モース理論入門　第9回

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石川剛郎

§16 A finite dimensional approximation to Ω^c

M: connected Riemannian manifold, $p, q \in M$

$\Omega = \Omega(M, p, q)$: space of piecewise smooth paths from p to q $[0,1] \to M$

$\rho : M \times M \to \mathbb{R}$ the distance function on M induced from Riemannian metric i.e

$$\rho(p,q) := \inf\{L(c) \mid c \in \Omega\}$$

Mの距離　$L(c) := \int_0^1 \|\dot{c}(t)\| dt$

$\omega, \omega' \in \Omega$　$s(t), s'(t)$ arclength

$$\left(s(t) := \int_0^t \|\dot{\omega}(t)\| dt, \quad \frac{ds}{dt} = \|\dot{\omega}(t)\| \right)$$

$$d(\omega, \omega') := \max_{0 \le t \le 1} \rho(\omega(t), \omega'(t)) + \left[\int_0^1 \left(\frac{ds}{dt} - \frac{ds'}{dt} \right)^2 dt \right]^{\frac{1}{2}}$$

Ωの距離

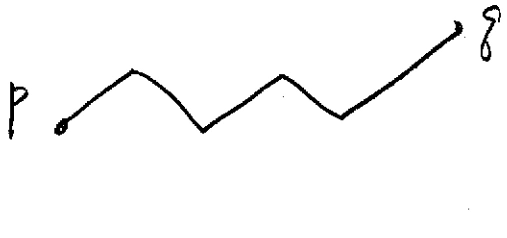
Then Energy functional $E : \Omega \to \mathbb{R}$ is continuous.

$c > 0$,　$\Omega^c := E^{-1}([0, c])$ closed in Ω

$\mathrm{Int}\,\Omega^c := E^{-1}([0, c))$ open in Ω

$0 = t_0 < t_1 < \cdots < t_k = 1$

$$\Omega(t_0, t_1, \cdots, t_k) := \left\{ \omega \in \Omega \;\middle|\; \begin{array}{l} (\omega(0) = p,\ \omega(1) = q) \\ \omega|[t_{i-1}, t_i] \text{ geodesic} \\ \quad (1 \le i \le k) \end{array} \right\}$$



折れているかもしれない場所指定

$\Omega(t_0,t_1,\cdots,t_k)^c := \Omega^c \cap \Omega(t_0,t_1,\cdots,t_k)$
"piecewise geodesics" with energy $\leq c$

$\mathrm{Int}\,\Omega(t_0,t_1,\cdots,t_k)^c := (\mathrm{Int}\,\Omega^c) \cap \Omega(t_0,t_1,\cdots,t_k)$
" $< c$

Lemma 16.1. M : complete Riemann manifold $\Omega^c \neq \emptyset$
$0 = t_0 < t_1 < \cdots < t_k = 1$ sufficiently fine (十分細かい)
$\Rightarrow$ $\mathrm{Int}\,\Omega(t_0,t_1,\cdots,t_k)^c =: B$ smooth manifold of dimension $n(k-1)$.

(!) $S = \{x \in M \mid \rho(x,p) \leq \sqrt{c}\}$ closed ball
compact ($\because$ M complete, Cor. 10.10)

$\forall \omega \in \Omega^c$, $\forall \tau \in [0,1]$

p $\rho(\omega(0), \omega(\tau)) \leq \sqrt{c}$

$\left(\because\ L(\omega_\tau)^2 \leq E(\omega_\tau) \leq c \quad L(\omega_\tau) \leq \sqrt{c}\right)$
$\omega_\tau = \omega|[0,\tau]$

Therefore $\omega([0,1]) \subseteq S$.

$\exists \varepsilon > 0$, $\forall x,y \in S$ $\rho(x,y) < \varepsilon$
$\Rightarrow \exists 1$ geodesic from x to y with $L < \varepsilon$ (Cor. 10.8).
smoothly depending on x, y.

Suppose $t_i - t_{i-1} < \dfrac{\varepsilon^2}{c}$ $(i = 1,2,\cdots,k)$.

Then

$$\rho(\omega(t_{i-1}),\omega(t_i))^2 \leq L(\underbrace{\omega|[t_{i-1},t_i]}_{\text{geodesic}})^2 = (t_i - t_{i-1})(E_{t_{i-1}}^{t_i}\omega)$$
$$\leq (t_i - t_{i-1})(E\omega) \leq (t_i - t_{i-1})c < \varepsilon^2$$

$\therefore$ $\rho(\omega(t_{i-1}),\omega(t_i)) < \varepsilon$

$$\varphi : \Omega(t_0,t_1,\cdots,t_k)^c \to M \times M \times \cdots \times M = M^{k-1}$$
$$\omega \longmapsto (\omega(t_1), \omega(t_2), \cdots, \omega(t_{k-1}))$$

continuous, injection.

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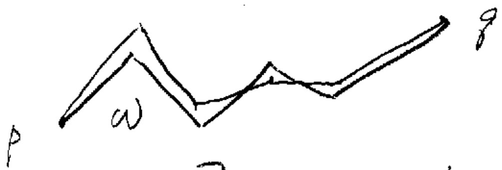
continuous, injective

homeomorphism onto the image $\varphi(\Omega(t_0, t_1, \cdots, t_k)^c)$



$\left(\forall \varepsilon > 0 \quad \varphi(\omega) \in {}^{\exists}W \right.$ 開近傍, $\left. \varphi(\omega') \in W \Rightarrow d(\omega', \omega) < \varepsilon \right)$

$\varphi(\mathrm{Int}\,\Omega(t_0, t_1, \cdots, t_k)^c) \subset M^{k-1}$ open

(${}^{\forall}\omega \in \mathrm{Int}\,\Omega(t_0, t_1, \cdots, t_k)^c$, $\varphi(\omega) \in {}^{\exists}W$ 開近傍, ${}^{\forall}(p_1, p_2, \cdots, p_{k-1}) \in W$

${}^{\exists}\omega' \in \mathrm{Int}\,\Omega(t_0, t_1, \cdots, t_k)^c$, $\varphi(\omega') = (p_1, p_2, \cdots, p_{k-1})$

ω' は $p, p_1, p_2, \cdots, p_{k-1}, q$ を測地線で結んだもの. $E(\omega')$ は $E(\omega) < c$

に十分近い $E(\omega') < c$)

$B := \mathrm{Int}\,\Omega(t_0, t_1, \cdots, t_k)^c$

has the structure of C^∞ manifold induced from $\varphi(B) \underset{\text{open}}{\subset} M^{k-1}$

$\dim B = n(k-1)$ //

$E' : B \to \mathbb{R} \qquad E' := E|B$

E' : C^∞ function

($E' \circ (\varphi|B)^{-1}$ depends $p_1, p_2, \cdots, p_{k-1}$ smoothly.)

$$E' \circ (\varphi|B)^{-1}(p_1, p_2, \cdots, p_{k-1}) = \sum_{i=1}^{k} \frac{\rho(p_i, p_{i-1})^2}{t_i - t_{i-1}}$$

Theorem 16.2 M : complete Riemann manifold, $c \in \mathbb{R}$
$\Omega^c \neq \phi$, $a < c$
$\Rightarrow$ • $B^a := (E')^{-1}([0, a])$ compact
deformation retract of $\Omega^a (= E^{-1}([0, a]))$.
• $C(E') (:= \{$ critical paths γ of $E' \})$
$= \{ \gamma : [0,1] \to M$ geodesic $\omega(0) = p$, $\omega(1) = q$, $L(\gamma) < \sqrt{c} \}$
• index of E'_{**} at γ = index of E_{**} at γ
• nullity of ——— = nullity of ———

※ (B, E') ; finite dimensional model of (Ω^a, E).

Proof of Th. 16.2 $\quad (S = \{ x \in M \mid \rho(x, p) \leq \sqrt{c} \})$

$$\varphi(B^a) = \{ (p_1, p_2, \cdots, p_{k-1}) \in S \times S \times \cdots \times S \mid \sum_{i=1}^{k} \frac{\rho(p_i, p_{i-1})^2}{t_i - t_{i-1}} \leq a \}$$

closed in $S \times S \times \cdots \times S$. S : compact (Cor 10.10)
$\therefore S \times S \times \cdots \times S$ compact $\therefore \varphi(B^a)$ compact, $\therefore B^a$ compact.

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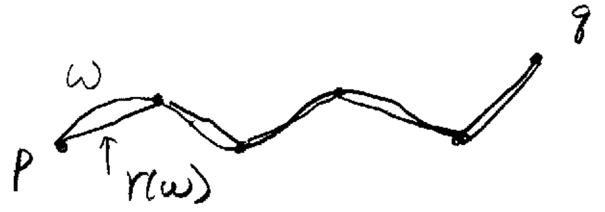
r : Int $\Omega^c \to B$ retraction
$r(\omega) \in B$: piecewise geodesics connecting $\omega(t_{i-1}), \omega(t_i)$



(r が well-defined になるように $0 = t_0 < t_1 < \cdots < t_k = 1$
をとっておいた.)

$r|_B = id_B$. $\quad r$: continuous

Define

$r_u : \mathrm{Int}\,\Omega^c \longrightarrow \mathrm{Int}\,\Omega^c \qquad (u \in [0,1])$

For $t_{i-1} \leq u \leq t_i$

$r_u(\omega)(t) = r(\omega)(t) \qquad (0 \leq t \leq t_{i-1})$

$r_u(\omega)|_{[t_{i-1}, u]}$: $\exists 1$ minimal geodesic from $\omega(t_{i-1})$ to $\omega(t_i)$.

$r_u(\omega)(t) = \omega(t) \qquad (u \leq t \leq 1)$

$r_0 = id|_{\mathrm{Int}\,\Omega^c}$, $r_1 = r$

$R : \mathrm{Int}\,\Omega^c \times [0,1] \longrightarrow \mathrm{Int}\,\Omega^c$ continuous.

$R(\omega, u) = r_u(\omega)$

$E(r_u(\omega)) \leq E(\omega)$ （エネルギーが非増加）

$\therefore$ B^a : deformation retract of Ω^a.

$C(E|_{\mathrm{Int}\,\Omega^c}) = \{$ geodesics γ with $E(\gamma) < c \}$

$\qquad = C(E')$

(Th. 12.2 : First variation formula)

γ : geodesic

$T_\gamma B \cong T_\gamma \Omega(t_0, t_1, \cdots, t_k)$

$(v_1, \cdots, v_{k-1}) \longmapsto$ Jacobi field J with

$J(t_i) = v_i \quad (i = 1, \cdots, k-1)$

By Lemma 15.4 （講義ノート 8.4 ）

the proof of Th. 16.2 is completed //

Theorem 16, 3
M: complete Riemann $p, q \in M$, $a > 0$
p, q are not conjugate along any geodesic of length $\leq \sqrt{a}$
$\Rightarrow$ Ω^a has homotopy type of finite CW-complex with one cell e^λ (of dim. λ) corresponding just to each geodesics $\in \Omega^a$ of index λ.

nullity $= 0$

(!) $E: \Omega^a \to \mathbb{R}$ has only non-degenerate critical points by Th. 14.1 (講義ノート 7-6)
So does $E': B^a \to \mathbb{R}$ by Th. 16.2.
Then by Th. 3.5 (講義ノート 2-3)
we have the conclusion

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