Singular Legendre curves and Goursat systems

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The classification problem of Goursat systems is reduced to that of Legendre curves [1][4][2]. Conversely, the classification of Goursat systems gives an insight to that of each individual Legendre curve. We describe their correspondence from general viewpoint using the results in [3].

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Let E be a germ of differential system of rank 2 on a manifold N of dimension n $(n \geq 2)$ at $x_0 \in N$. This means that $E \subseteq TN$ is a subbundle of rank 2. We call E a *Goursat system* if $E^2 := E + [E, E]$ is of rank 3, namely, of corank $n - 3$, and, setting $E =: E_{n-2}, E^2 =: E_{n-3}$, if $E_{i-1} := (E_i)^2$ is of corank $i - 1$, for $i = n - 2, n - 3, \dots, 2, 1$.

Fact([3]) For any $n \geq 3$, there exist an n-manifold $\mathbf{M}^n$ and a Goursat system $\mathbf{E}_{n-2}$ ("Monster Goursat") on $\mathbf{M}$ such that any Goursat germ (N^n, x_0, E) is isomorphic to $(\mathbf{M}, y_0, \mathbf{E})$ for some $y_0 \in \mathbf{M}$. The systems $(\mathbf{M}^n, \mathbf{E}_{n-2})$ are obtained by iterative prolongations from $\mathbf{M}^2 := \mathbf{R}^2, \mathbf{E}_0 := T\mathbf{R}^2$. Note that $\mathbf{M}^3 = PT^*\mathbf{R}^2$, the projective cotangent bundle of the plane with the canonical contact structure $\mathbf{E}_1$.

Let $n \geq 3$. Then there exists the intrinsic projection $\pi_{n,3} : \mathbf{M}^n \to \mathbf{M}^3$. The projection $\pi \circ \gamma$ of an integral curve-germ $\gamma : (\mathbf{R}, 0) \to \mathbf{M}^n$ to $\mathbf{E}_{n-2}$ $(\dot{\gamma}(t) \in \mathbf{E}_{n-2}, (t \in (\mathbf{R}, 0)))$ is a Legendre curve in $PT^*\mathbf{R}^2$. Recall that a curve-germ $c : (\mathbf{R}, 0) \to PT^*\mathbf{R}^2$ is called a *Legendre curve* if the velocity vector $\dot{c}(t) \in \mathbf{E}_1$, for $t \in (\mathbf{R}, 0)$. Here we do not assume c is an immersion. A Legendre curve is called *of finite type* if it is of finite type as a space curve-germ, i.e. if it is determined by its finite jet up to diffeomorphisms. If $c : (\mathbf{R}, 0) \to \mathbf{M}^3$ is a Legendre curve of finite type, there exists a unique integral lifting $\tilde{c} : (\mathbf{R}, 0) \to \mathbf{M}^n$ to $\mathbf{E}_{n-2}$ with $\pi \circ \tilde{c} = c$ for all $n \geq 3$.

Now we introduce an important invariant of Goursat systems: Let (N^n, x_0, E) be a germ of Goursat system of rank 2. Then there exists intrinsically the projection $\pi : (N^n, x_0) \to \mathbf{M}^3$ making E is the prolongation of the contact $\mathbf{E}_1$. Then we define intrinsically

$$C_E = \left\{ \pi \circ \gamma \;\middle|\; \begin{array}{l} \gamma : (\mathbf{R}, 0) \to \mathbf{M}^n \text{ integral to } \mathbf{E}_{n-2}, \\ \gamma(0) = x_0, \pi \circ \gamma \, \text{is of finite type} \end{array} \right\} / \sim,$$

where $\sim$ means the contactomorphism equivalence: For two Legendre curves $c, c' : (\mathbf{R}, 0) \to PT^*\mathbf{R}^2$, we set $c \sim c'$ if if there exist a diffeomorphism $\sigma : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ and a contactomorphism $\tau : (PT^*\mathbf{R}^2, c(0)) \to (PT^*\mathbf{R}^2, c'(0))$ such that $\tau \circ c = c' \circ \sigma$.

Theorem 1: (Legendre invariants of Goursat systems). *Let* (N^n, x_0, E) *and* (N'^n, x'_0, E') *be germs of Goursat systems of rank two. Then the following conditions are equivalent to each other:* (i) $(N, x_0, E) \cong (N', x'_0, E')$. (ii) $C_E = C_{E'}$. (iii) $C_E \cap C_{E'} \neq \emptyset$.

Let $c : (\mathbf{R}, 0) \to PT^*\mathbf{R}^2$ be a Legendre curve of finite type. Then we define G_c to be the set of isomorphism classes of germs of Goursat systems E satisfying $[c] \in C_E$. Actually G_c is realized, up to isomorphisms, by the prolongation-deprolongation infinite sequence of Goursat systems which is obtained by lifting of c.

Theorem 2: (Goursat-Legendre duality.) *Let* $c : (\mathbf{R}, 0) \to PT^*\mathbf{R}^2$ *be a Legendre curve of finite type. Then the contactomorphism class* $[c]$ *of* c *is recovered from the sequence* G_c*: In fact we have* $\bigcap_{[E] \in G_c} C_E = \{[c]\}$. *Conversely, let* E *be a germ of Goursat system of rank two. Assume* E *is not the contact system. Then the isomorphism class* $[E$ *of* E *is recovered from* C_E*: In fact we have* $\sup\{n \mid \#\pi_n \left(\bigcup_{[c] \in C_E} G_c \right) = 1\} = d(E)$, *where* $d(E)$ *denote the dimension of the base manifold of* E, *and* π_n *means the operation of taking isomorphism classes of Goursat systems over* n*-dimensional manifold. Moreover we have* $\pi_{d(E)} \left(\bigcup_{[c] \in C_E} G_c \right) = \{[E]\}$.

References

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