モース理論入門　第5回

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M: connected Riemannian manifold.

smooth curve $\gamma : I \to M$ is a <u>geodesic</u> (測地線)

$\overset{\text{def}}{\Longleftrightarrow}$ $\nabla_{\dot\gamma}\dot\gamma = 0$ 　 ↑interval (区間) 　 $\left(\dot\gamma = \frac{d\gamma}{dt}\right)$

$\left(\frac{D}{dt}\frac{d\gamma}{dt}\right)$

◎ γ geodesic $\Rightarrow$ $\|\dot\gamma\|$ constant

(∵) $\frac{d}{dt}(\|\dot\gamma\|^2) = \frac{d}{dt}\langle\dot\gamma, \dot\gamma\rangle = 2\langle\nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle = 0$ //

$$s(t) := \int_{t_0}^{t} \|\dot\gamma(t)\|\,dt + s(t_0)$$ ← Take $s(t_0)$ for a fixed $t_0 \in I$

arclength function

If $\|\dot\gamma\| = 0$, then s is constant.

If $\|\dot\gamma\| = c \neq 0$ 　 $s(t) = c(t - t_0) + s(t_0)$

$s = {}^{\exists}a\,t + {}^{\exists}b$ 　 $a, b \in \mathbb{R}$ 　 $a \neq 0$.

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local coord. $x^1, \cdots, x^n$ of M

$$\gamma(t) = (x^1(t), \cdots, x^n(t))$$

$$\dot\gamma(t) = \sum_{i=1}^{n} (x^i)' \frac{\partial}{\partial x^i}$$

$$\nabla_{\dot\gamma}\dot\gamma = \sum_i (x^i)'' \frac{\partial}{\partial x^i} + \sum (x^i)' \nabla_{\dot\gamma}\frac{\partial}{\partial x^i}$$

$$= \sum_i (x^i)'' \frac{\partial}{\partial x^i} + \sum_{i,j} (x^i)'(x^j)' \nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}$$

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}$$

$\Gamma_{ji}^k(x(t))$

$$\therefore \nabla_{\dot\gamma}\dot\gamma = \sum_k \left(\frac{d^2x^k}{dt^2} + \sum_{i,j} \frac{dx^i}{dt} \overset{\frac{dx^j}{dt}}{\cancel{x^j}} \Gamma_{ji}^k \right)\frac{\partial}{\partial x^k}$$

$$\gamma : \text{geodesic} \iff \frac{d^2x^k}{dt^2} + \sum_{i,j} \Gamma_{ji}^k \overset{\frac{dx^j}{dt}}{\cancel{x^j}} \frac{dx^i}{dt} = 0$$

non-linear 2nd order ordinary differential equation

Lemma 10.2. M : Riemannian manifold
$\forall p_0 \in M$, $p_0 \in {}^{\exists}U$ open neighborhood $\exists \varepsilon > 0$ such that
$\forall p \in U$, $\forall v \in T_pM$ with $\|v\| < \varepsilon$, $\exists 1$ $\gamma_v : (-2, 2) \to M$
geodesic satisfying

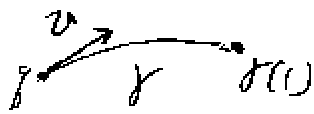
$$\gamma_v(0) = p, \quad \frac{d\gamma_v}{dt}(0) = v$$

◎ Exponential map (指数写像)

$q \in M$ $\quad v \in T_qM$ $\quad \|v\|$ sufficiently small

$\Rightarrow$ $\exists 1$ geodesic $\gamma : [0,1] \to M$

such that $\gamma(0) = q$, $\frac{d\gamma}{dt}(0) = v$ $\quad$ ($\because$ 10.2)



$$\exp_q(v) := \gamma(1) \in M$$

$$\exp_q : \{ v \in T_qM \mid \|v\| < \varepsilon \} \longrightarrow M$$ <u>exponential map</u>

Remark : $\gamma(t) = \exp_q(tv)$ holds.

Definition (M, g) <u>geodesically complete</u>
$\forall q \in M$, $\forall v \in T_q M$, $\exp_q(v)$ is defined.

By the uniqueness of geodesics by initial condition, $\forall$ geodesic $\gamma_0 : [a,b] \to M$ is (uniquely) extended to a geodesic $\gamma : \mathbb{R} \to M$ if M is geodesically complete.

Tangent bundle:

$$TM = \{(p, v) \mid p \in M,\ v \in T_p M\}$$

$(U; u^1, \cdots, u^n)$ local coord. of M

$q \in U$, $T_q M \ni v$

$$v = \sum_{i=1}^{n} t^i \frac{\partial}{\partial u^i}\Big|_q$$

$(TU = U \times \mathbb{R}^n, u^1, \cdots, u^n, t^1, \cdots, t^n)$ local coord. of TM

Lemma 10.3 $\forall p \in M$, $\exists W$ open neighborhood of p, $\exists \varepsilon > 0$

(1) $\forall q_1, q_2 \in W$ are connected by $\exists 1$ geodesic of length $< \varepsilon$.

(2) For geodesic $t \mapsto \exp_{q_1}(tv)$ connecting q_1, q_2, the initial condition $(q_1, v) \in TM$ depends smoothly on (q_1, q_2)

(3) $\forall q \in W$, $\exp_q : \{v \in T_q M \mid \|v\| < \varepsilon\} \to \underset{\substack{\cup \\ W}}{U_q} \subset M$ is a diffeomorphism.

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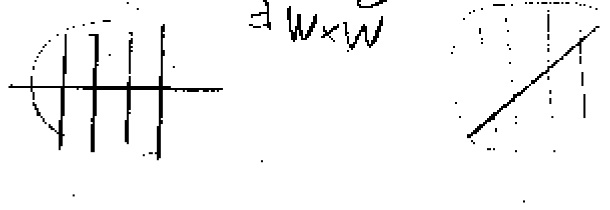
(1) $\forall p \in M$, $(p, 0) \in \exists V \underset{\text{open}}{\subset} TM$, $F : V \to M \times M$

$F(q, v) := (q, \exp_q(v))$, $F(p, 0) = (p, p)$

inverse map theorem: $(p, 0) \in \exists V' \underset{\text{open}}{\subset} V$,

$F|V' : V' \to \underset{\substack{\cup \\ \exists W \times W}}{F(V')} \subset M \times M$, $\exists p \in \exists W \underset{\text{open}}{\subset} M$



Lemma 10,3

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Theorem 10,4 $p \in M$ $\overset{\exists}{p \in W} \subset M$, $^{\exists}\varepsilon > 0$
$q_0, q_1 \in W$, $\gamma : [0,1] \to M$ geodesic connecting q_0, q_1 in M with length $< \varepsilon$, $\omega : [0,1] \to M$ any piecewise smooth path connecting q_0, q_1. Then

$$\int_0^1 \|\dot\gamma\| dt \leqq \int_0^1 \|\dot\omega\| dt.$$

The equality holds if and only if $\omega([0,1]) = \gamma([0,1])$

◎ A geodesic is a "local minimizer".

Lemma 10.5 In U_q as in Lemma 10.3, the geodesic through q intersect orthogonally with $\{\exp_q(v) \mid v \in T_qM,\ \|v\| = \text{const}\}$

(証) $^{\forall}v : I \to \{v \in T_qM \mid \|v\| = 1\} \subset T_qM$ curve

We show

$t \mapsto \exp_q(r_0 v(t))$ $\quad 0 < r_0 \leq \varepsilon$ $\quad$ and

$r \mapsto \exp_q(r\, v(t_0))$ $\quad t_0 \in I$

are intersect orthogonally at $\exp_q(r_0 v(t_0))$.

$f(r,t) := \exp_q(r\, v(t))$

It is sufficient to show $\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle = 0$

$$\frac{\partial}{\partial r}\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle = \left\langle \underbrace{\nabla_{\frac{\partial f}{\partial r}} \frac{\partial f}{\partial r}}_{0}, \frac{\partial f}{\partial t} \right\rangle + \left\langle \frac{\partial f}{\partial r}, \nabla_{\frac{\partial f}{\partial r}} \frac{\partial f}{\partial t} \right\rangle$$

$$= \left\langle \frac{\partial f}{\partial r}, \nabla_{\frac{\partial f}{\partial r}} \frac{\partial f}{\partial t} \right\rangle = \frac{1}{2}\frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial r} \right\rangle = \frac{1}{2}\frac{\partial}{\partial t}\|v(t)\|^2 = 0$$

$\nabla_{\frac{\partial f}{\partial r}} \frac{\partial f}{\partial t} = \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial r}$ (Lemma 8.7)

$\frac{\partial f}{\partial t}(0,t) = 0$ $\quad\therefore \left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle = 0$

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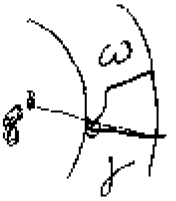
Lemma 10.6 $\omega : [a,b] \to U_q \setminus \{q\}$

$\omega(t) = \exp_q(r(t)v(t))$ $\quad 0 < r(t) < \varepsilon \quad \|v(t)\| = 1$

$\Rightarrow \int_a^b \|\dot\omega\| dt \geq |r(b) - r(a)| = \int_a^b \|\dot\gamma\| dt$



(!) $f(r,t) := \exp_q(r v(t))$, $\omega(t) = f(r(t), t)$

$\dot\omega = \frac{\partial f}{\partial r}\dot r(t) + \frac{\partial f}{\partial t}$

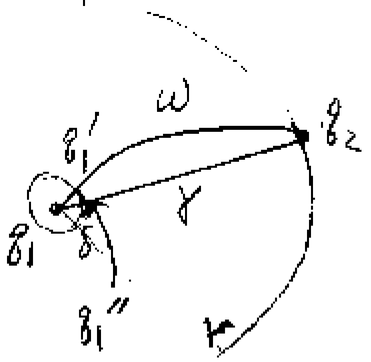
$\|\dot\omega\|^2 = |\dot r(t)|^2 + \|\frac{\partial f}{\partial t}\|^2 \geq |\dot r(t)|^2$ $\quad$ ($\because$ Lemma 10.5)

$\left(= \Leftrightarrow \frac{\partial f}{\partial t} = 0 \Leftrightarrow \dot v = 0 \right)$

$\int_a^b \|\dot\omega\| dt \geq \int_a^b |\dot r(t)| dt \geq |r(b) - r(a)|$

$=$ $\Leftrightarrow$ $r(t)$ monotone, $v(t)$ constant //

Proof of Th. 10.4



ωのq_1'からq_2の長さ

$\geq$ γのq_1''からq_2の長さ

Lemma 10.6

$= r - \delta$

ωのq_1からq_2の長さ $\geq r =$ γのq_1からq_2の長さ

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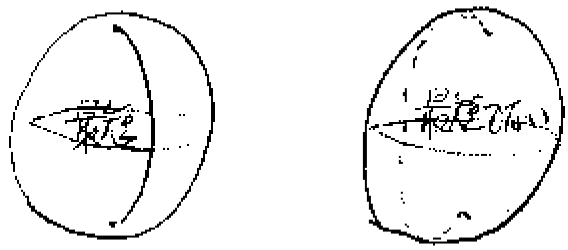
Definition. A geodesic $\gamma:[a,b]\to M$ is called a minimal (極小, 最短) geodesic or a minimizer (最短路)

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall \omega:[\alpha,\beta]\to M$ piecewise smooth path

$\omega(\alpha)=\gamma(a),\quad \omega(\beta)=\gamma(b)$

$$\Rightarrow \underbrace{\int_\alpha^\beta \|\dot\omega\|\,dt}_{\text{length of }\omega} \geq \underbrace{\int_a^b \|\dot\gamma\|\,dt}_{\text{length of }\gamma}$$

Remark. 志賀浩二先生は minimal を極小と訳しているが，それよりは最小あるいは(長さについてなので)最短と訳すのがよい。



(M,g) connected Riemannian manifold, $p,q\in M$

$$\rho(p,q):=\inf\left\{\int_\alpha^\beta \|\dot\omega\|\,dt \;\middle|\; \begin{array}{l}\omega:[\alpha,\beta]\to M \text{ piecewis smooth}\\ \omega(\alpha)=p,\ \omega(\beta)=q\end{array}\right\}$$

(M,ρ) metric space which gives the same topology on M (as a manifold)

Cor. 10.8 (of Th. 10.4) $\forall K\subset M$ compact, $\exists \delta>0$
$\forall p,q\in K$ with $\rho(p,q)\leq\delta$, $\exists 1$ minimal geodesic
$\gamma:[a,b]\to M$ $\quad\gamma(a)=p,\ \gamma(b)=q$.
γ depend smoothly on p,q (see Lemma 10.3)

Theorem 10.9 (Hopf-Rinow) M: connected Riemannian, M: geodesically complete $\Rightarrow$
$\forall p, q \in M$ $\exists!$ γ geodesic connecting p and q.

Corollary 10.10. M: geodesically complete
$\Rightarrow$ $\forall X \subseteq M$ bounded subset (for ρ), $\bar{X}$ (closure) is compact

M: geodesically complete $\Leftrightarrow$ M: complete as metric space

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$\Rightarrow$ diameter$(X) = d$, $\forall p \in X$ $\exp_p : T_pM \to M$
$X \subset \exp_p(\{v \in T_pM \mid \|v\| \le d\})$ compact
$\therefore$ $\bar{X}$: compact
$\forall$ Cauchy sequence converges. $\therefore$ M: complete
$\Leftarrow$ M: complete. By Lemma 10.3, M is geod. complete //

Example. $M = \mathbb{R}^n$ Euclidean space
geodesics are streight lines. geod. complete

. $M = S^n$ geodesics are great circles

$M = \mathbb{R}^2 \setminus \{0\}$

non-complete