$

Example 1.5.5 Recall that a smooth map-germ $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ is called a **cuspidal edge** if f is right-left equivalent to the germ $(u, v) \mapsto (u, \frac{1}{2}v^2, \frac{1}{6}v^3)$.

Since $d(\frac{1}{6}v^3) = \frac{1}{2}v^2 dv$ and $d(\frac{1}{2}v^2) = v dv$, we can take $(\nu_1, \nu_2, \nu_3) = (0, -\frac{1}{2}v, 1)$. Therefore any cuspidal edge is a front. Moreover $g = (u, \frac{1}{2}v^2, \frac{1}{6}v^3 + cu^{r-1}v)$ is not a frontal for any $r \geq 2$ and $c \neq 0$.

In what follows in this subsection, we treat only the case $n = 2, m = 3$. First we observe

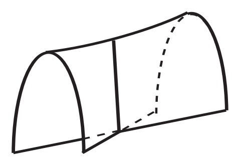
Lemma 1.5.6 (1) *Any* 1*-jet is a frontal jet.*
(2) *Any jet $z \in J^r(2, 3)$ of corank* 0 *i.e. immersive jet is a frontal jet. Moreover any germ f with $j^r f(0) = z$ is a frontal. (z is “absolutely frontal".)*

Proof (1) Any 1-jet is realised by a linear map, which is a frontal. (2) states simply that any immersion is a frontal. □

By simple calculations we have

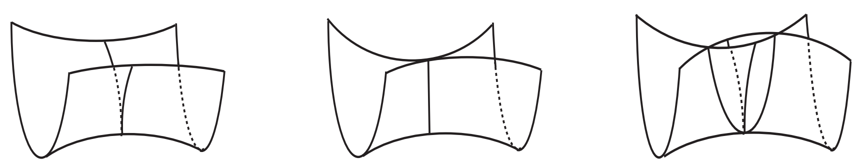
Proposition 1.5.7 *2-jets of corank 1 in $J^2(2,3)$ are classified up to $\mathcal{A}^2$-equivalence into the following 4-classes:* $j^2(u,v,0)(0)$, $j^2(u,\frac{1}{2}v^2,0)(0)$, $j^2(u,uv,0)(0)$, *and* $j^2(u,\frac{1}{2}v^2,uv)(0)$: Whitney umbrella. □

A 2-jet $z \in J^2(2,3)$ of corank 1 is a frontal jet if and only if z is not $\mathcal{A}^2$-equivalent to that of Whitney umbrella.



Whitney umbrella.

Proposition 1.5.8 *Let $z = j^3f(0) \in J^3(2,3)$. Suppose $j^2f(0)$ is $\mathcal{A}^2$-equivalent to $j^2(u,\frac{1}{2}v^2,0)(0)$ (2-jet of space fold). Then z is $\mathcal{A}^3$-equivalent to*
$j^3(u,\frac{1}{2}v^2,0)(0)$ *: frontal,*
$j^3(u,\frac{1}{2}v^2,\frac{1}{6}v^3)(0)$ *: 3-jet of cuspidal edge, frontal,*
$j^3(u,\frac{1}{2}v^2,\frac{1}{2}u^2v)(0)$ *: non-frontal, or*
$j^3(u,\frac{1}{2}v^2,\frac{1}{6}v^3 \pm \frac{1}{2}u^2v)(0)$ *: 3-jet of Mond's $S_1^\pm$([96]), non-frontal.* □



Bifurcation from Mond's S_1^+ to Mond's S_1^-.

Let $z = j^rf(0) \in J^r(2,3)$, $f(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v))$. We set, for $i,j \in \{1,2,3\}$,

$$\lambda_{ij} := f_{iu}f_{jv} - f_{iv}f_{ju}.$$

Note that the $(r-1)$-jet of λ_{ij} depends only on r-jet z of f.

Lemma 1.5.9 *Let $z = j^2f(0) \in J^2(2,3) = \mathbb{R}^{15}$ be a jet of corank 1. Then z is a frontal jet if and only if there exists a permutation (i,j,k) of $(1,2,3)$ such that* $\left(\lambda_{iju}\lambda_{ikv} - \lambda_{ijv}\lambda_{iku}\right)(0) = 0$. □

Lemma 1.5.10 *Let $z = j^2f(0) \in J^2(2,3)$ be a jet of corank 2. Then z is a frontal jet if and only if*

$$\begin{vmatrix} f_{1uu} & f_{1uv} & f_{1vv} \\ f_{2uu} & f_{2uv} & f_{2vv} \\ f_{3uu} & f_{3uv} & f_{3vv} \end{vmatrix}(0) = 0$$

□

Definition 1.5.11 A 2-jet $z \in J^2(2,3)$ is called a **space fold 2-jet** if z is $\mathcal{A}^2$-equivalent to $j^2(u, \frac{1}{2}v^2, 0)(0)$.

Proposition 1.5.12 *A 2-jet $z \in J^2(2,3)$ is a space fold 2-jet if and only if there exists a permutation (i,j,k) of $(1,2,3)$ such that $\left(f_{iu}\lambda_{ijv} - f_{iv}\lambda_{iju}\right)(0) \neq 0$, $\left(\lambda_{iju}\lambda_{ikv} - \lambda_{ijv}\lambda_{iku}\right)(0) = 0$,*

Let $f = (f_1, f_2, f_3) : (\mathbb{K}^2, p) \to \mathbb{K}^3$ be a smooth map-germs. we put, for $i, j \in \{i, 2, 3\}$, $\lambda_{ij} := f_{iu}f_{jv} - f_{iv}f_{ju}$.

Proposition 1.5.13 *Let $z = j^3 f(0) \in J^3(2,3)$. Suppose $j^2 f(0)$ is $\mathcal{A}^2$-equivalent to $j^2(u, \frac{1}{2}v^2, 0)(0)$ (space fold 2-jet). Then the 3-jet z is a frontal jet if and only if there exists a permutation (i,j,k) of $(1,2,3)$ such that*

$$\begin{aligned}&\left(\lambda_{ijv}\lambda_{iku}\lambda_{ijuv} + \lambda_{ijv}^2\lambda_{ikuu} + \lambda_{iju}\lambda_{ikv}\lambda_{ijuv} + \lambda_{iju}^2\lambda_{ikvv}\right)(0)\\ &= \left(\lambda_{ijv}\lambda_{ikv}\lambda_{ijuu} + 2\lambda_{iju}\lambda_{ijv}\lambda_{ikuv} + \lambda_{iju}\lambda_{iku}\lambda_{ijvv}\right)(0)\end{aligned}$$

□

We set $\eta_i = -f_{iv}\frac{\partial}{\partial u} + f_{iu}\frac{\partial}{\partial v}$.

By rewriting Kokubu-Rossman-Saji-Umehara-Yamada's recognition theorem of cuspidal edges ([85]), we have

Proposition 1.5.14 *Let $f : (\mathbb{K}^2, 0) \to \mathbb{K}^3$ be a frontal-germ. Then f is a cuspidal edge, i.e. right-left equivalent to the $(\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$, $(u, v) \mapsto (u, \frac{1}{2}v^2, \frac{1}{6}v^3)$, if and only if the 3-jets of f at 0 satisfies the following conditions :*
There exists a permutation (i,j,k) of $(1,2,3)$ such that $(*1i)$ $(f_{iu}, f_{iv})(0) \neq (0,0)$, $(*1ij)$ $\lambda_{ij}(0) = 0$, $(*1ik)$ $\lambda_{ik}(0) = 0$, $(*2ij)$ $\eta_i\lambda_{ij}(0) \neq 0$,
$(*3ij,jk)$ $\{(\eta_i\lambda_{ij})(\eta_i\eta_i\lambda_{jk}) - (\eta_i\lambda_{jk})(\eta_i\eta_i\lambda_{ij})\}(0) \neq 0$, or
$(*3ij,ik)$ $\{(\eta_i\lambda_{ij})(\eta_i\eta_i\lambda_{ik}) - (\eta_i\lambda_{ik})(\eta_i\eta_i\lambda_{ij})\}(0) \neq 0$. □

Proposition 1.5.15 *A 3-jet $z = j^3 f(0) \in J^3(2,3)$ is the jet of a cuspidal edge if and only if the following conditions $(*)$:*
There exists a permutation (i,j,k) of $(1,2,3)$ such that $(*1i)$ $(f_{iu}, f_{iv})(0) \neq (0,0)$, $(*1ij)$ $\lambda_{ij}(0) = 0$, $(*1ik)$ $\lambda_{ik}(0) = 0$, $(*2ij)$ $\eta_i\lambda_{ij}(0) \neq 0$, $(*2ijk)$ $\left(\lambda_{iju}\lambda_{ikv} - \lambda_{ijv}\lambda_{iku}\right)(0) = 0$,
$(*3ijk)$ $(\lambda_{ijv}\lambda_{iku}\lambda_{ijuv} + \lambda_{ijv}^2\lambda_{ikuu} + \lambda_{iju}\lambda_{ikv}\lambda_{ijuv} + \lambda_{iju}^2\lambda_{ikvv})(0)$
$= \left(\lambda_{ijv}\lambda_{ikv}\lambda_{ijuu} + 2\lambda_{iju}\lambda_{ijv}\lambda_{ikuv} + \lambda_{iju}\lambda_{iku}\lambda_{ijvv}\right)(0)$, and
$(*3ij,jk)$ $\{(\eta_i\lambda_{ij})(\eta_i\eta_i\lambda_{jk}) - (\eta_i\lambda_{jk})(\eta_i\eta_i\lambda_{ij})\}(0) \neq 0$, or
$(*3ij,ik)$ $\{(\eta_i\lambda_{ij})(\eta_i\eta_i\lambda_{ik}) - (\eta_i\lambda_{ik})(\eta_i\eta_i\lambda_{ij})\}(0) \neq 0$. □

Then we have

Corollary 1.5.16 *The subset of 3-jets of cuspidal edges in $J^3(2,3) = \mathbb{K}^{27}$,*

$$\{z = j^3 f(0) \in J^3(2,3) \mid f : (\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0) \text{ is any frontal-germ}\},$$

is a semi-algebraic set of $J^3(2,3)$*, which can be explicitly expressed as a union of at most* 12 *sets defined by systems of algebraic equalities and non-equalities of degree at most* 7. □

1.5.3 Cofrontals

Any frontal map-germ $f : (\mathbb{K}^n, a) \to (\mathbb{K}^m, b)$, $n \leq m$, has the characteristic property that the Jacobi ideal J_f is principal (Propositions 1.2.12, 1.3.23). Then it is natural to consider the class of map-germs $f : (\mathbb{K}^n, a) \to (\mathbb{K}^m, b)$ with principal Jacobi ideal in the case $n \geq m$,

A related notion has been introduced in [60], which we are going to review briefly. For details consult the original paper [60]. We treat only the case $\mathbb{K} = \mathbb{R}$.

Let N, M be smooth manifolds of dimension n and m respectively with $n \geq m$.

Definition 1.5.17 (Cofrontal map-germ, kernel field.) A map-germ $f : (N, a) \to (M, b)$ is called a **cofrontal** if there exists an integrable vector-subbundle $K = K_f$ of TN of corank m which satisfies the condition $(K_f)_x \subseteq \mathrm{Ker}(T_x f)$ for any $x \in (N, a)$.

Then K is called a **kernel field** or a **co-Legendre field** of the cofrontal f.

We have imposed the integrability condition on K in addition. Recall that a subbundle (distribution) $K \subset TN$ is called integrable if $[K, K] \subset K$ for the Lie bracket $[\ ,\ ]$. The kernel field is regarded as a section $K : (N, a) \to \mathrm{Gr}(n - m, TN)$ satisfying $(df)(K_x) = \{0\}$, $x \in (N, a)$.

If f is proper or fair (Definition 1.5.27), i.e. the singular locus of the confrontal f has no interior point nearby $a \in N$, then the integrability of the germ K follows automatically. In this case, moreover we have that K is uniquely determined by the cofrontal f (Lemma 1.5.29).

In some sense, cofrontal is the dual notion to frontal: Frontals are mappings such that the images of differentials are well-behaved, and cofrontals are mappings such that the kernels of differentials are well-behaved.

Definition 1.5.18 (Cofrontal mapping.) A global mapping $f : N \to M$ is called a **cofrontal** if all germs $f_a : (N, a) \to (M, f(a))$ of f at every $a \in N$ are cofrontal.

Important examples of cofrontals are obtained as mappings which are constant along Seifert fibres ([10], cf. Example 1.5.30).

We see that frontals and cofrontals are not a stable mapping except for the trivial cases, immersions and submersions and far from generic classes in the space of all C^∞ mappings. Nevertheless we see that they enjoy rather interesting properties to be studied. For example, we see that any smooth map is approximated by a frontal or a cofrontal in C^0-topology, at least if the source manifold is compact (Proposition 1.5.31).

Example 1.5.19 (1) Any immersion is a frontal. The Legendre lift is given by $\widetilde{f} := (T_t f(T_t N))_{t \in (N,a)}$. Any submersion is a cofrontal. The kernel field K is given by $K := (\mathrm{Ker} T_x f))_{x \in (N,a)}$.

(2) Any map-germ $(N, a) \to (M, b)$ between same dimensional manifolds ($n = m$) is a frontal and a cofrontal simultaneously. In fact the Legendre lift is given by $\widetilde{f}(t) := T_{f(t)}M, t \in (N, a)$ and the kernel field K is given by the zero-section of TN.

(3) Any constant map-germ $(N, a) \to (M, b)$ is a frontal if $n \leq m$ and a cofrontal if $n \geq m$. In fact we can take any family of n-planes along the germ as a Legendre lift and any subbundle $K \subset TN$ of rank $n - m$ as a kernel field.

We have that cofrontals have a common characteristic property to frontals (see also Propositions 1.2.12 and 1.3.23).

Proposition 1.5.20 (Criterion of cofrontality, [60], Lemma 2.3 of [61]) *Let $f : (N, a) \to (M, b)$ be a map-germ with $n = \dim(N) \geq m = \dim(M)$. If f is a cofrontal, then there exists a germ of submersion $\pi : (N, a) \to (\overline{N}, \overline{a})$ to an m-dimensional manifold $\overline{N}$ and a smooth map-germ $\overline{f} : (\overline{N}, \overline{a}) \to (M, b)$ such that $f = \overline{f} \circ \pi$. Moreover the Jacobi ideal J_f of f is principal, i.e. it is generated by one element. In fact J_f is generated by $\lambda = \pi^*(\overline{\lambda})$ for the Jacobian determinant $\overline{\lambda}$ of $\overline{f}$.*

Conversely, if the Jacobi ideal J_f is principal and the singular locus

$$S(f) = \{x \in (N, a) \mid \mathrm{rank}(T_x f : T_x N \to T_{f(x)}M) < m\}$$

of f is nowhere dense in (N, a), then f is a cofrontal.

Definition 1.5.21 (Reductions of cofrontals.) We call $\overline{f}$ a **reduction** of the cofrontal-germ f. A germ of cofrontal $f : (N, a) \to (M, b)$ is called **reduction-finite** if a reduction $\overline{f} : (\overline{N}, \overline{a}) \to (M, b)$ of f is $\mathcal{K}$-finite (or finite briefly), i.e. the dimension of $Q_{\overline{f}} := \mathcal{O}_{\overline{N},\overline{a}}/\overline{f}^*(m_b)$ is finite, where $\overline{f}^* : \mathcal{O}_{M,b} \to \mathcal{O}_{\overline{N},\overline{a}}$ is the $\mathbb{R}$-algebra homomorphism defined by $\overline{f}^*(h) = h \circ \overline{f}$, and $m_b \subset \mathcal{O}_{M,b}$ is the maximal ideal of function-germs vanishing at b (see [93][40][118][9]).

The notion of reductions can be considered as the dual notion to openings (§1.4).

Remark 1.5.22 If $\overline{f} : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ is $\mathcal{K}$-finite, then the zero set of $\overline{f}$ is isolated and any nearby germ of $\overline{f}$ has the same property. The number of fibres of $\overline{f}$ is uniformly bounded by $\dim(Q_{\overline{f}})$ (Propositions 2.2, 2,4 of Ch.VII in [40], see also [22][84]).

Regarding the importance of Proposition 1.5.20 we repeat its proof here.

Proof of Proposition 1.5.20. Let f be a cofrontal and K be a kernel field of f. Since K is integrable subbundle of TN of rank $n - m$, there exists a submersion $\pi : (N, a) \to (\mathbb{R}^m, 0)$ such that $K_x = \mathrm{Ker}(\pi_* : T_x N \to T_{\pi(x)}\mathbb{R}^m)$ for any $x \in (N, a)$, i.e. π-fibres form the foliation induced by K. Take any curve $\gamma : (\mathbb{R}, 0) \to N$ in a fibre of π. Then $(f \circ \gamma)'(t) = (T_{\gamma(t)}f)(\gamma'(t)) = 0$. Therefore f is constant along the curve γ. Hence f is constant on π-fibres. Then there exists a map-germ $\overline{f} : (\mathbb{R}^m, 0) \to (M, b)$ such that $f = \overline{f} \circ \pi$. Take a smooth section $s : (\mathbb{R}^m, 0) \to (N, a)$. Then $\overline{f} = \overline{f} \circ \pi \circ s = f \circ s$. Therefore $\overline{f}$ is a smooth map-germ.

Take a system of local coordinates $x_1, \dots, x_m, x_{m+1}, \dots, x_n$ of N around a such that π is given by $\pi(x_1, \dots, x_m, x_{m+1}, \dots, x_n) = (x_1, \dots, x_m)$, and therefore K_x is generated by $\partial/\partial x_{m+1}, \dots, \partial/\partial x_n$ in T_xN. Then f is expressed as

$$f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_m), \dots, f_m(x_1, \dots, x_m)).$$

Then J_f is generated by one element

$$\det(\partial f_i/\partial x_j)_{1\le i,j\le m} = \pi^*(\det(\partial \overline{f}_i/\partial x_j)_{1\le i,j\le m}) = \pi^*(\overline{\lambda})$$

and therefore J_f is a principal ideal in $\mathcal{O}_{N,a}$.

Conversely suppose J_f is a principal ideal generated by one element $\lambda \in J_f$ and $S(f)$ is nowhere dense. Denote by Γ the set of subsets $I \subseteq \{1, 2, \dots, n\}$ with $\#(I) = m$. For a map-germ $f : (N, a) \to (M, b), n \ge m$ and $I \in \Gamma$, we set $D_I = \det(\partial f_i/\partial x_j)_{1\le i\le m, j\in I}$ for some coordinates $x_1, \dots, x_n$ of (N, a) and $y_1, \dots, y_m$ of (M, b) with $f_i = y_i \circ f$. For any $I \in \Gamma$, there exists $h_I \in \mathcal{O}_a$ such that $D_I = k_I\lambda$. Since $S(f)$ is nowhere dense, there exists $I_0 \in \Gamma$ such that $D_{I_0} \ne 0$. Since $\lambda \in J_f$, there exists $\ell_I \in \mathcal{O}_a$ for any $I \in \Gamma$ such that $\lambda = \sum_{I\in\Gamma} \ell_I D_I$. Then $(1 - \sum_{I\in\Gamma} \ell_I k_I)\lambda = 0$. If $k_I(a) = 0$ for any $I \in \Gamma$, then $1 - \sum_{I\in\Gamma} \ell_I k_I$ is invertible in $\mathcal{O}_a$, therefore $\lambda = 0$ and then we have $J_f = 0$. This contradicts to the assumption that $S(f)$ is nowhere dense. Hence there exists $I_0 \in \Gamma$ such that $(\ell_{I_0} k_{I_0})(a) \ne 0$. Then $k_{I_0}(a) \ne 0$. Therefore J_f is generated by D_{I_0}. Hence $D_I = h_I D_{I_0}$ for any $I \in \Gamma$ with $h_{I_0}(a) = 1$. Then the Plücker-Grassmann coordinates $(h_I)_{I\in\Gamma}$ give a smooth section $K : (\mathbb{R}^n, a) \to \mathrm{Gr}(n - m, TN) \cong \mathrm{Gr}(m, T^*\mathbb{R}^n)$, which is regarded as a subbundle $K \subseteq TN$ of rank $n-m$ and $K_x \subseteq \mathrm{Ker}(T_x f)$ for any $x \in (N, a)$. Moreover K_x coincides with $\mathrm{Ker}(T_x f)$ for $x \in (N \setminus S(f), a)$ and therefore K is integrable outside of $S(f)$. Since $S(f)$ is nowhere dense, K is integrable. Thus f is a cofrontal map-germ with the kernel field K. □

Corollary 1.5.23 *Let $f : (N, a) \to (M, b)$ be a map-germ. Suppose f is analytic and $J_f \ne 0$. Then f is a frontal or a cofrontal if and only if J_f is a principal ideal.*

Proof By Proposition1.3.23 and Lemma 1.5.20, if f is a frontal or a cofrontal, then J_f is principal. If J_f is principal, $J_f \ne 0$ and f is analytic, then $S(f)$ is nowhere dense. Thus f is a frontal if $n \le m$ or a cofrontal if $n \ge m$. □

Example 1.5.24 Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be the map-germ given by $f(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2, 0)$. Then f is analytic and $J_f = 0$ is principal. However f is not a cofrontal. In fact, suppose f is a cofrontal and K a kernel field of f of rank 1. Let

$$\xi(x) = \xi_1(x)\partial/\partial x_1 + \xi_2(x)\partial/\partial x_2 + \xi_3(x)\partial/\partial x_3,\ \xi(0) \ne 0,$$

be a generator of K. Then $\xi_1(x)x_1 + \xi_2(x)x_2 + \xi_3(x)x_3$ is identically zero in a neighborhood of 0 in $\mathbb{R}^3$. In particular we have $\xi_1(x_1, 0, 0)x_1 = 0$ and therefore $\xi_1(x_1, 0, 0) = 0$, so $\xi_1(0, 0, 0) = 0$. Similarly we have also $\xi_2(0, 0, 0) = 0$ and $\xi_3(0, 0, 0) = 0$. This leads a contradiction.

Definition 1.5.25 (Jacobians of frontals and cofrontals.) Let $f : (N, a) \to (M, b)$ be a frontal or a cofrontal. Then a generator $\lambda \in \mathcal{O}_a$ of J_f is called a **Jacobian** (or a **singularity identifier**) of the cofrontal f, which is uniquely determined from f up to multiplication of a unit in $\mathcal{O}_a$.

Remark 1.5.26 Let $f : (N, a) \to (M, b)$ be a cofrontal and K a kernel field of f. Set

$$K_x^{\perp} := \{\alpha \in T_x^* N \mid \alpha(v) = 0 \text{ for any } v \in K_x\}.$$

Then $K^{\perp}$ is a germ of subbundle of the cotangent bundle T^*N of rank m. Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be a local frame of $K^{\perp}$. Then there is a unique $\lambda \in \mathcal{O}_a$ such that

$$df_1 \wedge df_2 \wedge \cdots \wedge df_m = \lambda \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m.$$

Then λ generates J_f and therefore λ is a Jacobian of the cofrontal f.

Definition 1.5.27 (Fair frontals and cofrontals.) A frontal or a cofrontal $f : (N, a) \to (M, b)$ is called **fair** or **proper** if the singular locus $S(f)$ is nowhere dense in (N, a).

Remark 1.5.28 A cofrontal f is fair if and only if a reduction $\overline{f}$ (Definition 1.5.21) is fair. In fact if $f = \overline{f} \circ \pi$ for a submersion-germ $\pi : (N, a) \to (\mathbb{R}^m, 0)$, we have $S(f) = \pi^{-1}(S(\overline{f}))$, and therefore $S(f)$ is nowhere dense in (N, a) if and only if $S(\overline{f})$ is nowhere dense in $(\mathbb{R}^m, 0)$. If a cofrontal f is reduction-finite (Definition 1.5.21), then f is fair, since a reduction $\overline{f}$ is $\mathcal{K}$-finite so is necessarily fair.

Lemma 1.5.29 *Let $f : (N, a) \to (M, b)$ be a fair cofrontal or* $\dim(N) = \dim(M)$. *Then the kernel filed K of f is uniquely determined and the reduction $\overline{f}$ of f (Definition 1.5.21) is uniquely determined up to right equivalence.* □

Example 1.5.30 (1) Any submersion is a cofrontal. Any immersion is a frontal.

(2) Any constant mapping $N \to M$ is a cofrontal of a frontal depending on $\dim(N) \geq \dim(M)$ or $\dim(N) \leq \dim(M)$.

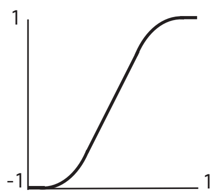
(3) Let $\mathcal{F}$ be a foliation of codimension m on a manifold N of dimension n. If a mapping $f : N^n \to M^m$ is constant on any leaf of $\mathcal{F}$, then f is a cofrontal.

(4) As a motivating example from symplectic geometry, consider a Lagrangian foliation $\mathcal{L}$ on a symplectic manifold N^{2n} and a system of functions $f_1, \dots, f_n$ on N. Then $f = (f_1, \dots, f_n) : N \to \mathbb{R}^n$ is a cofrontal if f is constant along each leaf of $\mathcal{L}$.

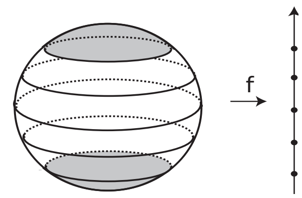
We observe “unfair" (co)frontal maps are not so restrictive in topological or homotopical sense. The following can be shown by the standard theory of stratifications.

Proposition 1.5.31 (C^0-approximation. [60]) *Let N be compact. Then any smooth (C^∞) map $f : N \to M$ is C^0-approximated by a frontal or a cofrontal $g : N \to M$, i.e., for any open neighborhood $\mathcal{U}$ of f, for C^0-topology on the space $C^\infty(N, M)$ of all C^∞ mappings, there exists a frontal or a cofrontal g which belongs to $\mathcal{U}$. Moreover any smooth map $f : N \to M$ is homotopic to a frontal or a cofrontal $g : N \to M$.* □

Example 1.5.32 Let $S^2 \subset \mathbb{R}^3$ be the unit sphere and $g : S^2 \to \mathbb{R}$ the height function, i.e. $g(x_1, x_2, x_3) = x_3$. Then g is not a cofrontal. Let $\varepsilon > 0$. Let $\varphi : [-1, 1] \to [-1, 1]$ be any smooth map satisfying that $\varphi(y) = -1(-1 \leq y - 1 + \varepsilon), \varphi(y) = 1(1 - \varepsilon \leq y \leq 1)$, and that φ is a diffeomorphism from $(-1 + \varepsilon, 1 - \varepsilon)$ to $(-1, 1)$. Then $f = \varphi \circ g$ is a cofrontal. See the figure: In the right picture, f restricted to the north (resp. south) gray part is constant.



graph of φ.



A cofrontal on the sphere.

Note that f can be taken to be arbitrarily near g in C^0-topology.

Similar construction can be applied to any proper Morse function $g : N \to \mathbb{R}$ and we have a cofrontal which is a C^0-approximation to g.

Contrary to the case of “unfair" cofrontals, the following lemma show that the sauce space of a *fair* cofrontal must be very restrictive.

Lemma 1.5.33 ([60]) *Let N be compact, $f : N \to M$ a fair cofrontal and K the kernel field of f. Let $\mathcal{F}$ be the foliation induced by the integrable subbundle K of TN of rank $n - m$. Then the closure of any leaf of $\mathcal{F}$ is nowhere dense in N.* □

A classification results of cofrontals of fibre-dimension one is given in [60].

1.6 Problems

1.6.1 Problems on frontals

It is interesting to study frontals from differential topology, though we have not discussed on global results on frontals in §1.2. For example, we may impose, as a problem, to study topological embeddings of a given manifold into Euclidean space as frontal hypersurfaces.

Problem 1.6.1 (Global frontal mappings to Euclidean space.) Find a condition on a C^∞ manifold N for the existence of a proper frontal mapping $f : N \to \mathbb{R}^{n+1}$. Is it able to take f to be of integral corank ≤ 1 (and frontally stable) ? Is it possible to take f to be a topological embedding ?

See also Definition 1.6.9.

In a short but important paper [37], A. B. Givental gave a conjecture on singular Lagrangian surfaces in the symplectic $\mathbb{R}^4 = T^*\mathbb{R}^2$ that any isotropic map from a surface to $\mathbb{R}^4$ is approximated by an isotropic map of corank ≤ 1 everywhere on the surface.

V. M. Zakalyukin gave a substantial contribution to the conjecture in [130]. However, as far as author knows, Givental's conjecture is not solved yet. The conjecture is generalised in higher dimensional cases naturally. The conjecture for Lagrangian (isotropic) maps in symplectic geometry have Legendre (integral) counter-part in contact geometry as follows:

Conjecture 1.6.2 (Frontal-Legendre version of generalised Givental's conjecture)
(1) Let $f : (\mathbb{K}^n, a) \to (\mathbb{K}^{n+1}, b)$ be a frontal-germ of integral corank ≥ 2, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then there exists a frontal deformation $F : (\mathbb{K}^n \times \mathbb{K}, (a, 0)) \to (\mathbb{K}^{n+1}, b)$ of f, $F = (f_u)$ such that $\mathrm{corank}(\widetilde{f_u}) \leq 1$, for any sufficiently small $u \neq 0$, for a representative of F.
(2) Let $f : N \to \mathbb{R}^{n+1}$ be a frontal mapping. Suppose f has a global Legendre lift $\widetilde{f} : N \to PT^*\mathbb{R}^{n+1}$. Then, for any neighborhood U of $\widetilde{f}$ for Whitney C^∞ topology ([40]), f is approximated by a frontal $f' : N \to \mathbb{R}^{n+1}$ such that f' has a Legendre lift $\widetilde{f'}$ is of corank ≤ 1 and $\widetilde{f'} \in U$.

In other words, Conjecture 1.6.2 (1) claims that it is possible to *eliminate singularities of integral corank* ≥ 2 by a frontal deformation. The conjecture (2) claims that *any frontally stable frontal is of integral corank* ≤ 1 *necessarily*. See also [38, 39].

Problem 1.6.3 Prove or disprove the generalised Givental's conjecture of frontal-Legendre version in any form as above.

In Theorem 1.2.33, we have given the characterisation on frontal stability of frontals of integral corank ≤ 1.

Problem 1.6.4 (Characterisation of frontal stability in general.) Give the characterisation of frontal stability in the case of integral corank ≥ 2.

In the papers [100, 101], the notion of frontalisations is introduced. It is interesting to find an intrinsic formulation of "frontalisations" of map-germs.

Problem 1.6.5 (Geometric formulation of frontalisations.) Give any intrinsic formulation of frontalisations.

1.6.2 Problems on general frontals

In §1.3.3, we have discussed on frontal stability for frontal multi-germs in the case of corank ≤ 1.

Problem 1.6.6 (Characterisation of frontal stability via multi-transversality.) Describe Theorem 1.3.17 on the infinitesimal characterisation of frontal stability in general case via multi-transversality.

Problem 1.6.7 (General characterisation of frontal stability.) Give a characterisation of frontal stability of frontal germs $(\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ in the case of corank ≥ 2 with $m \geq n + 2$.

In Example 1.3.3, we have introduced two frontals the **product of cusps** and the **complex cusp**. The complex cusp $\mathbb{C} = \mathbb{R}^2 \to \mathbb{C}^2 = \mathbb{R}^4$ is frontally stable in holomorphic category $\mathbb{K} = \mathbb{C}$. The author assumes also the product of cusps possesses a kind of stability. Then the following problem concerns on their frontal stability within frontals $\mathbb{R}^2 \to \mathbb{R}^4$.

Problem 1.6.8 (Frontal stability of the product of cusps and the complex cusps.) Prove or disprove that the product of cusps and the complex cusp are frontally stable.

Compare with evolutions of fronts $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$. It is known a projection of the product of cusps (resp. the complex cusp) to $\mathbb{R}^3$ is never frontally stable. See [3].

Here we propose a definition of *global* frontal stability in general cases.

Definition 1.6.9 (Global frontal stability of generalised frontals.) Let N, M be real C^∞ manifolds of dimension n, m respectively with $n \leq m$. Let $f : N \to M$ be a C^∞ frontal mapping with $n \leq m$. (see Definition 1.3.7). Then f is called **frontally stable** if there exist an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of N, Legendre lifts $\widetilde{f_\lambda} : U_\lambda \to \mathrm{Gr}(n, TM)$ of $f|_{U_\lambda}$, $r \in \mathbb{N}$ and open neighbourhoods W_λ of $j^r \widetilde{f_\lambda}$ in $J^r(U_\lambda, \mathrm{Gr}(n, TM))$, $\lambda \in \Lambda$, such that if a frontal map $g : N \to M$ satisfies that for any point $t_0 \in N$, there exist an open neighborhood V_0 of t with $V_0 \subset U_\lambda$ for some $\lambda \in \Lambda$, and a Legendre lift $\widetilde{g}$ with $j^r \widetilde{g}(V_0) \subset U_\lambda$, then g is right-left equivalent to f.

Problem 1.6.10 (Characterisation of global frontal stability.) Give a characterisation of *global* frontal stability of proper (= inverse image of any compact set is compact) frontal mapping in infinitesimal languages and/or in terms of multi-transversality.

Problem 1.6.11 (Existence of global frontally stable frontals.) Given C^∞ manifolds N, M, find a condition on the existence of a frontally stable C^∞ mapping $f : N \to M$.

Also it interesting to ask on frontals from a differential geometric aspect. A symmetric positive semi-definite $(0, 2)$-tensor field on a manifold is called a **singular metric**.

Problem 1.6.12 (Realisations of singular metrics by frontals). Given a germ of singular metric g on $(\mathbb{R}^n, a)$, find a frontal $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ and a Riemannian (resp. Euclidean) metric G on $(\mathbb{R}^m, b)$ such that the pull-back metric f^*G is equal to g. Is it possible take $m = n$ or $m = n + 1$, etc. ? What is the minimum of embedding dimensions m ? Is it possible to take the frontal f *frontally stable* for the geometric realisation ?

Let $f : N \to M$ be a frontal from an n-dimensional manifold N to an m-dimensional manifold with $n \leq m$. Suppose there exists a global Legendre lift $\widetilde{f} : N \to \mathrm{Gr}(n, TM)$. Consider the pull-back bundle

$$f^*TM := \{(t, v) \mid t \in N, v \in T_{f(t)}M\}$$

over N by f. Then we have the subbundle

$$D := \{(t, v) \in f^*TM \mid v \in \widetilde{f}(t)\}$$

of f^*TM of rank n. Moreover we have the bundle homomorphism $df : TN \to D$ defined by $df(t, w) := (t, df(w))$ for any $(t, w) \in TN$. The related notion of coherent tangent bundles is introduced in [113] in the context of differential geometry of singular surfaces. See also [111]. Then we present

Problem 1.6.13 (Realisation of vector bundles by frontals). Characterise a bundle homomorphism $\varphi : TN \to D$ to a vector bundle D over N of rank $\dim(N)$ which is realised by a frontal $f : N \to M$ such that $\varphi = df$. Consider the similar problem when D is endowed with a metric.

1.6.3 Problems on openings

Here we mention several problems related to openings.

In §1.4, the notion of extendability of unfoldings is introduced (Definition 1.4.22). In Corollary 1.4.23, we have seen that if corank of f is at most one, then any unfolding of f is extendable. In §1.4.3, we have seen that there exist non-extendable unfoldings for map-germs of corank ≥ 2.

Problem 1.6.14 (Characterisation of extendable unfoldings). Given a map-germ of corank ≥ 2, find a sufficient and/or necessary condition that an unfolding of the map-germ is extendable.

The existence of versal openings of a smooth map-germ of corank ≥ 2 in C^∞ is still open.

Problem 1.6.15 (Existence of versal openings in general.) Study on the existence of versal openings of germs of corank ≥ 2 in the real C^∞ case.

We have given calculations of ramification modules of several typical examples of map-germs in §1.4.3. It is natural to expect these considerations will be useful to show the frontal stability of typical examples of frontals. For instance we propose

Problem 1.6.16 (Frontal stability and opennings.) Prove or dis-prove that the product of cusps (resp. the complex cusps) $(\mathbb{K}^2, 0) \to (\mathbb{K}^4, 0)$ (Example 1.3.3) is frontally stable from the aspect of openings, if possible.

1.6.4 Problems on other topics

Related to the study of parametrised Lagrangian maps, we consider the following construction.

Let $g : (\mathbb{K}^n, A) \to (\mathbb{K}^m, b)$ be a map-germ. Denote by $g^* : \mathcal{O}_{(\mathbb{K}^m,b)} \to \mathcal{O}_{(\mathbb{K}^n,A)}$, the induced $\mathbb{K}$-algebra homomorphism by g. Recall that the $g^*\mathcal{O}_{(\mathbb{K}^m,b)}$-subalgebra $\mathcal{R}_g$ of $\mathcal{O}_{(\mathbb{K}^n,A)}$ is defined by

$$\mathcal{R}_g = \{k \in \mathcal{O}_{(\mathbb{K}^n,A)} \mid dk \in \sum_{i=1}^{m} \mathcal{O}_{(\mathbb{K}^n,A)} \cdot dg_i\}.$$

We set $\mathcal{R}_g^{(-1)} = \mathcal{O}_{(\mathbb{K}^n,A)}$, $\mathcal{R}_g^{(0)} = \mathcal{R}_g$ and inductively,

$$\mathcal{R}_g^{(i)} = \{k \in \mathcal{O}_{(\mathbb{K}^n,A)} \mid dk \in \sum_{i=1}^{p} \mathcal{R}_g^{(i-1)} \cdot dg_i\},$$

$(i = 1, 2, \dots)$. Then we have the sequence of $g^*\mathcal{O}_{\mathbb{K}^m,b}$-subalgebras,

$$\mathcal{O}_{(\mathbb{K}^n,A)} = \mathcal{R}_g^{(-1)} \supseteq \mathcal{R}_g^{(0)} \supseteq \mathcal{R}_g^{(1)} \supseteq \cdots \supseteq g^*\mathcal{O}_{\mathbb{K}^m,b}.$$

Problem 1.6.17 (Characterisation of composite differentiable functions.) Given a map-germ, characterise composite differentiable functions in terms of restricted ramification modules as above. Is $\bigcap_{i=1}^{\infty} \mathcal{R}_g^{(i)}$ equal to $g^*\mathcal{O}_{(\mathbb{K}^m,B)}$?

In §1.5.2 we have given a result on jets which are realised by frontal map-germs, in particular, studied on the set of 3-jets in $J^3(\mathbb{K}^2, \mathbb{K}^3)$ which are realised by cuspidal edges. Then we pose

Problem 1.6.18 (Characterisation of frontal jets.) Characterise frontal jets explicitly. Namely, provide an explicit criterion whether a jet is frontal or not.

Problem 1.6.19 (Description of 4-jets of swallowtails,) Describe 4-jets which are realised by swallowtails.

Related to Proposition 1.5.4, we propose the following

Conjecture 1.6.20 (Existence of "dis-frontalisations"). Let $f : (\mathbb{K}^n, a) \to (\mathbb{K}^m, b)$ be a non-immersive frontal-germ and $s \in \mathbb{N}$. Then there exist $r \in \mathbb{N}$ and a smooth map-germ $g : (\mathbb{K}^n, a) \to (\mathbb{K}^m, b)$ such that $s \leq r$, $j^r g(a) = j^r f(a)$ and that g is *not* a frontal.

Problem 1.6.21 Prove or dis-prove the above Conjecture 1.6.20.

As is described in §1.5.3, the study on structures of cofrontal map-germs is reduced to the case of mappings between equi-dimensional manifolds.

Problem 1.6.22 (Classification of cofrontal singularities.) Apply the classification results of map-germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$, in particular in the case $m = 2$ (see [120, 107, 109, 83] for instance), to classifications of cofrontals.

In §1.2 and §1.3, we have studied on stability of frontals under deformations through frontals. Then we propose

Problem 1.6.23 (Cofrontal deformations.) Study on deformations of cofrontals and the stability of cofrontals.

In this survey we treat cofrontals only in the real case $\mathbb{K} = \mathbb{R}$ and give a global classification result of cofrontal mappings with one-dimensional fibres, $n = m + 1$.

Problem 1.6.24 (Cofrontals with higher dimensional fibres.) Study global cofrontals in cases $n \geq m + 2$.

For the local study on cofrontals in the complex case $\mathbb{K} = \mathbb{C}$ goes in parallel to that in the real case. Then we naturally ask

Problem 1.6.25 (Global complex cofrontal mappings.) Study on global complex analytic cofrontal mappings.

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Index