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On Topology of Real Algebraic Functions

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Several preliminary observations on the topological structures of real algebraic functions are explained.

The theory of real algebraic functions is one of centers of mathematics. On the other hand, topological study on real algebraic functions can be regarded as a natural extension of investigations on Hilbert's 16th problem.

In §1, a general problem is treated. As an example of concrete investigations, a problem on topology of real pencils of hypersurfaces is studed in §2. Several remarks and problems are collected in §3.

1. Topological types of algebraic functions.

1.1. Let P^n denote the complex projective n-space $\mathbb{C}P^n$. Consider a non-zero polynomial with complex coefficients:

$$\mathbb{F} = \mathbb{F}(x_0, x_1, \ldots, x_n;\ u_0, u_1, \ldots, u_n),$$

which is homogeneous of degree d, r with respect to $x = (x_0, \ldots, x_n)$, $u = (u_0, \ldots, u_n)$ respectively. The equation $\mathbb{F} = 0$ defines a (possibly reducible) subvariety $A = A[\mathbb{F}]$ of $P^n \times P^m$ of codimension one.

Our central object is the following diagram:

$$[\mathbb{F}] \qquad \begin{array}{ccc} A & \xrightarrow{\varphi} & P^m \\ {\scriptstyle\pi}\downarrow & & \\ P^n & & \end{array}$$

where $\pi = \pi[\mathbb{F}]$, $\varphi = \varphi[\mathbb{F}]$ are restrictions to $A \subset P^n \times P^m$ of the projections to P^n, P^m respectively. This diagram is regarded as a generalization (of a compactification) of a polynomial mapping of corank at most one.

Fundamental Problem: For fixed $(n,d;m,r)$, investigate the topological structures of diagrams $[\mathbb{F}]$.

An equivalence for diagrams $[\mathbb{F}]$, $[\mathbb{G}]$ is a pair $(H;h,h')$ of "isomorphisms" such that the diagram

$$\begin{array}{ccccc} P^n & \xleftarrow{\pi[\mathbb{F}]} & A[\mathbb{F}] & \xrightarrow{\varphi[\mathbb{F}]} & P^m \\ h\downarrow & & H\downarrow & & \downarrow h' \\ P^n & \xleftarrow[\pi[\mathbb{G}]]{} & A[\mathbb{G}] & \xrightarrow[\pi[\mathbb{G}]]{} & P^m \end{array}$$

commutes. Here $H = (h \times h')|A$.

The <u>topological equivalence</u> is defined by taking h, h' to be homeomorphisms.

The number of topological equivalence classes is continuum if $d \geqq 3$ and $r \geqq 3$. In fact, the family

$$(u_0x_0-u_1x_1)(u_0x_0-au_1x_1)(u_0x_0-bu_1x_1)$$

contains a continuous family of topological equivalence classes (cf. [2]).

1.2. Let us concentrate the case $m = 1$. Put $(u_0, u_1) = (\lambda, \mu)$ and

$$\mathbb{F} = \lambda^r F_0 + \lambda^{r-1}\mu F_1 + \dots + \lambda\mu^{r-1}F_{r-1} + \mu^r F_r,$$

where $F_i = F_i(x_0, x_1, \dots, x_n)$ are homogeneous polynomials of degree d $(i = 0, \dots, r)$.

Then the diagram $[\mathbb{F}]$ in (1.1) is called an <u>algebraic function</u> of type $(n,d;r)$.

To state results, we prepare several notions.

A <u>"marked" partition type</u> $J = \langle j_0; j_1, \dots, j_p\rangle$ of r is the equivalence class of a partition $r = j_0 + j_1 + \dots + j_p$ by positive integers $j_0, j_1, \dots, j_p$. Here two partitions

$$r = j_0 + j_1 + \dots + j_p = k_0 + k_1 + \dots + k_q$$

are equivalent if $j_0 = k_0$, $q = p$ and for some permutation σ of $\{1, \dots, p\}$, $k_{\sigma(i)} = j_i$ $(i = 1, \dots, p)$.

Put $B = B[\mathbb{F}] = \{[x] \in P^n \mid F_0(x) = F_1(x) = \dots = F_r(x) = 0\}$. For each point $a = ([x], [\lambda:\mu]) \in A[\mathbb{F}]$ with $[x] \notin B[\mathbb{F}]$, we denote by j_a the intersection number of $A[\mathbb{F}]$ and the line $[x] \times P^1$. Define $J_a = \langle j_a; j_{a_1}, \dots, j_{a_p}\rangle$ if the fiber $\pi^{-1}[x]$ consists of $p+1$ points $\{a, a_1, \dots, a_p\}$.

Then a decomposition $A[\mathbb{F}] = \pi^{-1}B[\mathbb{F}] \cup (\bigcup_J R_{\mathbb{F}}(J))$ is obtained, where $R_{\mathbb{F}}(J) = \{a \in A \mid J_a = J\}$ for each marked partition type J of r.

The ramification data of $\mathbb{F}$ are $\pi^{-1}(B[\mathbb{F}])$ and $R_{\mathbb{F}}(J)$ for any marked partition type J of r.

An <u>isotopy</u> of algebraic functions $[\mathbb{F}], [\mathbb{G}]$ of type $(n,d;r)$ is a pair (H_t, h'_t) of continuous one-parameter families of homeomorphisms of $P^n \times P^1$, P^1 respectively with parameter $t \in [0,1]$ such that

(i) H_t covers h'_t $(t \in [0,1])$.

(ii) $H_0 = id_{P^n \times P^1}$, $h'_0 = id_{P^1}$.

(iii) $H_1(A[\mathbb{F}]) = A[\mathbb{G}]$.

(iv) H_1 preserves ramification data, that is, H_1 maps $\pi^{-1}(B[\mathbb{F}])$ to $\pi^{-1}(B[\mathbb{G}])$ and $R_{\mathbb{F}}(J)$ to $R_{\mathbb{G}}(J)$ for any marked partition type J of r.

If there exists an isotopy of $[\mathbb{F}]$ and $[\mathbb{G}]$, then they are called <u>isotopic.</u>

Theorem 1. For each $(n,d;r)$, the set of isotopy classes of algebraic functions of type $(n,d;r)$ is finite.

Though this result seems essentially in [3], we give an outline of proof.

Let $\mathcal{H}$ denote the vector space of homogeneous polynomials $\mathbb{F} = \mathbb{F}(x_0,\ldots,x_n;\lambda,\mu)$ of degree d,r with respect to x, $u = (\lambda,\mu)$ respectively, and $P\mathcal{H}$ the projectification of $\mathcal{H}$; $P\mathcal{H} = (\mathcal{H}-0)/\mathbb{C}^*$.

Put $\mathcal{A} = \{([\mathbb{F}],[x],[u]) \in P\mathcal{H} \times P^n \times P^1 \mid \mathbb{F}(x,u) = 0\}$ and $R(J) = \{([\mathbb{F}],[x],[u]) \in \mathcal{A} \mid ([x],[u]) \in R_{\mathbb{F}}(J)\}$ for each marked partition type J of r. Each $R(J)$ is constructable in $\mathcal{A}$. Set $\mathcal{R} = \{R(J)\}$. Further put $\mathcal{B} = \{([\mathbb{F}],[x],[u]) \in \mathcal{A} \mid F_i(x,u)=0$ $(i = 0,\ldots,r)\}$. Thus a decomposition $\mathcal{A} = \mathcal{B} \cup (\cup \mathcal{R})$ is obtained.

Consider the diagram

$$\mathcal{A} \xrightarrow[\Phi]{\qquad} P\mathcal{H} \times P^1 \xrightarrow[\Gamma]{\qquad} P\mathcal{H},$$

where Φ, Γ are projections.

It suffices to construct a stratification $(X_{\mathcal{A}}, X_{P\mathcal{H}\times P^1}, X_{P\mathcal{H}})$ of $(\Phi;\Gamma)$ such that

(i) $X_{\mathcal{A}}$, $X_{P\mathcal{H}\times P^1}$ and $X_{P\mathcal{H}}$ are Whitney stratifications of $\mathcal{A}$, $P\mathcal{H}\times P^1$ and $P\mathcal{H}$ respectively.

(ii) For each $Z \in X_{P\mathcal{H}}$, $(X_{\mathcal{A}}, X_{P\mathcal{H}\times P^1})$ induces a Thom stratification $(X_{\mathcal{A}} \mid (\Phi\circ\Gamma)^{-1}Z,\ X_{P\mathcal{H}\times P^1} \mid \Gamma^{-1}Z)$ of $(\Phi;\Gamma) : (\Phi\circ\Gamma)^{-1}Z \longrightarrow \Gamma^{-1}Z \to Z$.

(iii) $\mathcal{B}$ and every $R(J) \in \mathcal{R}$ are stratified subsets of $X_{\mathcal{A}}$.

For this, the existence of "bonnes stratifications" is essential (cf.[3]).

1.3. Next assume $\mathbb{F}$ is a real polynomial (and $m = 1$). Then the diagram $[\mathbb{F}]$ has a natural real structure induced by the complex conjugations τ_n, τ_1 of P^n, P^1 respectively:

$$\begin{array}{ccccc}
P^n & \xleftarrow{\ \pi\ } & A[\mathbb{F}] & \xrightarrow{\ \varphi\ } & P^1 \\
\downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \tau_n\times\tau_1} & & \downarrow{\scriptstyle \tau_1} \\
P^n & \xleftarrow{\ \pi\ } & A[\mathbb{F}] & \xrightarrow{\ \varphi\ } & P^1
\end{array}$$

An isotopy (H_t, h'_t) of real algebraic functions $[\mathbb{F}]$, $[\mathbb{G}]$ is <u>equivariant</u> if $(\tau_n\times\tau_1)\circ H_t = H_t\circ(\tau_n\times\tau_1)$ for any $t\in[0,1]$.

If $[\mathbb{F}]$ and $[\mathbb{G}]$ have an equivariant isotopy, then they are called <u>equivariantly isotopic.</u>

Theorem 2. For each $(n,d;r)$, the set of equivariant isotopy classes of real algebraic functions of type (n,d;r) is finite.

For the proof, instead of the isotopy lemma, the equivariant isotopy lamma is used.

1.4. Put

$$S[\mathbb{F}] = \{([x],[u]) \in P^n \times P^1 \mid (\partial \mathbb{F}/\partial x_i)(x,u) = 0 \quad (i=0,\ldots,n)\},$$

$$T[\mathbb{F}] = \{([x],[u]) \in P^n \times P^1 \mid (\partial \mathbb{F}/\partial \lambda)(x,u) = (\partial \mathbb{F}/\partial \mu)(x,u) = 0\}.$$

$A[\mathbb{F}]$ is non-singular if and only if $S[\mathbb{F}] \cap T[\mathbb{F}] = \emptyset$.

Suppose $A[\mathbb{F}]$ is non-singular. Then $\varphi = \varphi[\mathbb{F}]: A[\mathbb{F}] \longrightarrow P^1$ is called <u>nice</u> if each critical point of φ is non-dedenerate and φ restricted to the critical locus is injective.

Let $\mathcal{H}$ denotes the vector space of polynomials $\mathbb{F} = \mathbb{F}(x,u)$ of type (n,d;r).

Proposition 1. The set of $[\mathbb{F}] \in P\mathcal{H}$ such that $A[\mathbb{F}]$ is non-singular and $\varphi[\mathbb{F}]$ is nice is a non-void Zariski open set in $P\mathcal{H}$ invariant under the complex conjugation of $P\mathcal{H}$.

Proof. Each $\mathbb{F} \in \mathcal{H}$ defines a map $\mathcal{A}'_{\mathbb{F}}: \mathbb{C}^2 \longrightarrow H$, where H is the vector space of homogeneous polynomials $F(x_0,\ldots,x_n)$ of degree d; $H = H^0(P^n, \mathcal{O}(-d))$. Let Z denote the set of $\mathbb{F} \in P\mathcal{H}$ such that $\mathcal{A}'^{-1}_{\mathbb{F}}(0) \neq 0$, or, $\mathcal{A}'^{-1}_{\mathbb{F}}(0) = 0$ and its projectification $\mathcal{A}_{\mathbb{F}}: P^1 \longrightarrow PH$ is transverse to the locus of singular hypersurfaces $D \subset PH$. Then Z is Zariski closed, and, $[\mathbb{F}] \in P\mathcal{H} - Z$ if and only if $A[\mathbb{F}]$ is non-singular and $\varphi[\mathbb{F}]$ is nice.

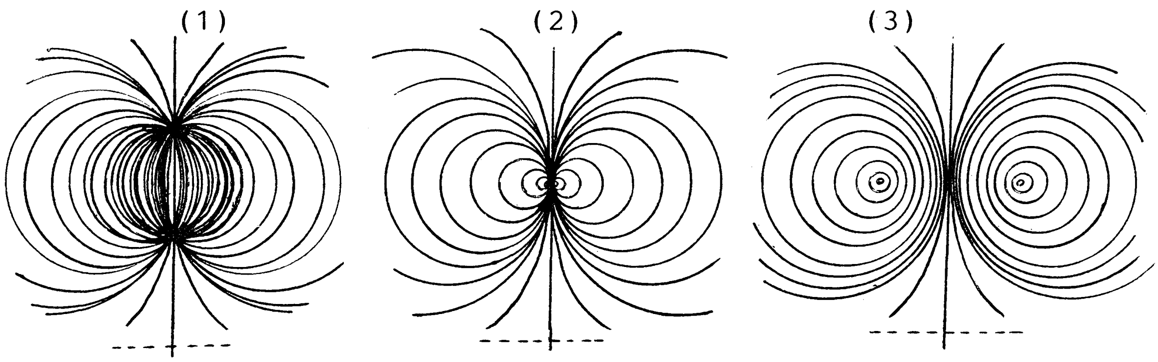
Proposition 2. If $A[\mathbb{F}]$ is non-singular and $\varphi[\mathbb{F}]$ has only isolated critical points, then $\sum_{x\in A} \mu_x(\varphi) = r(n+1)(d-1)^n$. Here μ denotes the Milnor number.

Proof. For $r = 1$, Proposotion 2 is proved in [5]. Then, for any r, Proposition 2 is a consequence of Bezout's theorem.

2. The number of singular points in a pencil of real plane curves

2.1. A (real) algebraic function $[\mathbb{F}]$ of type $(n,d;1)$ is called a (real) pencil of hypersurfaces in P^n of degree d.

Example 1. Three pencils of plane conics (n=1, d=2):



The numbers of singular points in pencils (1), (2), (3) are 1,2,3 respectively.

In [5], the domain of numbers of singular points in pencils of real plane curves of fixed degree is determined under a certain genericity condition.

2.2. ([5]) In real algebraic geometry, it is effective to estimate numerical invariants of a real algebraic object (e.g. a real algebraic variety, a real algebraic mapping, a real family of algebraic varieties) by those of its complexification. Such estimatates take the following form:

$$(*) \qquad \text{(real quantity)} \leqq \text{(complex quantity)}.$$

For a natural family of real algebraic objects, the right hand sides of inequalities (*) frequently turn out a constant. Then it is important to ask the uniform estimate

$$(*') \qquad \text{(real quantity)} \leqq \text{(constant)}$$

obtained by such a process is sharp or not.

Let $[\mathbb{F}] = [\lambda F_0 + \mu F_1]$ be a pencil of hypersurfaces in P^n of degree d:

$$[\mathbb{F}] \qquad \begin{array}{ccc} A & \xrightarrow{\varphi} & P^1 \\ \downarrow{\scriptstyle \pi} & & \\ P^n & & \end{array}$$

A pencil $[\mathbb{F}]$ is <u>generic</u> (resp. <u>of finite singularity)</u> if $A[\mathbb{F}]$ is non-singular and $\varphi[\mathbb{F}]$ has only non-degenerate (resp. isolated) critical points. (In Example 1, (1),(3) are generic, and (2) is not generic.)

If a pencil $[\mathbb{F}]$ is of finite singularity type, then the sum of Milnor numbers of φ at points in A is equal to $(n+1)(d-1)^n$.

Especially, for a generic real pencil $[\mathbb{F}]$ of hypersurfaces in P^n of degree d, the number s of real singular points of $[\mathbb{F}]$ satisfies the following inequality:

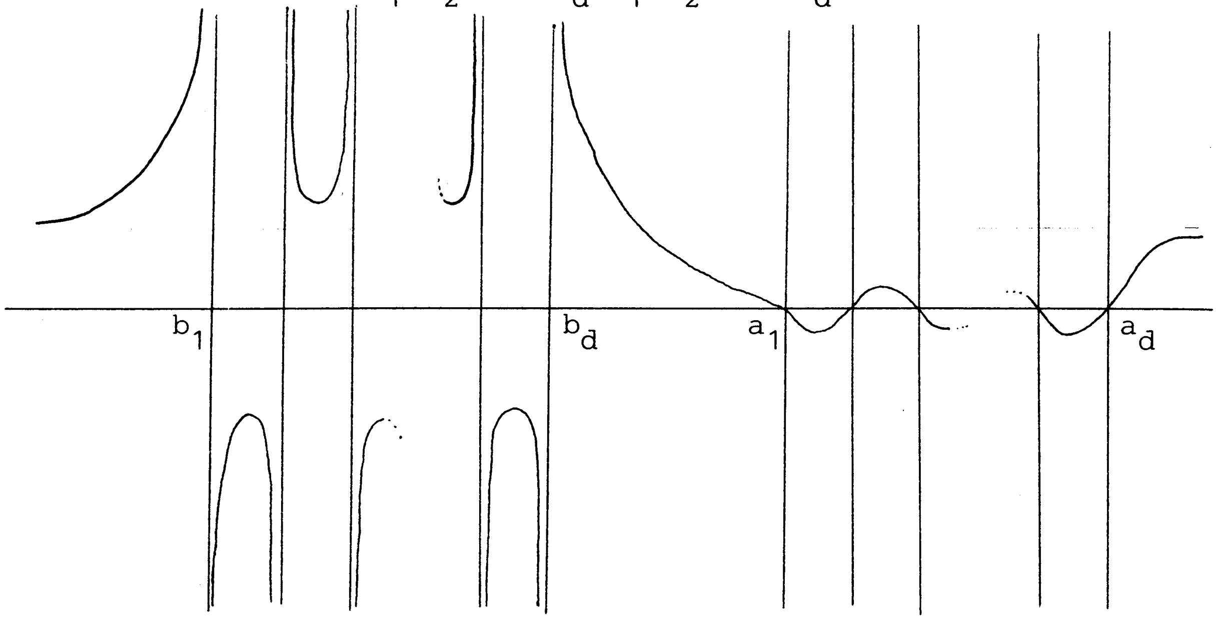
$$(*'') \qquad s \leqq (n+1)(d-1)^n.$$

The sharpness of (*'') for $n = 1$ is straightforward. In fact, the pencil

$$\mathbb{F} = \lambda \prod_{i=1}^{d} (x_0 - a_i x_1) + \mu \prod_{i=1}^{d} (x_0 - b_i x_1)$$

has $2(d-1)$ real singular points if

$$b_1 < b_2 < \ldots < b_d < a_1 < a_2 < \ldots < a_d.$$



For $n = 2$, the following is proved in [5]:

Theorem 3. ([5]) There exists a generic pencil of real plane algebraic curves of degree d with exactly s real singular points if and only if the non-negative integer s satisfies $s \leqq 3(d-1)^2$ and $s \equiv d-1 \pmod{2}$.

To show the existence of a generic pencil with $3(d-1)^2$ real singular points, it is sufficient to construct a pair of M-curves satisfying some prescribed topological conditions.

A pair $([F],[G])$ of real plane curves of degree d is a M-pair if the following conditions are satisfied:

(i) $[F]$ and $[G]$ are M-curves.

(ii) The real parts of curves $F = 0$ and $G = 0$ intersect transversely in $\mathbb{R}P^2$ at d^2 points.

(iii) The union $FG = 0$ of curves has $2g$ empty ovals.

Proposition 3. Assume $([F],[G])$ is a M-pair and assume the pencil $[\lambda F + \mu G]$ is generic. Then $s = 3(d-1)^2$.

For each d $(d=1,2,\ldots)$, a M-pair of degree d is constructed, based on Harnack's mathod [4].

By Theorem 3, the domain of the numbers s of real critical points of generic real rational functions $h = f(x,y)/g(x,y)$, $(\deg f = \deg g = d)$ on $\mathbb{R}^2$ is determined. The condition is

$$0 \leqq s \leqq 3(d-1)^2, \quad s \equiv d-1 \pmod{2}.$$

2.3.

Theorem 4. ([6]) There exists a generic real rational function $h = f(x,y)/g(x,y)$ $(\deg f = d, \deg g = d')$ on $\mathbb{R}^2$ with $d \neq d'$ with exactly s real critical points if and only if the non-negative integer s satisfies

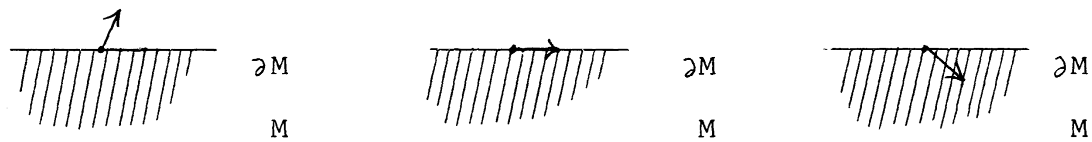
$$s \leqq (d-1)^2 + (d'-1)^2 + dd' - 1,$$
$$s \equiv (d-1)(d'-1) \pmod{2}.$$

3. Supplements.

3.1. A simple formula for indices of vector fields over a manifold with boundary and its application.

Let M be a n-dimensional C^∞ manifold with boundary ∂M, v a C^∞ vector field over M. Assume each singular points of v is isolated and not on ∂M.

Let us classify the vector $v(x)$ for a point $x \in \partial M$ into the following three classes:



external tangential internal

First put $M = M_0$, $\partial M = T_0$. Next put

$$M_1 = \{x \in T_0 \mid \nu(x) \text{ is external or tangential}\},$$

$$T_1 = \{x \in T_0 \mid v(x) \text{ is tangential}\}.$$

If M_1 is a C^∞ manifold with boundary T_1, then put

$$M_2 = \{x \in T_1 \mid v(x) \text{ is external or tangential w.r.t. } M_1\} = (M_1)_1,$$

$$T_2 = \{x \in T_1 \mid v(x) \text{ is tangential}\} = (T_1)_1.$$

Inductively, if M_k is a C^∞ manifold with boundary T_k, then put $M_{k+1} = (M_k)_1$, $T_{k+1} = (T_k)_1$.

Assumption: M_k is a ($n-k$ dimensional) C^∞ manifold with boundary T_k ($k = 1,2,\ldots,n-1$).

Let $S(v)$ denote the set of singular points of v. For each $x \in S(v)$, $\mathrm{ind}_x v \in \mathbb{Z}$ is defined. Put $\mathrm{ind}\, v = \sum_{x \in S(x)} \mathrm{ind}_x v$.

Theorem 5. Suppose M is compact. Then, under the above assumption,

$$\mathrm{ind}\, v = \sum_{i=0}^{n} (-1)^i \chi(M_i) \qquad \ldots(*).$$

Remark 1. (1) If $\partial M = \emptyset$, then $\mathrm{ind}\, v = \chi(M)$ (Poincaré-Hopf's theorem).

(2) By the formula (*), $\mathrm{ind}\, v$ is computed from only the behavior of v along ∂M.

(3) The above assumption is "generic" ([1],[8]).

The proof of Lemma 1 is achieved by the induction on n.

Let P be a closed C^∞ manifold and E a line bundle over P. For two sections $F, G \in \Gamma(E)$, put

$$V(F) = \{x \in P \mid F(x) = 0 \in E_x\}, \quad B = V(F) \cap V(G),$$

and consider the rational function $[F:G]: M - B \longrightarrow \mathbb{R}P^1$, defined by $[F:G](x) = [F(x):G(x)]$.

Take F, G "generic" and decompose $M - (V(F) \cup V(G))$ into connected components $\{U_i\}$. Put $D_i = \mathrm{Cl}\, U_i$ and

$$\partial D_i^+ = \{x \in \partial D_i - B \mid \mathrm{grad}[F:G](x) \text{ is external}\}.$$

Proposition 4. The number of critical points of $[F:G]$ is not less than $\sum_i |\chi(D_i) - \chi(\partial D_i^+)|$.

3.2. Isotopy classification of real loci of real algebraic functions of type (n,d;r).

A <u>real isotopy</u> of two real algebraic functions $[\mathbb{F}]$, $[\mathbb{G}]$ of type (n,d;r) is a pair (H_t, h'_t) of one-parameter continuous families of homeomorphisms of $\mathbb{RP}^n \times \mathbb{RP}^1$, $\mathbb{RP}^1$ respectively such that

(i) H_t covers h'_t.

(ii) $H_0 = \mathrm{id}$, $h'_0 = \mathrm{id}$.

(iii) $H_1(\mathbb{R}A[\mathbb{F}]) = \mathbb{R}A[\mathbb{G}]$

(iv) H_1 maps the ramification data of $\mathbb{R}\pi[\mathbb{F}]: \mathbb{R}A[\mathbb{F}] \longrightarrow \mathbb{RP}^n$ to that of $\mathbb{R}\pi[\mathbb{G}]: \mathbb{R}A[\mathbb{G}] \longrightarrow \mathbb{RP}^n$.

(To define the ramification data of $\mathbb{R}\pi$, marked partition types of non-negative integers s satisfying $s \leq r$, $s \equiv r$ (mod. 2) are used.)

For example, the number of real isotopy classes of real algebraic functions $[\mathbb{F}]$ of type (2,2;1) such that $A[\mathbb{F}]$ is non-singular and $\varphi[\mathbb{F}]$ is a Morse function is equal to three. The classes are distinguished by the numbers of real base points.

Related to the results in §2, we have a general problem: Determine the homological possibilities (over $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$) of real singular loci of φ and real ramification loci of π for "generic" real algebraic functions of fixed type.

It will be needed to develop "real intersection theory".

3.3. After choosing an orientation of $\mathbb{RP}^1$, denote s_i (resp. b) the number of real singular points of index i ($i = 0,1,2$) (resp. real base points) of a generic pencil of real plane curves of degree d. Then

$$b \leqq d^2, \quad b \equiv d \pmod{2},$$

$$s = s_0+s_1+s_2 \leqq 3(d-1)^2, \quad s \equiv d-1 \pmod{2},$$

$$1-b = s_0-s_1+s_2.$$

Do these determine the domain of pairs $(s_0, s_1, s_2; b)$ or (s,b) for generic pencils of real plane curves of degree d?

3.4. Let $[\mathbb{F}] = [\lambda F_0 + \mu F_1]$ be a pencil of real plane cubics. If $\mathbb{R}A$ is non-singular and $\mathbb{R}\varphi : \mathbb{R}A \longrightarrow \mathbb{R}P^1$ is a Morse function, then the real isotopy type of $[\mathbb{F}]$ is determined by the numbers b, s and the position of critical values on $\mathbb{R}P^1 = S^1$ marked by their indices.

By (3.3) and Harnack's theorem, logical possibilities of real isotopy types are followings, omitting the types obtained by changing the orientation of S^1:

b = 9, s = 12.
[010111111111]
[011101111111]
[011111011111]
[011211111111]
[011112111111]
[011111121111]
[011111111211]
[011111111112]

b = 9, s = 10
[0111111111]

b = 9, s = 8
[11111111]

b = 7, s = 12
[010101111111]
[010110111111]
[010111110111]
[011101110111]
[010112111111]
[010111121111]
[010111111211]
[010111111112]
[011201111111]
[011101121111]
[011101111211]
[011101111112]
[011211011111]
[011112011111]

b = 7, s = 10.
[0101111111]
[0111011111]
[0112111111]
[0111121111]
[0111111211]
[0111111112]

b = 7, s = 8.
[01111111]

b = 7, s = 6.
[111111]

b = 5, s = 12.
[010101011111]
[010101110111]
[010111010111]
[010101121111]
[010101111211]
[010101111112]
[010112011111]
[011011011211]
[010111011112]
[010112110111]
[010111120111]
[010111110112]
[010112121111]
[010112111211]
[010112111112]
[010111121211]
[010111121112]
[010111111212]
[011201121111]
[011201111211]
[011201111112]
[011101121211]
[011101121112]
[011101111212]
[011212011111]

b = 5, s = 12.
[011211011211]
[011211011112]

b = 5, s = 10.
[0101011111]
[0101110111]
[0101121111]
[0101111211]
[0101111112]
[0112011111]
[0111011211]
[0111011112]

b = 5, s = 8.
[01011111]
[01110111]
[01121111]
[01111211]
[01111112]

b = 5, s = 6.
[111111]

b = 5, s = 4.
[1111]

b = 3, s = 12.
[010101010111]
[010112010111]
[010101011211]
[010101011112]
[010101120111]
[010101110112]
[010101121211]
[010101121112]
[010101111212]
[010112011211]
[010112011112]
[010111011212]

b = 3, s = 10.
[0101010111]
[0101011211]
[0101011112]
[0101120111]
[0101110112]
[0101121211]
[0101121112]
[0101111212]
[0112011211]
[0112011112]
[0111011212]

b = 3, s = 8.
[01011211]
[01011112]
[01120111]

b = 3, s = 6.
[010111]
[011211]
[011112]

b = 3, s = 4.
[0111]

b = 3, s = 2.
[11]

b = 1, s = 12.
[010101010101]
[010101010112]
[010101011212]
[010101120112]
[010112010112]
[010101121212]
[010112011212]
[011201011212]
[011201120112]

b = 1, s = 10.
[0101010101]
[0101010112]
[0101011212]
[0101120112]

b = 1, s = 8.
[01010101]
[01010112]
[01011212]
[01120112]

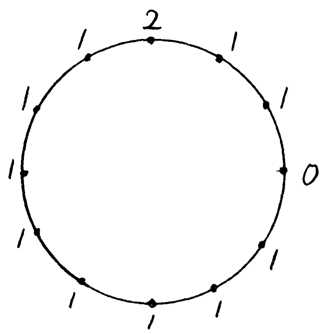
b = 1, s = 6.
[010101]
[010112]

b=1, s=4.
[0101]
[0112]

b=1, s=2.
[01]

b=1, s=0.
[]

Here,for example, [011211111111] means the following diagram:



3.5. Let H (resp. $\mathbb{R}$H) denotes the vector space of (real) homogeneous polynomials $F = F(x_0, x_1, \ldots, x_n)$ of degree d. Let $D_1, D_2 \subset H$ denotes the set of F such that the hypersurface defined by F has singularities with total multiplicities $\geqq 1$, $\geqq 2$, respectively.

For each smooth map $f: S^1 \longrightarrow \mathbb{R}H - D_2$ transverse to $\mathbb{R}(D_1 - D_2)$, put

$$M_f = \{([x],[t]) \in \mathbb{R}P^n \times S^1 \mid f(t)(x) = 0\},$$

and denote by $\varphi_f: M_f \longrightarrow S^1$ the projection.

Consider the following two notions:

(i) φ_f and φ_g are equivalent if there exist a smooth manifold M with boundary $\partial M = M_f \amalg M_g$ and a smooth map $\varphi: (M, M_f, M_g) \longrightarrow (S^1 \times I, S^1 \times 0, S^1 \times 1)$ such that $\varphi | M_f = \varphi_f$, $\varphi | M_g = \varphi_g$, any critical points of φ is of fold type and the set $D(\varphi)$ of critical values is a smooth submanifold of $S^1 \times I$.

(ii) φ_f and φ_g are equivalent if thete exists a smooth map $\rho: S^1 \times I \longrightarrow \mathbb{R}H - D_2$ such that ρ is transverse to $\mathbb{R}(D_1 - D_2)$, $\rho | S^1 \times 0 = f$ and $\rho | S^1 \times 1 = g$.

Do they differ exactly?

3.6. For a generic real algebraic functions of type (n,d;r), the number s of real critical points satisfies $s \leqq r(n+1)(d-1)^n$. Is this inequality sharp in general? Perhaps the constructions in [7] are powerful.

3.7. A substitute for Arnold's conjecture on estimates of the numbers of connected components of complements of real projective hypersurfaces. This relates to the estimates on the numbers of rigid isotopy classes of non-singular real projective hypersurfaces of fixed degree.

3.8. A general method to prove the sharpness of inequalities (*') in 2.2.

3.9. Are the numbers of topological right-left equivalence classes of polynomial mappings $\mathbb{C}^n \longrightarrow \mathbb{C}^m$ of corank at most one countable?

3.10. Let us take up an another compactification of $\mathbb{R}^2$ and $\mathbb{C}^2$.

Let $A = A[\mathbb{F}]$ be a curve in $P^1 \times P^1$ defined by $\mathbb{F} = \mathbb{F}(x_0, x_1; u_0, u_1)$ of degree (d,r). Suppose $\mathbb{F}$ is real and consider the real part $\mathbb{R}A \subset \mathbb{R}P^1 \times \mathbb{R}P^1$. What to do first is to study isotopy types of $\mathbb{R}A$ in $\mathbb{R}P^1 \times \mathbb{R}P^1 = T^2$.

Lemma. The number of connected components of $\mathbb{R}A$ does not exceed $1 + (d-1)(r-1)$.

Is this estimate sharp?

The isotopy type of an embedding $i: S^1 \longrightarrow T^2$ is determined by $i_*: H_1(S^1) = \mathbb{Z} \longrightarrow H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$. Estimate the number of non-trivial components of $\mathbb{R}A$.

3.11. Reconstruct the theory of real manifolds (resp. real mappings) as the theory of "real-complex manifolds" (resp. "real-complex mappings").

Please teach the author related informations.

The contents in §1 closely relate to the recent work of Nakai [9].

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