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DEVELOPABLE HYPERSURFACES AND HOMOGENEOUS SPACES IN A REAL PROJECTIVE SPACE

(submitted by B.N.Shapukov)

Dedicated to Professor Fuichi Uchida on his 60th birthday.

Abstract. We present new examples of non-singular developable hypersurfaces, which are algebraic and homogeneous, in real projective spaces. Moreover we give a characterization of compact homogeneous developable hypersurfaces, using the theory of isoparametric hypersurfaces.

0. Introduction

A C^∞ hypersurface M in the n-dimensional real projective space $\mathbf{R}P^n$ is called **developable** if its Gauss map

$$\gamma : M \to \mathrm{Gr}(n, \mathbf{R}^{n+1}) \cong \mathrm{Gr}(1, (\mathbf{R}^{n+1})^*) = \mathbf{R}P^{n*}$$

defined by $\gamma(x) = T_x\hat{M} \subset \mathbf{R}^{n+1}$ $(x \in M)$ has $\mathrm{rank}(\gamma) < \dim(M) = n - 1$. Here, we mean by $\hat{M}$ the n-dimensional submanifold of $\mathbf{R}^{n+1} - \{\mathbf{0}\}$, corresponding to $M \subset \mathbf{R}P^n$, and therefore $T_x\hat{M}$ is a linear hyperplane in $\mathbf{R}^{n+1}$. Moreover we mean by $\mathbf{R}P^{n*}$ the dual projective space, and by $\mathrm{rank}(\gamma)$ the maximum of the rank of differential maps $\gamma_* : T_xM \to T_x\mathbf{R}P^{n*}$ $(x \in M)$ of γ.

Remark that the developability is a notion of projective geometry; the image of a developable hypersurface under a projective transformation is again developable.

In this paper we treat mainly developable hypersurfaces. See [FW][W] for developable submanifolds of arbitrary codimension.

It is well-known, as classical examples of developable surfaces in the three dimensional space, cylinders, cones and tangent developables of space curves

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[Cay][I]: Among them, only the planes have no singularities in the projective space. Observing the singularities of developable hypersurfaces, we expect, also in the general case, that non-singular compact developable hypersurfaces are strictly restrictive. In fact, it is known that a non-singular complex algebraic developable hypersurface in $\mathbf{C}P^n$ is necessarily a projective hyperplane ([GH][W][L1]). Also in a real projective space, we have the following restriction, via the geometrical investigation of homogeneous Monge-Ampère equations based on projective duality:

Theorem 1 ([IM]). *For a compact developable C^∞ hypersurface M in $\mathbf{R}P^n$, the maximal rank $r = \mathrm{rank}(\gamma)$ of the Gauss map $\gamma : M \to \mathbf{R}P^{n*}$ is an even integer and satisfies the inequality $n < (1/2)r(r+3)$, provided $r \neq 0$. In particular, if $r < 2$, then M is necessarily a projective hyperplane of $\mathbf{R}P^n$. Any compact developable C^∞ hypersurfaces in $\mathbf{R}P^3$ or $\mathbf{R}P^5$ are projective hyperplanes.*

The rank condition appeared in Theorem 1 is essential in the real case; in fact there exist non-trivial examples of compact developable hypersurfaces in real projective spaces:

Proposition 2. *For $n = 4, 7, 13, 25$, there exists a real algebraic cubic non-singular developable hypersurface in $\mathbf{R}P^n$. These developable hypersurfaces have the structure of homogeneous spaces of groups $SO(3), SU(3), Sp(3), F_4$, respectively. Their projective duals are linear projections of Veronese embeddings of projective planes $\mathbf{K}P^2$, for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$ (the Cayley's octonians, octanions or octonions). Each of these real algebraic developable hypersurfaces admits deformations to C^∞ developable hypersurfaces with $2, 3, 5, 9$ functional parameters, or more rigorously, with the space of sections of normal bundles to $\mathbf{K}P^2 \subset \mathbf{R}P^n$ as the infinitesimal space of C^∞ developable deformations.*

Notice that it is classically known that a properly embedded developable hypersurface in $\mathbf{R}^n$ of $\mathrm{rank}(\gamma) \leq 1$ is necessarily a cylinder (Hartman-Nirenberg's theorem [HN][Ste][Sto]). Similar result is known for $\mathbf{C}^n$ by Abe [Ab]. For this direction, see the survey [B]. The first example of non-cylindrical C^∞ developable hypersurfaces in $\mathbf{R}^4$ is given by Sacksteder [Sac]:

$$M = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid x_4 = x_1 \cos x_3 + x_2 \sin x_3\}.$$

Mori [Mo] gives an example of families of non-cylindrical developable hypersurfaces in $\mathbf{R}^4$, in connection with the study of deformable submanifolds. On the other hand, Akivis [Ak] proves the existence of C^∞ complete developable hypersurfaces in $\mathbf{R}P^4$ which is not a projective hyperplane, using the theory of differential systems. (See also [AG] Ch. 4, for the method of construction). However notice that it is not given any concrete examples there. Recently, Fischer and Wu ([FW][W]) study developable submanifolds in $\mathbf{C}P^n, \mathbf{C}^n$ and $\mathbf{R}^n$ of higher codimension. In [W], it is introduced an (unpublished) example of non-cylindrical real algebraic developable hypersurfaces in $\mathbf{R}^4$ by

Bourgain:

$$M = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid x_1 x_4^2 + x_2(x_4 - 1) + x_3(x_4 - 2) = 0\}.$$

Then, M is non-singular in $\mathbf{R}^4$ and even in $\mathbf{C}^4$ after complexification, while the Zariski closure $\overline{M} \subset \mathbf{R}P^4$ of M has singularities in $\mathbf{R}P^4$. (The singular loci is an $\mathbf{R}P^2$ in the projective hyperplane at infinity).

In general, developable submanifolds has the **Monge-Ampère foliation** so that the tangent spaces to the submanifold are constant along each leaf. For instance, in the case of F_4 in Proposition 2, the Gauss mapping is a submersion and the Monge-Ampère foliation is given by the fiberwise $\mathbf{Z}/2\mathbf{Z}$ quotient of the fibration

$$\mathbf{O}P^1 \cong \mathrm{Spin}(9)/\mathrm{Spin}(8) \to F_4/\mathrm{Spin}(8) \to F_4/\mathrm{Spin}(9) \cong \mathbf{O}P^2,$$

arising from the filtration $F_4 \supset \mathrm{Spin}(9) \supset \mathrm{Spin}(8)$. Remark that there exists natural identification $\mathbf{O}P^1 \cong S^8$, and the antipodal map induces the involution on $\mathbf{O}P^1$.

Now we give a characterization of compact homogeneous developable hypersurfaces under some assumption, by means of metric geometry.

We say M and $M' \subset \mathbf{R}P^n$ are **projectively equivalent** if there is a projective transformation $\varphi : \mathbf{R}P^n \to \mathbf{R}P^n$ with $\varphi(M) = M'$.

A C^∞ hypersurface $M \subset \mathbf{R}P^n$ is called a **Cartan hypersurface** if, $n = 4, 7, 13$ or 25, and M is projectively equivalent to one of examples in Proposition 2. Since the developability is projectively invariant, Cartan hypersurfaces are developable.

Remark that a Cartan hypersurface in $\mathbf{R}P^n$ is a G-orbit for a compact Lie subgroup $G \subset \mathrm{GL}(n+1, \mathbf{R})$. In general, we call a (projectively) homogeneous submanifold $M \subset \mathbf{R}P^n$ **of compact type** if M is a G-orbit for a compact Lie subgroup G of $\mathrm{GL}(n+1, \mathbf{R})$, under the action on $\mathbf{R}P^n$ induced by the natural linear action on $\mathbf{R}^{n+1}$. If M is of compact type, then M is compact.

In this paper we show the following result:

Theorem 3. *Let M be a connected homogeneous developable hypersurface of compact type. Then M is a projective hyperplane or a Cartan hypersurface.*

By Theorem 3, we know that it is impossible to find other non-trivial examples of developable hypersurfaces than Cartan hypersurfaces, among quotients of compact groups of linear transformations.

In the next section, we recall the notion of projective duality and the second fundamental form of submanifold in a projective space. In §3, we give a direct proof of Proposition 2 within the framework of projective geometry. The main tool for the construction of Proposition 2 is the real projective-contact geometry [M1][M2] over Jordan algebras. In the last section, we prove Theorem 3 using metric geometry. The proof shows clearly the power of the theory of isoparametric hypersurfaces. However we wish to find alternative proof of Theorem 3 within the framework of projective geometry, since it would be most natural one and may work for just compact projectively homogeneous developable hypersurfaces.

It is interesting to ask the connection between Cartan hypersurfaces and the classification of Severi varieties in the projective spaces over algebraically closed field of characteristic zero, for instance, over $\mathbf{C}$, by Zak [Z] (cf. [FL][LV, p.15]). See also [L1][L2][Kaj] for complex projective geometry on the second fundamental forms and degenerate secant varieties, related to homogeneous spaces and Clifford algebras.

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1. Projective duality and second fundamental forms

Let $M \subset \mathbf{R}P^n$ be a submanifold of dimension m, $(m < n)$. Consider the projective conormal bundle of M:

$$\widetilde{M} = \{(p, q) \in \mathbf{R}P^n \times \mathbf{R}P^{n*} \mid p \in M,\ T_pM \subset q^\vee\},$$

where $q^\vee$ is the projective hyperplane of $\mathbf{R}P^n$ determined by $q \in \mathbf{R}P^{n*}$, and we identify T_pM as the corresponding m-dimensional plane through p in $\mathbf{R}P^n$. Then we see $\widetilde{M}$ is a C^∞ submanifold in $\mathbf{R}P^n \times \mathbf{R}P^{n*}$ of dimension $n-1$. Let $\rho : \widetilde{M} \to \mathbf{R}P^n$ (resp. $\rho' : \widetilde{M} \to \mathbf{R}P^{n*}$) denotes the projection to the first (second) component. Then $\rho(\widetilde{M}) = M$ and $\rho'(\widetilde{M}) = M^\vee$ is the projective dual of M.

We call M is **developable** if the Gauss map $\gamma : M \to \mathrm{Gr}(m+1, \mathbf{R}^{n+1})$, defined by $\gamma(x) = T_x\hat{M}$ satisfies $\mathrm{rank}(\gamma) < \dim M$.

If M is developable and $\mathrm{rank}(\gamma) = r$, then there exists an $(m-r)$-dimensional foliation on $\Omega = \{x \in M \mid \mathrm{rank}_x(\gamma) = r\}$, which we call Monge-Ampère foliation [D]. Moreover in this case, $M^\vee$ is ruled by r-parameter $(n-m-1)$-planes, and $\mathrm{rank}(\rho') < \dim \widetilde{M} = n-1$.

Remark that, if $m = n-1$, then ρ is a diffeomorphism onto its image and the Gauss map is decomposed as $\gamma = \rho' \circ \rho^{-1}$.

Let $g : W \to \mathbf{R}P^{n*}$ be an immersion. For $x \in W$, the **second fundamental form** of g at x is a linear family of quadratic forms (Hessians) on T_xW parametrized by conormal vector space $N^* = (T_{g(x)}\mathbf{R}P^{n*}/g_*(T_xW))^*$ to g at x:

$$II^* : N^* \to S^2(T_x^*M) \text{ (the symmetric product)},$$

defined as follows: Let $u \in N^*$. Take an affine function v on an affine neighborhood of $g(x)$ such that the corresponding cotangent vector $T_{g(x)}\mathbf{R}P^{n*} \to \mathbf{R}$ vanishes on $g_*(T_xW)$ and represents the conormal vector u. Then we define $II^*(u)$ to be the Hessian at x of the composition $v \circ g$. Then $II^*(u)$ is independent of the choice of v and depends only on u (cf. [L1],[L2],[Sas]).

Now we recall the following fundamental result [IM], which we are going to use for showing Proposition 2:

Lemma 4. *For an immersed submanifold W of $\mathbf{R}P^{n*}$ of codim ≥ 2, the following conditions are equivalent to each other:*

(i) *W is a projective dual of a properly immersed hypersurface in $\mathbf{R}P^n$.*

(ii) *The second fundamental form at each point of W does not contain any singular quadratic forms.*

(iii) *For any projective hyperplane $H \subset \mathbf{R}P^{n*}$, each singular point of the hyperplane section $W \cap H$ on W is non-degenerate.*

Proof. The condition (i) is equivalent to that $\rho : \widetilde{W} \to \mathbf{R}P^n$ is an immersion. For a local equation

$$y_{r+1} = \varphi_{r+1}(y_1, \dots, y_r), \ \dots, \ y_n = \varphi_n(y_1, \dots, y_r)$$

of W, $\widetilde{W}$ is defined by $F = \partial F/\partial y_1 = \cdots = \partial F/\partial y_r = 0$, where

$$F(X; y_1, \dots, y_r) = X_0\varphi_n + \cdots + X_{n-r-1}\varphi_{r+1} + X_{n-r}y_r + \cdots + X_{n+1}y_1 + X_n,$$

for a homogeneous coordinates $(X_0, X_1, \dots, X_n)$ of $\mathbf{R}P^n$. Then ρ is an immersion on $\widetilde{W}$ if and only if the second fundamental form

$$II^*(X_0, \dots, X_{n-r-1}) = \sum_{k=0}^{n-r-1} X_k \left(\frac{\partial^2 \varphi_{n-k}}{\partial y_i \partial y_j} \right)_{1 \leq i,j \leq r}$$

does not represent a singular matrix, provided $(X_0, \dots, X_{n-r-1}) \neq (0, \dots, 0)$. This condition is equivalent to (ii). The equivalence (ii) $\Leftrightarrow$ (iii) is clear.

2. Proof of Proposition 2

First we show the construction of a cubic non-singular developable hypersurface M in $\mathbf{R}P^4$. For this, first we construct the projective dual $M^\vee$, then M is obtained as the dual of $M^\vee$.

Define $\varphi : \mathbf{R}P^2 \to \mathbf{R}P^{4*}$ by

$$\varphi([u, v, w]) = [\frac{1}{2}(u^2 - v^2), \frac{1}{2}(v^2 - w^2), uv, vw, wu],$$

which is an embedding obtained after a linear projection of the Veronese embedding $\psi : \mathbf{R}P^2 \to \mathbf{R}P^{5*}$ defined by

$$\psi([u, v]) = [\frac{1}{2}u^2, \frac{1}{2}v^2, \frac{1}{2}w^2, uv, vw, wu].$$

(See [Sas] Example 3, and [CDK]). Then we set $M^\vee = \varphi(\mathbf{R}P^2)$. Further we set

$$F = X_0\frac{1}{2}(u^2 - v^2) + X_1\frac{1}{2}(v^2 - w^2) + X_2uv + X_3vw + X_4wu.$$

Then the ρ-projection of the projective conormal bundle $\widetilde{M^\vee}$ of $M^\vee$ is obtained by eliminating u, v, w from

$$F = \frac{\partial F}{\partial u} = \frac{\partial F}{\partial v} = \frac{\partial F}{\partial w} = 0.$$

Then we have

$$\begin{vmatrix} X_0 & X_2 & X_4 \\ X_2 & -X_0 + X_1 & X_3 \\ X_4 & X_3 & -X_1 \end{vmatrix} = 0,$$

which is the equation of required $M \subset \mathbf{R}P^4$.

In fact, M is the projectivization of the set of real symmetric matrices of determinant zero and of trace zero. Since $SO(3)$ acts on M transitively, we see M is non-singular and $M \cong SO(3)/H$, where H is the subgroup of $SO(3)$ of order 8:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In general, we set $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$. Then $\dim_{\mathbf{R}} \mathbf{K} = 2^{i-1},\ i = 1, 2, 3, 4.$ Consider

$$\mathcal{J} = \{A \in M_3(\mathbf{K}) \mid A^* = A\},$$

the space of "Hermitian" matrices of size 3 ([H][Y]). Each element A of $\mathcal{J}$ has the form

$$A = \begin{pmatrix} \xi_1 & z_1 & z_3 \\ \bar{z_1} & \xi_2 & z_2 \\ \bar{z_3} & \bar{z_2} & \xi_3 \end{pmatrix},\ \xi_j \in \mathbf{R},\ z_j \in \mathbf{K},\ j = 1, 2, 3.$$

We see

$$\dim_{\mathbf{R}} \mathcal{J} = 3 \cdot 2^{i-1} + 3 = 6, 9, 15, 27.$$

For $A, B \in \mathcal{J}$, we define the **Jordan product**

$$A \circ B = \frac{1}{2}(AB + BA) \in \mathcal{J}.$$

Moreover we set $\mathrm{tr}A = \xi_1 + \xi_2 + \xi_3 \in \mathbf{R}$ and

$$\det A = \xi_1\xi_2\xi_3 + 2\mathrm{Re}((z_2\bar{z}_3)z_1) - \xi_1 z_2\bar{z}_2 - \xi_2 z_3\bar{z}_3 - \xi_3 z_1\bar{z}_1 \in \mathbf{R},$$

for $A \in \mathcal{J}$. The bilinear form $\mathrm{tr}(A \circ B)$ on the real vector space $\mathcal{J}$ is positive definite and induces the isomorphism between $\mathcal{J}$ and its dual vector space $\mathcal{J}^*$.

Set

$$\Sigma = \{A \in \mathcal{J} \mid \det A = 0\}.$$

Then the projectivization $P\Sigma \subset P\mathcal{J} = (\mathcal{J} - O)/\mathbf{R}^\times \cong \mathbf{R}P^{3\cdot 2^{i-1}+2}$ is a real cubic hypersurface. Setting

$$\mathcal{J}_0 = \{A \in \mathcal{J} \mid \mathrm{tr}A = 0\},$$

we will see

$$M = P\mathcal{J}_0 \cap P\Sigma \subset P\mathcal{J}_0 = \mathbf{R}P^4, \mathbf{R}P^7, \mathbf{R}P^{13}, \mathbf{R}P^{25},$$

is a non-singular real cubic developable hypersurface. The projective dual $M^\vee = \mathbf{K}P^2$ is embedded in $P\mathcal{J}_0^* = \mathbf{R}P^{4*}, \mathbf{R}P^{7*}, \mathbf{R}P^{13*}, \mathbf{R}P^{25*}$, as a linear projection of the Veronese embedding of $\mathbf{K}P^2$ in $P\mathcal{J} \cong P\mathcal{J}^*$. Remark that $\mathrm{rank}(\gamma) = 2, 4, 8, 16$ and the dimension of the Monge-Ampère foliation is $1, 2, 4, 8$, respectively.

Recall that the projective plane over $\mathbf{K}$ is defined by

$$\begin{aligned}\mathbf{K}P^2 &= \{X \in \mathcal{J} \mid X^2 = X,\ \mathrm{tr}X = 1\},\\ &= \{\mathbf{x}\mathbf{x}^* \mid\ {}^t\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{K}^3 - \mathbf{0}, \|\mathbf{x}\| = 1, x_1(x_2x_3) = (x_1x_2)x_3\}\end{aligned}$$

which is embedded in $P\mathcal{J}$. The embedding $\mathbf{K}P^2 \hookrightarrow P\mathcal{J}$ is called the **Verenose embedding** [F1][F2][H, Lemma 14.90][L2][Z]. This definition fits in the ordinary one in cases $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ by the correspondence

$$\mathbf{K}P^2 \ni [x_1, x_2, x_3] = [{}^t\mathbf{x}] \mapsto \frac{1}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^*.$$

In cases $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$, we set $G = \mathrm{O}(3), \mathrm{U}(3), \mathrm{Sp}(3)$. Then G acts on $\mathcal{J}$ by $f(A) = P^{-1}AP, (f = P \in G)$. In the case $\mathbf{K} = \mathbf{O}$, we take as G the exceptional simple Lie group

$$F_4 = \{f : \mathcal{J} \to \mathcal{J},\ \mathbf{R}\text{-linear isomorphism } \mid f(A \circ B) = f(A) \circ f(B)\}.$$

Then G preserves the Jordan product, the trace and the determinant, so G naturally acts on $P\mathcal{J}_0$, $P\Sigma$, so on $M = P\mathcal{J}_0 \cap P\Sigma$, as well as it acts on $\mathbf{O}P^2$. Furthermore G acts on M transitively. In fact, for $A \in \mathcal{J}$, there exists $f \in G$ such that $f(A) = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}$, for some $\xi_1, \xi_2, \xi_3 \in \mathbf{R}$. Moreover the diagonals are permuted freely by an element of G. Then, an $A \in \mathcal{J}_0 \cap \Sigma$ is transformed into $f(A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & -\xi \end{pmatrix}$, by some $f \in G$, for some $\xi \in \mathbf{R}$. (See, for $\mathbf{K} = \mathbf{O}$, [H] Page 313, [Y] Page 35). Also the action of G on $\mathbf{K}P^2$ is transitive. ([H] Theorem 14.99, [Y] Theorem 2.21).

Now set

$$Q = \{([A],[B]) \in P\mathcal{J} \times P\mathcal{J} \mid \mathrm{tr}(A \circ B) = 0\},$$

the incidence hypersurface of projective duality ([Sch][IM]). Then G acts on Q naturally by $f([A],[B]) = ([f(A)],[f(B)])$. Since the action on M is transitive, the action on $\widetilde{M}$ is also transitive. Here we remark that $\widetilde{M}$ projects diffeomorphically to M by ρ. Then the key fact is the following:

Lemma 5. *The projective conormal bundle of* $\mathbf{K}P^2 \subset P\mathcal{J}^*$ *is described by*

$$\widetilde{\mathbf{K}P^2} = PT^*_{\mathbf{K}P^2}P\mathcal{J}^* = \{([A],[X]) \in Q \mid X \in \mathbf{K}P^2,\ A \circ X = O\}.$$

Moreover its projection $S = \rho(PT^*_{\mathbf{K}P^2}P\mathcal{J}^*)$ *by* $\rho : PT^*_{\mathbf{K}P^2}P\mathcal{J}^* \to P\mathcal{J}$ *onto the first component coincides with*

$$P\Sigma = \{[A] \in P\mathcal{J} \mid \det A = 0\}.$$

Proof. We show for $\mathbf{K} = \mathbf{O}$; other cases are treated similarly. Let $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{K}^3 - \mathbf{0}$. Write $x_i = \sum_{j=0}^{7} x_{ij}e_j$, $i = 1,2,3$, with the standard basis $e_0 = 1, e_1, \dots, e_7$ and $x_{ij} \in \mathbf{R}$. Then the linear subspace $T_{\mathbf{x}_0}\mathbf{O}P^2 \subset \mathcal{J}$ of the tangent space to $\mathbf{O}P^2$ at $\mathbf{x}_0 = {}^t(1,0,0)$ is generated over $\mathbf{R}$ by

$$\frac{\partial}{\partial x_{10}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{\partial}{\partial x_{20}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{\partial}{\partial x_{2i}} = \begin{pmatrix} 0 & -e_i & 0 \\ e_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\frac{\partial}{\partial x_{30}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \frac{\partial}{\partial x_{3i}} = \begin{pmatrix} 0 & 0 & -e_i \\ 0 & 0 & 0 \\ e_i & 0 & 0 \end{pmatrix}, \quad 1 \le j \le 7,$$

while $\dfrac{\partial}{\partial x_{1j}} = O, 1 \le j \le 7$. Set $A = \begin{pmatrix} \xi_1 & w_1 & w_3 \\ \bar{w}_1 & \xi_2 & w_2 \\ \bar{w}_3 & \bar{w}_2 & \xi_3 \end{pmatrix}$. Then the condition that A annihilates $T_{\mathbf{x}_0}\mathbf{O}P^2$ via the inner product $\mathrm{tr}(A \circ B)$, namely that $\mathrm{tr}(A \circ \dfrac{\partial}{\partial x_{ij}}) = O, i = 1,2,3, 0 \le j \le 7$, is equivalent to that $\xi_1 = 0, w_1 = 0, w_3 = 0$. This is equivalent to that

$$A \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2\xi_1 & w_1 & w_3 \\ \bar{w}_1 & 0 & 0 \\ \bar{w}_3 & 0 & 0 \end{pmatrix}$$

equals to O. By the transitivity we have the first half. The second half follows from the following Lemma.

Lemma 6. *For $A \in \mathcal{J}$, (1) $A \circ X = O$, for some $X \in \mathbf{K}P^2$, if and only* (2) $\det A = 0$.

Proof. (1) $\Rightarrow$ (2): Choose $f \in G$ such that $f(X) = X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, since $f(A) \circ X_0 = f(A \circ X) = O$, we see $\det A = \det f(A) = 0$. (2) $\Rightarrow$ (1): Take $f \in G$ such that

$$f(A) = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix},$$

for some $\xi_1, \xi_2, \xi_3 \in \mathbf{R}$. Then $\det f(A) = \det A = 0$, so $\xi_1\xi_2\xi_3 = 0$, thus $\xi_i = 0$, for some i. Changing f if necessary, we may assume $\xi_1 = 0$. Then $A \circ f^{-1}(X_0) = f(A) \circ X_0 = O$. $\square$

Thus we see the projective dual of the hyperplane section $M = S \cap P\mathcal{J}_0 \subset P\mathcal{J}_0$ is the linear projection of $\mathbf{K}P^2 \subset P\mathcal{J} \cong P\mathcal{J}^*$ from the point in $P\mathcal{J}^*$ corresponding to the hyperplane $P\mathcal{J}_0 \subset P\mathcal{J}$.

Set

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathcal{J}_0 \cap \Sigma, \quad \text{and,} \quad X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbf{O}P^2 \subset \mathcal{J}.$$

Then $([A_0], [X_0]) \in \widetilde{M}$. Let $\mathbf{K} = \mathbf{O}$ and $G = F_4$. Then the isotropy group for $[X_0] \in P\mathcal{J}$ of the F_4-action is isomorphic to Spin(9) ([H] Theorem 14.99, [Y] Theorem 2.10). Further the isotropy group for $A_0 \in \mathcal{J}$ of the F_4-action on $\mathcal{J}$ is

$$\{f \in F_4 \mid f(E_i) = E_i,\ i = 1, 2, 3\},$$

which is isomorphic to Spin(8). Here E_i is the 3×3 matrix with (i, i)-element 1 which is the only non-zero element. (So $E_1 = X_0, A_0 = E_2 - E_3$). ([H] Page 313, [Y] Theorem 2.7). Then the isotropy group for $[A_0]$ in M is isomorphic to a $\mathbf{Z}/2\mathbf{Z}$-extension of Spin(8). Thus we see that the Monge-Ampère foliation is in fact a fibration $\gamma : M \to \mathbf{O}P^2$ described as in §0.

Similarly we have, in cases $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$, that the Monge-Ampère foliation is given by the fibration $\gamma : M \to \mathbf{K}P^2$ which is described as the fiberwise $\mathbf{Z}/2\mathbf{Z}$-quotient with respect to the antipodal involution of $\mathbf{K}P^1 \cong S^{2^{i-1}}$($i = 1, 2, 3$) of the following fibration: For $\mathbf{K} = \mathbf{R}$,

$$\begin{aligned} \mathbf{R}P^1 \cong O(2)/O(1) \times O(1) \to O(3)/O(1) \times O(1) \times O(1) \to \\ \to O(3)/O(2) \times O(1) \cong \mathbf{R}P^2, \end{aligned}$$

For $\mathbf{K} = \mathbf{C}$,

$$\begin{aligned} \mathbf{C}P^1 \cong U(2)/U(1) \times U(1) \to U(3)/U(1) \times U(1) \times U(1) \to \\ \to U(3)/U(2) \times U(1) \cong \mathbf{C}P^2, \end{aligned}$$

and for $\mathbf{K} = \mathbf{H}$,

$$\mathbf{H}P^1 \cong \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{Sp}(3)/\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \\ \to \mathrm{Sp}(3)/\mathrm{Sp}(2) \times \mathrm{Sp}(1) \cong \mathbf{H}P^2.$$

In particular, $M \in \mathbf{R}P^n$, $(n = 3 \cdot 2^{i-1} + 1,\ i = 1, 2, 3, 4)$ is a homogeneous space of $SO(3), SU(3), \mathrm{Sp}(3)$ and F_4, respectively.

The last statement of Proposition 2 is clear, since the condition (ii) of Lemma 4 is an open condition for immersions $\mathbf{K}P^2 \to \mathbf{R}P^{n*}$.

3. Proof of Theorem 3

Let M be a connected developable homogeneous hypersurface of compact type. By definition, M is an orbit of a compact Lie subgroup G of $\mathrm{GL}(n+1, \mathbf{R})$. We take a G-invariant metric on $\mathbf{R}^{n+1}$. Then, after a projective transformation on $\mathbf{R}P^n$, we may assume $G \subset O(n+1)$ and now we use the metric geometry.

Let $\pi : S^n \to \mathbf{R}P^n$ be the natural double covering from the unit sphere $S^n \subset \mathbf{R}^{n+1}$ and endow with $\mathbf{R}P^n$ the induced metric from S^n.

A hypersurface in $\mathbf{R}P^n$ or S^n is called **isoparametric** if its principal curvatures are constant (see [CR]). The following fact is suggested by K. Tsukada to the author:

Lemma 7. *$M \subset \mathbf{R}P^n$ is developable if and only if 0 is a principal curvature of M at each point in M.*

Proof. The condition that M is developable is equivalent to that the shape operators are degenerate at any points $x \in M$. The latter is equivalent to that 0 is a principal curvature at each point in M. □

Now M is a G-orbit and G acts on $\mathbf{R}P^n$ isometrically. So M is a connected compact isoparametric hypersurface.

Actually, we are going to show the following, which completes the proof of Theorem 3.

Proposition 8. *Let $M \subset \mathbf{R}P^n$ be projectively equivalent to a connected compact isoparametric hypersurface in $\mathbf{R}P^n$. If M is developable, then M is a projective hyperplane or a Cartan hypersurface.*

Remark 9: It is known that there exist compact isoparametric hypersurfaces in spheres which are not (isometrically) homogeneous with $g = 4$ [OT]. Recently Reiko Miyaoka [Mi] has proved that every compact isoparametric hypersurfaces in spheres with $g = 6$ are necessarily homogeneous, as well as the cases $g \leq 3$.

Proof of Proposition 8. Take a connected component of $\pi^{-1}(M)$ and denote it by $\bar{M}$. Then $\bar{M}$ is a connected compact isoparametric hypersurface of S^n. Moreover, by Lemma 7, 0 is a principal curvature of M and therefore of $\bar{M}$.

Let $\lambda_1, \dots, \lambda_g$ be the disjoint principal curvatures of $\bar{M}$. Set $\lambda_i = \cot \theta_i, 0 < \theta_i < \pi$, with $\theta_1 < \theta_2 < \cdots < \theta_g$. Moreover denote by m_i the multiplicity of the principal curvature λ_i. Then, by the general theory of isoparametric hypersurfaces [CR], we have the following:

(a) $\theta_i = \theta_1 + \dfrac{i-1}{g}\pi, \quad 1 \le i \le g.$

(a') $m_i = m_{i+2}, \quad$ (subscripts mod. g).

(b) $g = 1, 2, 3, 4, 6, \quad$ (Münzner).

Now 0 is a principal curvature, so $\theta_k = \frac{\pi}{2}$, for some $k, 1 \le k \le g$. Then, by (a), g must be odd and $\{\theta_i\}_{1 \le i \le g}$ are symmetric with respect to $\frac{\pi}{2}$. Then, by (a'), all multiplicities are same. Further, we see that the mean curvature of $\bar{M}$ is identically zero, namely, $\bar{M}$ is minimal. Besides, by (b), we have $g = 1$ or 3.

When $g = 1$, $\bar{M}$ must be a great sphere in S^n. Then M is a projective hypersurface.

Let $g = 3$. É. Cartan [Car] gave the complete classification of isoparametric hypersurfaces with three distinct principal curvatures (with same multiplicity) in spheres, up to isometry. Those are given by level hypersurfaces of $f = \det|S^n : S^n \to \mathbf{R}$. Here $\det : \mathcal{J}_0 \to \mathbf{R}$ is the determinant and $S^n \subset \mathcal{J}_0$ is the unit sphere with respect to the metric $\mathrm{tr}(\,\cdot\,) : \mathcal{J}_0 \times \mathcal{J}_0 \to \mathbf{R}$. Among them, minimal hypersurfaces are only $f^{-1}(0) \in S^n$. Hence $\bar{M} \subset S^n$ is isometric to $f^{-1}(0) = \{A \in S^n \mid \det(A) = 0\}$. Now an isometry of S^n is induced by an element $O(n+1) \subset \mathrm{GL}(n+1, \mathbf{R})$. Thus M is projectively equivalent to an example of Proposition 2. Therefore M is a Cartan hypersurface. □

This completes the proof of Theorem 3. □

Remark 10: There are known several alternative proofs of Cartan's theorem [Car]: See, for instance, [Kar], [KK], [CO].

Remark 11: It is suggested by H. Tazaki to the author, that Theorem 3 can be shown directly, using the concrete classification of (isometrically) homogeneous hypersurfaces in spheres by Hsiang and Lawson [HL], (see also [OT], [Ya]), and the formula of principal curvatures due to Takagi and Takahashi [TT].

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