モース理論入門　第6回　6-1

石川剛郎

Part III The calculus of variations applied to geodesics
変分法

§11

M : smooth ($= C^\infty$) manifold , $p, q \in M$ fixed

$\Omega = \Omega(M; p, q) :=$

$$\{\omega \mid \omega : [0,1] \to M \text{ continuous piecewise smooth curve } \omega(0) = p,\ \omega(1) = q\}$$

Define the "tangent space $T_\omega \Omega$ by

$$T_\omega \Omega := \{W : [0,1] \to TM \text{ continuous, piecewise smooth vector field along } \omega,\ W(0) = 0,\ W(1) = 0\}$$

tangent bundle

$T_\omega \Omega$ vector space

$$(\lambda_1 W_1 + \lambda_2 W_2)(t) := \lambda_1 W_1(t) + \lambda_2 W_2(t)$$

(The space of variational vector fields of ω.)
変分ベクトル場

6-2

Definition:

$\omega \in \Omega(M;p,q)$ fixed

$\bar{\alpha} : (-\varepsilon, \varepsilon) \longrightarrow \Omega(M;p,q)$: variation of ω
変分

$\underset{\text{def}}{\Longleftrightarrow}$ 1) $\bar{\alpha}(0) = \omega$

2) For $\alpha : (-\varepsilon,\varepsilon) \times [0,1] \to M$ defined by

$$\alpha(u,t) := \underset{\Omega}{\underline{\bar{\alpha}(u)}}(t) \in M$$

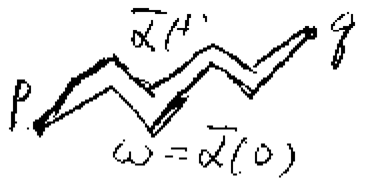
$\exists$ refinement (細分) of $[0,1]$ $\quad 0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$

α is C^∞ on $(-\varepsilon,\varepsilon) \times [t_{i-1}, t_i]$ $\quad (i = 1, \cdots, n)$

3) $\frac{\partial \alpha}{\partial u}(u,t)\big|_{u=0}$ continuos on $[0,1]$

(原文には仮定されていないが必要)

Remark 4) $\alpha(u,0) = p \qquad \alpha(u,1) = q \qquad (u \in (-\varepsilon,\varepsilon))$



Regard $\bar{\alpha} : (-\varepsilon,\varepsilon) \to \Omega(M;p,q)$ as a "smooth path" in Ω

Define $\frac{d\bar{\alpha}}{du}(0) : [0,1] \longrightarrow TM$ by

$$\frac{d\bar{\alpha}}{du}(0)(t) := \frac{\partial \alpha}{\partial u}(0,t) \qquad (\text{for } t \neq t_i)$$

non-smooth point

$\frac{d\bar{\alpha}}{du}(0) \in T_\omega \Omega$

variational vector field of the variation $\bar{\alpha}$.

Remark $\forall W \in T_\omega\Omega \ \exists \bar{\alpha}$ variation of ω such that $\frac{d\bar{\alpha}}{du}(0) = W$

(!) $\bar{\alpha}(u)(t) := \exp_{\omega(t)}(u W(t))$

$F : \Omega = \Omega(M; p, q) \to \mathbb{R}$ a function (a functional)

Definition: $\omega \in \Omega$ is a <u>critical path</u> (臨界道)

$\overset{\Longleftrightarrow}{\text{def}}$ $\left.\dfrac{dF(\bar{\alpha}(u))}{du}\right|_{u=0} = 0$ for $\forall$ variation $\bar{\alpha}$ of ω.

extremal ともいう

---

§12

M: Riemannian manifold with the metric g.

$v \in T_pM$, $\|v\| := \sqrt{g_p(v, v)} = \sqrt{\langle v, v \rangle}$

$\omega \in \Omega(M; p, q)$ (conti., piecewise smooth path from p to q)

$0 \le a < b \le 1$

$$E_a^b(\omega) := \int_a^b \|\dot{\omega}(t)\|^2 \, dt \qquad E := E_0^1$$

$\dot{\omega} = \dfrac{d\omega}{dt}$

energy functional

$$L_a^b(\omega) := \int_a^b \|\dot{\omega}(t)\| \, dt \qquad L := L_0^1$$

length functional

$$(L_a^b(\omega))^2 \le (b-a) E_a^b(\omega) \quad \text{(Schwarz inequality)}$$

$$= \iff \|\dot{\omega}(t)\| \text{ constant.}$$

$\gamma : [0,1] \to M$

$\gamma \in \Omega(M;p,q)$ minimal geodesic (minimizer)
最短測地線 (ノート5-6)

$$\Rightarrow \quad E(\gamma) = L(\gamma)^2 \leq L(\omega)^2 \leq E(\omega)$$

5-1 　 等号はωがminimizerのとき 　 等号はωのparameterが弧長に比例のとき 　 Schwarz

Lemma 12.1 M : complete, $p, q \in M$

5.6 $d = \rho(p,q)$ ($\Rightarrow \exists$ minimal geodesic connecting p and q of length d)

$\Rightarrow$ $E : \Omega(M;p,q) \to \mathbb{R}$ has the minimum d^2 on the set of minimal geodesics from p and q.

$$E^{-1}(d^2) = \{\text{minimal geodesics}\}$$

Find a criterion of critical paths of E ← energy

Theorem 12.2 (1st variational formula) $\omega \in \Omega(M,p,q)$
$\bar{\alpha} : (-\varepsilon, \varepsilon) \to \Omega$ variation of ω
$W_t := \frac{\partial \alpha}{\partial u}(0,t)$. Then

$$\frac{1}{2} \frac{dE(\bar{\alpha}(u))}{du}\Big|_{u=0} = -\sum_t \langle W_t, \Delta_t V\rangle - \int_0^1 \langle W_t, A_t\rangle\, dt$$

holds,

where $V_t := \dot{\omega}(t)$ velocity vector

$\Delta_t V := V_{t^+} - V_{t^-}$ discontinuity of V_t

$A_t := \frac{D}{dt}\frac{d\omega}{dt} \left(= \nabla_{\dot{\omega}} \dot{\omega}\right)$ acceleration vector of ω

(covariant derivative)

Proof of Th. 12.2 $\quad \alpha(u,t) = \bar{\alpha}(u)(t)$ smooth on subintervals $0 = t_0 < t_1 < \cdots < t_k = 1$

$$\frac{dE(\bar{\alpha}(u))}{du} = \frac{d}{du}\int_0^1 \left\langle \frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t} \right\rangle dt$$

$$= \int_0^1 \frac{\partial}{\partial u}\left\langle \frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t} \right\rangle dt$$

$$= \int_0^1 2\left\langle \nabla_{\frac{\partial\alpha}{\partial u}} \frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t} \right\rangle dt$$

$$\underset{\text{Lemma 8.7}}{=} \int_0^1 2\left\langle \nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial t} \right\rangle dt$$

On $[t_{i-1}, t_i]$

$$\int_{t_{i-1}}^{t_i} \left\langle \nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial t} \right\rangle dt$$

$$= \left\langle \frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial t} \right\rangle \Bigg|_{t=t_{i-1}^+}^{t=t_i^-} - \int_{t_{i-1}}^{t_i} \left\langle \frac{\partial\alpha}{\partial u}, \nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial t} \right\rangle dt$$

$$\therefore \frac{1}{2}\frac{\partial E(\bar{\alpha}(u))}{\partial u} = \sum_{i=0}^{k-1} \left\langle \frac{\partial\alpha}{\partial u}, V_{t_i^-} - V_{t_{i-1}^+} \right\rangle - \int_0^1 \left\langle \frac{\partial\alpha}{\partial u}, \nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial t} \right\rangle dt$$

$$= -\sum_{i=0}^{k-1} \left\langle \frac{\partial\alpha}{\partial u}, V_{t_{i-1}^+} - V_{t_i^-} \right\rangle - \int_0^1 \left\langle \frac{\partial\alpha}{\partial u}, \nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial t} \right\rangle dt$$

Let $u=0$.

$$\frac{1}{2}\frac{dE(\bar{\alpha}(u))}{du}\Bigg|_{u=0} = -\sum_t \langle W_t, \Delta_t V\rangle - \int_0^1 \langle W_t, A_t\rangle dt$$

//

Corollary 12.3 $\omega \in \Omega(M; p, q)$

ω: critical path for E $\Leftrightarrow$ ω: geodesic

(!) (ω geodesic $\Leftrightarrow$ $\nabla_{\dot\omega}\dot\omega = 0$)

($\Leftarrow$)

ω smooth, $\Delta_t V = 0$, $A_t = 0$

By Th 12.2. $\left.\frac{dE(\bar\alpha(u))}{du}\right|_{u=0} = 0$

$\therefore$ ω: critical path.

($\Rightarrow$)

Take a variation $\bar\alpha$ with variation vect. field $W_t = f(t)A_t$

s.t. $f(t) \geq 0$, $f(t) = 0 \Leftrightarrow t = t_i$

By Th 12.2, $\frac{1}{2}\left.\frac{dE(\bar\alpha(u))}{du}\right|_{u=0} = -\int_0^1 f(t)\langle A_t, A_t\rangle dt$

$\therefore$ $\omega|_{[t_i, t_{i+1}]}$ geodesic.

Take a variation with $W_{t_i} = \Delta_{t_i} V$

$\frac{1}{2}\left.\frac{dE(\bar\alpha(u))}{du}\right|_{u=0} = -\sum_t \langle \Delta_{t_i} V, \Delta_{t_i} V\rangle$

$\therefore$ $\Delta_t V = 0$ $\therefore$ ω : C^1.

By the uniqueness of differential equation on geodesics

ω is C^∞

//