モース理論入門 第7回

(7-1)

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§13

M : connected Riemannian manifold, $p, q \in M$

$\Omega = \Omega(M; p, q)$ the set of conti. piecewise smooth paths ($\omega : [0,1] \to M$) from p to q

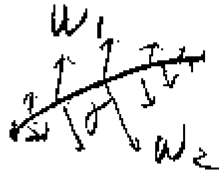
$\omega \in \Omega$, $T_\omega \Omega$ the set of conti, piecewise smooth vector fields along ω

Let $\gamma \in \Omega$ geodesic i.e. $D_{\dot\gamma} \dot\gamma = 0$ (the velocity vectors are parallel)

Then γ is a critical point of the Energy functional

$$E : \Omega \to \mathbb{R}, \quad E(\omega) = \int_0^1 \|\dot\omega\|^2 dt$$

Consider the Hessian of E at γ :



$$E_{**} : T_\gamma \Omega \times T_\gamma \Omega \to \mathbb{R}$$

To calculate E_{**}, take $W_1, W_2 \in T_\gamma \Omega$ and consider any two parameter variation

$$\bar\alpha : {}_{(0,0)\in} U (\subset \mathbb{R}^2) \longrightarrow \Omega(M; p, q)$$

of γ with $\alpha(u_1, u_2, t) = \bar\alpha(u_1, u_2)(t)$,

$\alpha(0,0,t) = \gamma(t)$, $\frac{\partial \alpha}{\partial u_1}(0,0,t) = W_1(t)$, $\frac{\partial \alpha}{\partial u_2}(0,0,t) = W_2(t)$

Then

$$E_{**}(W_1, W_2) := \frac{\partial^2 E(\bar\alpha(u_1, u_2))}{\partial u_1 \partial u_2}(0,0)$$

Theorem 13.1 (Second variation formula)

$$\frac{1}{2} \frac{\partial^2 E(\bar\alpha(u_1, u_2))}{\partial u_1 \partial u_2}(0,0) = -\sum_t \left\langle W_2(t), \Delta_t \frac{DW_1}{dt} \right\rangle - \int_0^1 \left\langle W_2(t), \frac{D^2 W_1}{dt^2} + (R(V, W_1)V)(t) \right\rangle dt$$

where $V = \dot\gamma$, $\Delta_t \frac{DW_1}{dt} = \frac{DW_1}{dt}(t^+) - \frac{DW_1}{dt}(t^-)$, $\left(\frac{DW_1}{dt} = D_{\dot\gamma} W_1 \right)$

"discontinuity"

(!) By (12.2), $\frac{1}{2}\frac{\partial E(\alpha)}{\partial u_2} = -\sum_t \langle \frac{\partial\alpha}{\partial u_2}, \Delta_t \frac{\partial\alpha}{\partial t}\rangle - \int_0^1 \langle \frac{\partial\alpha}{\partial u_2}, \nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial t}\rangle dt$

By (8.3),

$$\frac{1}{2}\frac{\partial^2 E(\alpha)}{\partial u_1 \partial u_2} = -\sum_t \langle \nabla_{\frac{\partial\alpha}{\partial u_1}} \frac{\partial\alpha}{\partial u_2}, \underbrace{\Delta_t \frac{\partial\alpha}{\partial t}}_{0}\rangle - \sum_t \langle \frac{\partial\alpha}{\partial u_2}, \nabla_{\frac{\partial\alpha}{\partial u_1}} (\Delta_t \frac{\partial\alpha}{\partial t})\rangle dt$$

$(u_1,u_2)=(0,0)$のとき

$$- \int_0^1 \langle \nabla_{\frac{\partial\alpha}{\partial u_1}} \frac{\partial\alpha}{\partial u_2}, \underbrace{\nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial t}}_{=0}\rangle dt - \int_0^1 \langle \frac{\partial\alpha}{\partial u_2}, \nabla_{\frac{\partial\alpha}{\partial u_1}} (\nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial t})\rangle dt$$

Set $(u_1,u_2)=(0,0)$ $(u_1,u_2)=(0,0)$のとき

$$\therefore \frac{1}{2}\frac{\partial^2 E(\alpha)}{\partial u_1 \partial u_2}(0,0) = -\sum_t \langle \frac{\partial\alpha}{\partial u_2}, \Delta_t (\nabla_{\frac{\partial\alpha}{\partial u_1}} \frac{\partial\alpha}{\partial t})\rangle dt \qquad (13.2)$$

$$- \int_0^1 \langle \frac{\partial\alpha}{\partial u_2}, \nabla_{\frac{\partial\alpha}{\partial u_1}} (\nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial t})\rangle dt$$

$$= -\sum_t \langle \frac{\partial\alpha}{\partial u_2}, \Delta_t (D_{\dot\gamma} W_1)\rangle dt$$

(8.7)
(9.2)

$$- \int_0^1 \langle W_2, \nabla_{\frac{\partial\alpha}{\partial t}} (\underbrace{\nabla_{\frac{\partial\alpha}{\partial u_1}} \frac{\partial\alpha}{\partial t}}_{= \nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial u_1}}) + R(\dot\gamma, W_1)\dot\gamma\rangle dt$$

$$(13.3)$$

$$= -\sum_t \langle W_2, \Delta_t (D_{\dot\gamma} W_1)\rangle dt$$

$$- \int_0^1 \langle W_2, D_{\dot\gamma}(D_{\dot\gamma} W_1) + R(\dot\gamma, W_1)\dot\gamma\rangle dt \quad //$$

Cor. 13.4 $E_{**}(W_1, W_2) = \frac{\partial^2 E(\alpha)}{\partial u_1 \partial u_2}(0,0)$ is well defined symmetric bilinear function.

(!) Th. 13.1 //

Remark 13.5

$$E_{**}(W,W) = \frac{d^2E(\bar{\alpha})}{du^2}(0) \quad \text{for}$$

$\bar{\alpha} : (-\varepsilon, \varepsilon) \to \Omega$, $\frac{d\bar{\alpha}}{du}(0) = W$.

↑ 1-param. variation

(∵) $\beta(u_1, u_2, t) := \alpha(u_1+u_2, t)$

$$\left.\frac{\partial^2 E(\bar{\beta})}{\partial u_1 \partial u_2}\right|_{(u_1,u_2)=(0,0)} = \left.\frac{\partial^2 E(\bar{\alpha})}{\partial u^2}\right|_{u=0}$$

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最短測地線

Lemma 13.6. $\gamma \in \Omega(M; p, q)$: minimal geodesic

$\Rightarrow$ E_{**} positive <u>semi</u> - difinite

$\text{index}_\gamma E_{**} = 0$

(∵) $E(\bar{\alpha}(u)) \underset{\uparrow \gamma \text{ minimal}}{\geq} E(\gamma) = E(\bar{\alpha}(0))$.

$\therefore \frac{d^2 E(\bar{\alpha})}{du^2}(0) \geq 0 \quad \therefore E_{**}(W,W) \geq 0$

$\forall W \in T_\gamma\Omega$ $\forall \bar{\alpha}$

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Example $p = q$, γ : constant geodesic

$\Rightarrow$ γ : minimal geodesic

Then $\text{index}_\gamma E_{**} = 0$

§14

M : Riemannian manifold $\dim M = n$

$\gamma : [a,b] \to M$ geodesic $(\nabla_{\dot\gamma}\dot\gamma = 0)$

$J : [a,b] \to TM$ vector field along γ, $J(t) \in T_{\gamma(t)}M$

Definition:

J : Jacobi field (ヤコビ場) of γ

$\underset{\text{def}}{\Longleftrightarrow}$ $\nabla_{\dot\gamma}^2 J + \underbrace{R}_{\text{curvature tensor}}(\dot\gamma, J)\dot\gamma$ … (∗) Jacobi equation

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$P_1, \cdots, P_n$ orthonormal parallel field along γ.

Let $J(t) = \sum_{i=1}^{n} {}^{\exists}f^i(t) P_i(t)$

$$\nabla_{\dot\gamma} J = \sum_i \left(\frac{df^i}{dt} P_i + f^i \overset{=0}{\nabla_{\dot\gamma} P_i} \right)$$

($\nabla_{\dot\gamma}$ は γ に沿って $\frac{d}{dt}$ で共変微分すること)

$$\nabla_{\dot\gamma}^2 J = \sum_i \nabla_{\dot\gamma}\left(\frac{df^i}{dt} P_i\right)$$

$$= \sum_i \frac{d^2 f^i}{dt^2} P_i + \frac{df^i}{dt} \overset{=0}{\nabla_{\dot\gamma} P_i}$$

$$R(\dot\gamma, J)\dot\gamma = \sum_{j=1}^{n} f^j R(\dot\gamma, P_j)\dot\gamma$$

$$(*) \Longleftrightarrow \sum_{i=1}^{n} \frac{d^2 f^i}{dt^2} P_i + \sum_{j=1}^{n} f^j R(\dot\gamma, P_j)\dot\gamma = 0$$

$$\Longleftrightarrow {}^{\forall}k \quad \langle \qquad , P_k \rangle = 0$$

$$\Longleftrightarrow {}^{\forall}k \quad \frac{d^2 f^k}{dt^2} + \sum_{j=1}^{n} f^j \underbrace{\langle R(\dot\gamma, P_j)\dot\gamma, P_k \rangle}_{= a_j^k} = 0$$

$$(*) \Leftrightarrow \frac{d^2 f^k}{dt^2} + \sum_{k=1}^{n} a_j^{\ k}(t) f^j(t) = 0 \quad (1 \leq k \leq n)$$

2nd order ODE

J is determined by $J(a)$, $\nabla_{\dot\gamma} J(a) \in T_{\gamma(a)} M$
initial data

i.e. $f^1(a), \cdots, f^n(a), \frac{df^1}{dt}(a), \cdots, \frac{df^n}{dt}(a)$.

初期値問題ではなく境界値問題を考える

Definition : $p, q \in M$ $\gamma : [a,b] \to M$ geodesic $a \neq b$

$\gamma(a) = p$, $\gamma(b) = q$, possibly $p = q$

p and q are <u>conjugate</u> along γ (共役点)

$\overset{\text{def}}{\Longleftrightarrow} \exists$ Jacobi field of γ with $J(a) = 0$, $J(b) = 0$.

multiplicity of γ for p & q

$$:= \dim \{ J : \text{Jacobi field of } \gamma \mid \underbrace{J(a) = 0,\ J(b) = 0}_{2n\text{個の条件}} \}$$

p, q conjugate $\Leftrightarrow$ multiplicity > 0

$\gamma \in \Omega = \Omega(M; p, q)$ geodesic $J : [0,1] \to M$

$E_{**} : T_\gamma \Omega \times T_\gamma \Omega \to \mathbb{R}$ Hessian

$N := \{ W_1 \in T_\gamma \Omega \mid E_{**}(W_1, W_2) = 0 \text{ for } {}^\forall W_2 \in T_\gamma \Omega \}$

linear subspace of $T_\gamma \Omega$

$\nu := \dim_{\mathbb{R}} N$ nullity of γ (退化次数)

Theorem 14.1

$N = \{ J : \text{Jacobi field of } \gamma \mid J(0)=0,\ J(1)=0 \}$

E_{**} degenerate $\Leftrightarrow$ $\gamma(0)=p$, $\gamma(1)=q$ are conjugate points along γ

nullity of γ (ν) = multipicity of p & q along γ

(:) $\supset$: γ smooth, Th13.1

$\subset$ Suppose $W_1 \in N$ and W_1 is smooth on subintervals of $0 = t_0 < t_1 < \cdots < t_k = 1$.

Using Th13.1, we have $W_1|[t_{i-1}, t_i]$ Jacobi field of $\gamma|[t_{i-1}, t_i]$, and $D_{\dot{\gamma}} W_1$ is continuous.

Then by the uniqueness of initial value problem of Jacobi equation, we see W_1 smooth and a Jacobi field of γ. //

Remark : $0 \leq \nu < n$

(:) By the condition $J(0)=0$, we have $\nu \leq n$.

$\exists$ Jacobi field $\underset{t\dot{\gamma}(t)}{J}$ s.t $J(0)=0$, $\underset{\dot{\gamma}(1)}{J(1)} \neq 0$. //

Example 1. M flat ($\Leftrightarrow R(X,Y)Z = 0$ for $\forall X,Y,Z$)

Then Jacobi equation $\frac{d^2 f^i}{dt^2} = 0$ f_i : linear

$\forall \gamma$ geodesic $\nexists$ conj. pts, E_{**} : non-degenerate.

Lemma 14.3 $\bar{\alpha} : (-\varepsilon, \varepsilon) \to \Omega$ variation of a geodesic γ by geodesics

$\Rightarrow$ $W(t) := \frac{\partial \alpha}{\partial u}(0,t)$ is a Jacobi field of γ

変分ベクトル場

(!) $\alpha(u,t) := \bar{\alpha}(u)(t)$

$$D_{\frac{\partial\alpha}{\partial t}} \frac{\partial \alpha}{\partial t} = 0.$$

$$0 = D_{\frac{\partial\alpha}{\partial u}}\left(D_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial t}\right) \underset{(9.2)}{=} D_{\frac{\partial\alpha}{\partial t}}\left(D_{\frac{\partial\alpha}{\partial u}} \frac{\partial\alpha}{\partial t}\right) + R\left(\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial u}\right)\frac{\partial\alpha}{\partial t}$$

$$\underset{(8.7)}{=} D_{\frac{\partial\alpha}{\partial t}}\left(D_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial u}\right) + R\left(\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial u}\right)\frac{\partial\alpha}{\partial t}$$

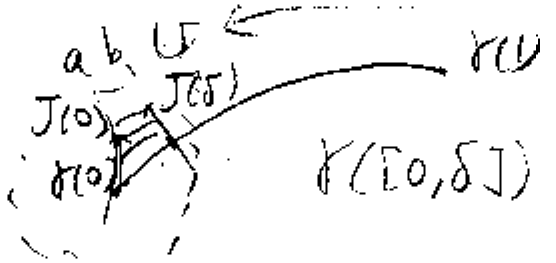
$\therefore$ $\left.\frac{\partial\alpha}{\partial u}\right|_{u=u_0}$ satisfies Jacobi equation of $\left.\alpha\right|_{u=u_0}$

$u_0 = 0$ $\quad \left.\frac{\partial\alpha}{\partial u}\right|_{u=0}$ is a Jacobi field of γ. //

Lemma 14.4 $\forall$ Jacobi field J of γ is obtained by a variation of γ by geodesics.

$\left(\exists\ \alpha : (-\varepsilon, \varepsilon) \times [0,1] \to M,\ \left.\alpha\right|_{u=\text{const}} \text{ geodesic}\right)$

(!)



$\gamma([0,\delta]) \subset U$

U : suff. small nbd of $\gamma(0)$
$\exists \delta > 0$

$J(0) \in T_{\gamma(0)}M \quad J(\delta) \in T_{\gamma(\delta)}M$

$\forall a, b : (-\varepsilon, \varepsilon) \to U$, $a(0) = \gamma(0)$ $\dot{a}(0) = J(0)$, $b(0) = \gamma(\delta)$, $\dot{b}(0) = J(\delta)$.

$\alpha : (-\varepsilon, \varepsilon) \times [0, \delta] \to M$ s.t. $\bar{\alpha}(u)$ is the unique (minimal) geodesic from $a(u)$ to $b(u)$.

$\mathcal{J}(\gamma)$: the space of Jacobi field along γ.
$\dim \mathcal{J}(\gamma) = 2n$.

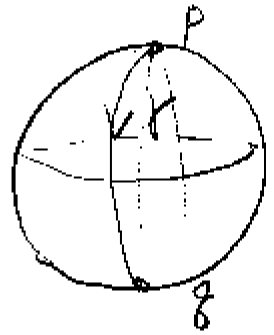
$$\underset{(2n)}{\mathcal{J}(\gamma)} \xrightarrow{k} \underset{(2n)}{\mathcal{J}(\gamma|[0,\delta])} \xrightarrow{l} \underset{(2n)}{T_{\gamma(0)}M \times T_{\gamma(\delta)}M}$$

$$J \longmapsto J|[0,\delta] \longmapsto (J(0), J(\delta))$$

k, l are surjective.
$\therefore$ k, l are isomorphism

Take ε sufficently small $\alpha : (-\varepsilon, \varepsilon) \times [0,\delta] \to M$
extends to a variation $\alpha : (-\varepsilon,\varepsilon) \times [0,1] \to M$
by geodesics with $\frac{\partial \alpha}{\partial u}\big|_{u=0} = J$.

Example 2 $M = S^n$, p, q anti-podal points



p and q are conj. of multiplicity $n-1$

rotations fixing p, q gives $n-1$ independent Jacobi fields.
by Lemma 14.3

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