モース理論入門　第8回　8-1

石川剛郎

§15 The index theorem (指標定理)

M: connected Riemannian manifold

$\gamma : [0,1] \to M$ geodesic

$E_{**} : T_\gamma\Omega \times T_\gamma\Omega \to \mathbb{R}$ Hessian ($\Leftarrow$ §13) of Energy functional

$$\begin{pmatrix}\text{index of } E_{**} \\ \text{at } \gamma\end{pmatrix} := \sup\left\{ \dim S \;\middle|\; \begin{array}{l} S \subset T_\gamma\Omega \text{ subspace} \\ E_{**}|_{S\times S} \text{ negative definite} \end{array} \right\}$$

$E_{**}(W,W) \leq 0$ for $\forall W \in S$

$= 0 \iff W = 0$

Theorem 15.1 (Morse)

index $= \sum_{0<t<1}$ multiplicity of $\gamma(0)$ & $\gamma(t)$ along γ

$\uparrow$ ノート 7-5 §14

and

index $< \infty$

Corollary 15.2 $p \in M$, $\gamma : [0,1] \to M$ geodesic

$\gamma(0) = p$

$\Rightarrow \#\{ t \mid 0<t<1,\ p \;\&\; \gamma(t) \text{ conjugate along } \gamma \} < \infty$

$\gamma : [0,1] \to M$ geodesic
$\exists$ subdivision of $[0,1]$
$0 = t_0 < t_1 < \cdots < t_k = 1$ such that
$\gamma|[t_{i-1}, t_i]$ is minimal (i.e. minimizer)
$\gamma([t_{i-1}, t_i]) \subset {}^{\exists}U$ open
$\forall p, q \in U$ $\exists$ minimal geodesic from p to q (∵ Lemma 10.3, $[0,1]$ compact)

Define $T_\gamma\Omega(t_0, t_1, \cdots, t_k) \subset T_\gamma\Omega$ by

$T_\gamma\Omega(t_0, t_1, \cdots, t_k) := \{ W \in T_\gamma\Omega \mid$

1) $W|[t_{i-1}, t_i]$ Jacobi field along $\gamma|[t_{i-1}, t_i]$

2) $W(0) = 0$, $W(1) = 0$ $\}$

$T' := \{ W \in T_\gamma\Omega \mid W(t_i) = 0 \quad (0 \le i \le k) \}$

Lemma 15.3 (i) $T_\gamma\Omega = T_\gamma\Omega(t_0, t_1, \cdots, t_k) \oplus T'$

(ii) $W_1 \subset T_\gamma\Omega(t_0, t_1, \cdots, t_k)$, $W_2 \subset T'$

$\Rightarrow E_{**}(W_1, W_2) = 0$

(iii) $E_{**}|T' \times T'$ positive definite

∵ (i) $W \in T_\gamma\Omega$. $\exists_1 W_1 \in T_\gamma\Omega(t_0, t_1, \cdots, t_k)$
s.t. $W_1(t_i) = W(t_i) \quad (0 \le i \le k)$
(ノート 7-8 の議論も参照)

$W - W_1 \in T'$
$\therefore T_\gamma\Omega = T_\gamma\Omega(t_0, t_1, \cdots, t_k) + T'$
If $W \in T_\gamma\Omega(t_0, t_1, \cdots, t_k) \cap T'$, then
$W = W_1 = 0$

┘

<Th.13.1

(ii) $\frac{1}{2}E_{**}(W_1, W_2)$

$$= -\sum_t \langle \underset{t_i\text{で}0}{W_2(t)}, \underset{t_i\text{以外で}0}{\Delta_t \frac{DW_1}{dt}} \rangle - \int_0^1 \langle W_2, \underbrace{\frac{D^2 W_1}{dt^2} + R(\dot\gamma, W_1)\dot\gamma}_{W_1\ \text{Jacobi で}0} \rangle dt$$

$= 0$

(iii) $E_{**}(W, W) = \frac{d^2 E\circ\bar\alpha}{du^2}(0)$,

where $\bar\alpha : (-\varepsilon, \varepsilon) \to \Omega$ is a variation of γ with the variation vector field W.

For $W \in T'$, take $\bar\alpha$ s.t. $\bar\alpha(u)(t_i) = \gamma(t_i)$

($\gamma(t_i)$を動かさない)

$\gamma|[t_{i-1}, t_i]$ minimal

$$\therefore\ E(\bar\alpha(u)) = \sum_i E_{t_{i-1}}^{t_i}(\bar\alpha(u)|_{[t_{i-1}, t_i]}) \geq \sum_i E_{t_{i-1}}^{t_i}(\gamma|_{[t_{i-1}, t_i]})$$

$$= E(\gamma) = E(\bar\alpha(0))$$

$\therefore\ \frac{d^2 E\circ\bar\alpha}{du^2}(0) \geq 0 \qquad \therefore\ E_{**}(W, W) \geq 0$

$E_{**}|_{T'\times T'}$ is positive semi-definite

Let us show $W \in T'$, $W \neq 0 \Rightarrow E_{**}(W, W) > 0$

Suppose $E_{**}(W, W) = 0$.

$\forall W_2 \in T'$, $\underline{\forall c \in \mathbb{R}}$

$0 \leq E_{**}(W + cW_2, W + cW_2) = 2c E_{**}(W, W_2) + c^2 E_{**}(W_2, W_2)$

$(\forall W_2,\ E_{**}(W_2, W_2) = 0) \Rightarrow E_{**}(W, W_2) = 0$

If $\exists W_2$ with $E_{**}(W_2, W_2) > 0$, then set $\begin{cases} a = E_{**}(W_2, W_2) > 0 \\ b = E_{**}(W, W_2) \end{cases}$

$0 \leq 2cb + c^2 a = a(c + \frac{b}{a})^2 - \frac{b^2}{a}$ hold for $\forall c \in \mathbb{R}$

Take $c = -\frac{b}{a}$, then $0 \leq -\frac{b^2}{a}$, $\therefore\ b (= E_{**}(W, W_2)) = 0$

By (ii), W belongs to null space of E_{**}. By Th.14.1, W: Jacobi.

$\therefore\ W \in T_\gamma\Omega(t_0, t_1, \cdots, t_k) \cap T'$ $\therefore\ W = 0$ //

Lemma 15.4

index E_{**} at γ

$= \text{index } E_{**}|_{T_\gamma\Omega(t_0,t_1,\cdots,t_k)\times T_\gamma\Omega(t_0,t_1,\cdots,t_k)}$

$< \infty$

∵ $S \subset T_\gamma\Omega$, $E_{**}|_{S\times S}$ negative definite

$\Rightarrow S \subset T_\gamma\Omega(t_0,t_1,\cdots,t_k)$ by Lemma 15.3

This implies the first equality.

Since $T_\gamma\Omega(t_0,t_1,\cdots,t_k)$ is of finite dimensional, we have index is finite.

To show Theorem 15.1, set

$\gamma_\tau := \gamma|_{[0,\tau]}$

$\lambda(\tau) :=$ index of E_{**} along γ_τ $(0<\tau\leq 1)$

Then we claim:

(1) $\tau \leq \tau' \Rightarrow \lambda(\tau) \leq \lambda(\tau')$ (広義単調増加)

∵ $\exists S \subset T_{\gamma_\tau}\Omega$, $\dim S = \lambda(\tau)$ and $(E_0^\tau)_{**}|_{S\times S}$ negative definite.

Extend (each element of) S by 0 to $S' \subset T_{\gamma_{\tau'}}\Omega$

Then $\dim S' = \lambda(\tau)$; $(E_0^{\tau'})_{**}|_{S'\times S'}$ neg. definite

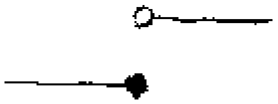
∴ $\lambda(\tau) \leq \lambda(\tau')$ //

Th. 13.1

(2) $\lambda(\tau) = 0$ for sufficiently small $\tau > 0$

∵ γ_τ: minimal, Lemma 13.6 //

(3) $\lambda(\tau-\varepsilon)=\lambda(\tau)$ for sufficiently small $\varepsilon>0$

( continuous from left 左連続

$\lambda:(0,1]\to\mathbb{R}_{\geq 0}$ に注意)

∵ Let $t_i<\tau\leq t_{i+1}$. Lemma 15.4

$t_0 (=0)$ を含めているので

$\lambda(\tau) = \text{index } E_{**}|T_{\gamma_\tau}\Omega(t_0,t_1,\dots,t_i,\tau)$

$T_\gamma\Omega(t_0,t_1,\cdots,t_i,\tau) \cong T_{\gamma(t_1)}M\times\cdots\times T_{\gamma(t_i)}M$

$H_\tau := E_{0**}^{\tau}|T_{\gamma(t_1)}M\times\cdots\times T_{\gamma(t_i)}M$ depends on τ continuously.

If H_τ is neg. def. on S, $\dim S=\lambda(\tau)$, then

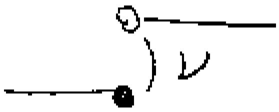
$H_{\tau-\varepsilon}$ is neg. def on S for suff. small $\varepsilon>0$

$\therefore \lambda(\tau-\varepsilon)\geq\lambda(\tau)$. By (1), $\lambda(\tau-\varepsilon)=\lambda(\tau)$.

(4) $\nu :=$ nullity of $(E_0^\tau)_{**}$

退化次数

$\lambda(\tau+\varepsilon)=\lambda(\tau)+\nu$ for suff. small $\varepsilon>0$

( $\gamma(0)$ & $\gamma(\tau)$ conj. points multiplicity ν

Th 14.1)

∵ $\Sigma := T_{\gamma(t_1)}M\times\cdots\times T_{\gamma(t_i)}M$. $\dim\Sigma = in$

$H_\tau|\Sigma$ index $\lambda(\tau)$, nullity ν. $\exists S\subset\Sigma$, $H_\tau|S\times S$

positive def. $\dim S = in-\lambda(\tau)-\nu$

$H_{\tau+\varepsilon}$ positive def. on S.

$\therefore \lambda(\tau+\varepsilon)\leq in-(in-\lambda(\tau)-\nu)=\lambda(\tau)+\nu$

Let us show $\lambda(\tau+\varepsilon)\geq\lambda(\tau)+\nu$

$W_1,\cdots,W_{\lambda(\tau)}\in T_{\gamma_\tau}\Omega$ $\gamma_\tau:[0,\tau]\to M$ ($\leftarrow \lambda(\tau)$ の定義)

$(E_0^\tau)_{**}(W_i,W_j)$ negative def.

$J_1,\cdots,J_\nu\in T_{\gamma_\tau}\Omega$ linearly indep ($\leftarrow$ Th. 14.1)

$\Leftrightarrow \nabla_{\dot\gamma_\tau}J_1(\tau),\cdots,\nabla_{\dot\gamma_\tau}J_\nu(\tau)$ lin. indep.

$\exists X_1, \cdots, X_\nu \in T_{\gamma_{\tau+\varepsilon}}\Omega$ s.t.

$$\left(\langle D_{\dot\gamma_\tau} J_h(\tau), X_k(\tau) \rangle \right)_{\substack{1 \le h \le \nu \\ 1 \le k \le \nu}}$$

unit matrix

Extend W_i, J_h by 0 on $[\tau, \tau+\varepsilon]$.

By the 2nd variation formula (Th. 13.1)

$$(E_0^{\tau+\varepsilon})_{**}(J_h, W_i) = 0, \quad (E_0^{\tau+\varepsilon})_{**}(J_h, X_k) = 2\delta_{hk},$$

$W_1, \cdots, W_{\lambda(\tau)}$, $c^{-1}J_1 - cX_1, \cdots, c^{-1}J_\nu - cX_\nu$

: vector fields along $\gamma_{\tau+\varepsilon}$.

- These are linearly independent, and $(E_0^{\tau+\varepsilon})_{**}$ is <u>negative definite</u> on the subspace generated by them.

The representation matrix of $(E_0^{\tau+\varepsilon})_{**}$ is given by

$$P = \begin{pmatrix} \left((E_0^{\tau})_{**}(W_i, W_j) \right) & cA \\ c^{t}A & -4I + c^2 B \end{pmatrix}$$

for some matrices A, B.

If c is suff. small, P is negative definite.

Let $\sum_i a_i W_i + \sum b_j (c^{-1}J_j - cX_j) = 0$

Then $0 = (a, b)\, P \begin{pmatrix} a \\ b \end{pmatrix}$

$\therefore$ $a = 0$, $b = 0$

Hence $\lambda(\tau+\varepsilon) \geq \lambda(\tau) + \varepsilon$ //

Proof of Th. 15.1

(2) (3) (4) Lemma 15.4 //